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On Durrmeyer Variant of Mittag-Leffler Operators

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Abstract

We introduce a Durrmeyer version of the positive linear operators known as Mittag-Leffler operators. We analyze their main characteristics, such as the computation of moments, and provide convergence estimates with the traditional modulus of continuity techniques. Additionally, we study *A*-statistical convergence, and we examine the convergence rate of Lipschitz spaces.

Keywords: Mittag-Leffler function; Positive Linear Operators; Rate of Convergences; Statistical Convergences

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1 Introduction

A unique function that generalizes the exponential function, the Mittag-Leffler function is essential to many mathematics, physics, and engineering branches. This function, which bears the name of the Swedish mathematician Gösta Mittag-Leffler, is especially important for studying complex analysis and fractional calculus. The following is the definition of the Mittag-Leffler function:

$$MLF_{\Upsilon}(w) = \sum_{v=0}^{\infty} \frac{w^v}{\Gamma(1+\Upsilon v)}, \quad (\Upsilon \in \mathbb{C} \text{ with } Re(\Upsilon) > 0, w \in \mathbb{C}). \tag{1}$$

The Mittag-Leffler function extends the idea of exponential growth and decay to non-integer orders, making it crucial in modeling systems that display memory or hereditary features. The function has asymptotic features that are well-defined and helpful for examining the long-term behavior of fractional differential equation solutions. The Mittag-Leffler function can be reduced to more straightforward forms, such as the polynomial and exponential functions, given certain values of $\Upsilon = 0$. When standard integer-order derivatives are insufficient to describe a variety of events in physics, biology, and economics, the Mittag-Leffler function provides a basic answer. In materials science, the function is used to explain the viscoelastic behavior of materials, capturing the time-dependent reaction of materials under stress. The Mittag-Leffler function in control systems offers insights into stability and performance and helps with the study and design of systems with fractional dynamics. Applications of the function may be found in statistical mechanics and quantum mechanics, namely in the investigation of anomalous diffusion and other non-classical processes.

In many scientific fields, the Mittag-Leffler function is an effective mathematical tool that provides a solid foundation for comprehending complicated systems that behave in ways that are not integer order. Its importance in fractional calculus and its many applications highlight the need for more investigation and use in theoretical and practical study. The Mittag-Leffler function is still a crucial tool for expanding our knowledge of dynamic systems in a variety of disciplines as the science develops. The Mittag-Leffler function ((1)) was expanded to include two indices in 1905 by Wiman [1, 2]. It was defined as follows:

$$MLF_{\Upsilon,\varrho}(w) = \sum_{v=0}^{\infty} \frac{w^v}{\Gamma(\varrho+\Upsilon v)}, \quad (\Upsilon, \varrho \in \mathbb{C} \text{ with } Re(\Upsilon) > 0 \;\&\; Re(\varrho) > 0, w \in \mathbb{C}). \tag{2}$$

Because of its wide variety of applications in several domains, the Mittag-Leffler function and its modifications have garnered substantial interest from mathematicians, scientists, and engineers during the past 30 years. These include diffusion-like diffusive transpory, electric networks, fluid movemeny, rheology, probability, and statistical distribution theory. Readers are referred to prior important publications [3, 4, 5, 6] and others for a thorough discussion of the numerous properties, generalizations, and applications of this function.

Additionally, in the setting of a singular integral equation, Prabhakar [7] generalized the Mittag-Leffler function for three parameters as a kernel function in 1971. This generalization is described as follows:

$$MLF^{\eta}_{\Upsilon,\varrho}(w) = \sum_{v=0}^{\infty} \frac{(\eta)_n}{\Gamma(\varrho+\Upsilon v)} w^v, \quad (\Upsilon, \varrho \in \mathbb{C} \text{ with } Re(\Upsilon) > 0 \text{ and } Re(\varrho) > 0, \eta > 0, w \in \mathbb{C}), \tag{3}$$

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where $(\eta)_n = \eta(\eta+1)\ldots(\eta+n-1)$.
The Prabhakar function is the usual name for the three-parameter Mittag-Leffler function. The traditional Mittag-Leffler function MLF_Υ is obtained when both ϱ and η equal 1, whereas the two-parameter Mittag-Leffler function $MLF_{\Upsilon,\varrho}$ is generated by setting $\eta = 1$. Notably, the Mittag-Leffler function MLF_Υ easily generalizes the exponential function e^x.
The term-wise Laplace transform of $x^{\varrho-1}MLF^{\eta}_{\Upsilon,\varrho}(-\lambda x^\Upsilon)$, under the conditions $\mathrm{Re}(s) > 0$, $\mathrm{Re}(\varrho) > 0$, and $\lambda \in \mathbb{C}$ such that $|\lambda s^{-\Upsilon}| < 1$, yields the following sum:

$$\int_0^\infty e^{-sx}x^{\varrho-1}MLF^{\eta}_{\Upsilon,\varrho}(-\lambda x^\Upsilon)dx = \frac{s^{\Upsilon\eta-\varrho}}{(\lambda+s^\Upsilon)^\eta}. \tag{4}$$

A number of characteristics of Mittag-Leffler functions have been identified by Gorenflo et al. [8] and other sources, such as the survey by Giusti et al. [9], many of which are related to fractional calculus and its applications.
The Mellin-Barnes integral may be used to define the three-parameter Mittag-Leffler function, much like other functions of the Mittag-Leffler type. This is covered in [8, Chapter 3]. With $\mathrm{Re}(\eta) > 0$, let $\Upsilon \in [0,\infty)$ and $\varrho, \eta \in \mathbb{C}$. The following person provides the representation:

$$MFL^{\eta}_{\Upsilon,\varrho}(z) = \frac{1}{\eta(z)}\frac{1}{2\pi}\int_L \frac{\Gamma(s)\Gamma(\eta-s)}{\Gamma(\varrho-\Upsilon s)}(-z)^{-s}ds, \tag{5}$$

when $|\arg(z)| < \pi$. With $0 < c < \mathrm{Re}(\eta)$, the contour of integration stretches from $c-i\infty$ to $c+i\infty$. All of the integrand's poles are separated by this contour at $s = -\nu$, where $\nu \in \mathbb{N}\cup\{0\}$, and to the right at $s = n+\eta$, where $n \in \mathbb{N}\cup\{0\}$.
We can determine the Mellin transform of the generalized Mittag-Leffler function provided in (3) by applying the Mellin inversion formula.

$$\int_0^\infty t^{s-1}MLF^{\eta}_{\Upsilon,\varrho}(-wt)dt = \frac{\Gamma(s)\Gamma(\eta-s)}{w^s\Gamma(\eta)\Gamma(\varrho-s\Upsilon)}. \tag{6}$$

The modified Mittag-Leffler function (2) used by Özarslan [10] to approximate certain class of function and to discussed A-statistical approximation theorem for the following operators (7). For sequence of real number (b_n), $\varrho > 0$ is fixed. For $n \in \mathbb{N}$, the Mittag-Leffler operators defined by

$$L_n^{(\varrho)}(\psi,\theta) = \frac{1}{MLF_{1,\varrho}\left(\frac{n\theta}{b_n}\right)}\sum_{\nu=0}^\infty \psi\left(\frac{\nu b_n}{n}\right)\frac{(n\theta)^\nu}{b_n^\nu\Gamma(\nu+\varrho)} = \sum_{\nu=0}^\infty p_{n,\nu}(\theta,\varrho)\psi\left(\frac{\nu b_n}{n}\right), \tag{7}$$

where $p_{n,\nu}(\theta,\varrho) = \frac{1}{MLF_{1,\varrho}\left(\frac{n\theta}{b_n}\right)}\frac{(n\theta)^\nu}{b_n^\nu\Gamma(\nu+\varrho)}$. Let $\psi \in E := \left\{\psi \in C[0,\infty) : \lim_{n\to\infty}\frac{\psi(\theta)}{1+\theta^2} < \infty\right\}$, where the space of continuous functions defined on $[0,\infty)$ is shown by $C[0,\infty)$. It is crucial to remember that the Banach lattice E has the norm

$$||\psi||_2 := \frac{|\psi(\theta)|}{1+\theta^2}.$$

According to [11, 12], the operators specified in (7) reduce to the well-known Szasz-Mirakjan operators when $\varrho = 1$.
Özarslan and Aktuglu [13] presented the Mittag-Leffler operators' Kantorovich variation in 2016 as defined below:

$$P_m^{(\varrho)}(\psi,\theta) = \frac{1}{b_m MLF_{1,\varrho}\left(\frac{m\theta}{b_m}\right)}\sum_{\nu=0}^\infty\left[\int_{\nu b_m/m}^{(\nu+1)b_m/m}\psi(s)\,ds\right]\frac{(m\theta)^\nu}{b_m^\nu\Gamma(\nu+\varrho)}, m \in \mathbb{N}. \tag{8}$$

In particular $\varrho = 1$, the positive linear operators (8) convert into the operators studied in [14]. One should note thay, q-analogue of Mittag-Leffler operators (7) was introduced and studied by İcöz and Cekim [15], An integral-type generalization of Mittag-Leffler operators using beta basis functions and Szász-Mirakjan basis functions was investigated in [16] and [17] respectively. The Kantorovich-Mittag-Leffler operators based on q-integers were established in [18]. Some more positive linear operators in the approximation theory were studied in [19, 20, 21, 22, 23]. Motivated by these works, we introduced the following Durrmeyer variant of Mittag-Leffler operators as follow:

$$D_n^{(\varrho)}(\psi,\theta) = \frac{n\Gamma(\varrho)}{MLF_{1,\varrho}(n\theta)}\sum_{\nu=0}^\infty\frac{(n\theta)^\nu}{\Gamma(\nu+\varrho)}\int_0^\infty\frac{MLF^{\varrho}_{1,\varrho}(-nt)(nt)^\nu}{\Gamma(\nu+\varrho)}\psi(t)dt \tag{9}$$

$$= \sum_{\nu=0}^\infty p_{n,\nu}(\theta,\varrho)\frac{\langle q_{n,\nu}(\theta,\varrho),\psi\rangle}{\langle p_{n,\nu}(\theta,\varrho),1\rangle}, \tag{10}$$

provided the integral is exist. Here, $q_{n,\nu}(\theta,\varrho) = \dfrac{MLF^{\varrho}_{1,\varrho}(-n\theta)(n\theta)^\nu}{\Gamma(\nu+\varrho)}$, $p_{n,\nu}(\theta,\varrho) = \dfrac{n\Gamma(\varrho)}{MLF_{1,\varrho}(n\theta)}\dfrac{(n\theta)^\nu}{\Gamma(\nu+\varrho)}$, and $\langle g,h\rangle = \displaystyle\int_0^\infty g(t)h(t)dt$.

It can be observed that when $\varrho = \eta = 1$, the operators described in (9) coincide with those examined in [24, 25]. The following lemmas will be essential for our subsequent discussion.

Lemma 1.1. *For any $\theta \geq 0$, $\varrho > 1$, and $n \in \mathbb{N}$, let $\phi_\theta^2(y) = (y-\theta)^2$. With this, we obtain*

1. $L_n^{(\varrho)}(1;\theta) = 1$;
2. $\left|L_n^{(\varrho)}(y;\theta) - \theta\right| \leq \frac{|1-\varrho|}{n}$;
3. $\left|L_n^{(\varrho)}(y^2;\theta) - \theta^2\right| \leq \frac{(2|1-\varrho|+1)}{n}\theta + \frac{2(1-\varrho)^2 + |1-\varrho| + |1-\varrho||\varrho-2|}{n^2}$;
4. $\left|L_n^{(\varrho)}(\phi_\theta^2;\theta) - \theta^2\right| \leq \frac{(4|1-\varrho|+1)}{n}\theta + \frac{2(1-\varrho)^2 + |1-\varrho| + |1-\varrho||\varrho-2|}{n^2}$.

Direct calculations allow us to prove the following lemma:

Lemma 1.2. *For any $\theta \geq 0$, $\varrho > 1$, and $n \in \mathbb{N}$, let $\phi_\theta^2(y) = (y-\theta)^2$. With this, we obtain*

1. $\left|D_n^{(\varrho)}(1;\theta) - 1\right| \leq 0$;
2. $\left|D_n^{(\varrho)}(y,\theta) - \theta\right| \leq \frac{|1-\varrho|+1}{n}$;
3. $\left|D_n^{(\varrho)}(y^2;\theta) - \theta^2\right| \leq \frac{2(|1-\varrho|+2)}{n}\theta + \frac{2(1-\varrho)^2 + 4|1-\varrho| + |1-\varrho||\varrho-2| + 5}{n^2}$;
4. $D_n^{(\varrho)}(\phi_\theta^2;\theta) \leq \frac{2(|1-\varrho|+3)}{n}\theta + \frac{2(1-\varrho)^2 + 4|1-\varrho| + |1-\varrho||\varrho-2| + 5}{n^2}$.

Proof. Using the identity (6), we get

$$\begin{aligned} D_n^{(\varrho)}(1,\theta) &= \frac{n\Gamma(\varrho)}{MLF_{1,\varrho}(n\theta)} \sum_{v=0}^{\infty} \frac{(n\theta)^v}{\Gamma(v+\varrho)} \int_0^\infty \frac{MLF_{1,\varrho}^{\varrho}(-nt)(nt)^v}{\Gamma(v+\varrho)} dy, \\ &= \frac{1}{MLF_{1,\varrho}(n\theta)} \sum_{v=0}^{\infty} \frac{(n\theta)^v}{\Gamma(v+\varrho)} \frac{\Gamma(v+1)}{\Gamma(v+\varrho)}, \end{aligned}$$

Utilizing the property that the gamma function is monotonically increasing for positive real numbers, we obtain

$$\left|D_n^{(\varrho)}(1,\theta)\right| \leq \left|L_n^{(\varrho)}(1,\theta)\right| = 1. \tag{11}$$

Hence, $\left|D_n^{(\varrho)}(1,\theta) - 1\right| \leq 0$. Now, Using the identity (6), we get

$$D_n^{(\varrho)}(y;\theta) = \frac{1}{nMLF_{1,\varrho}(n\theta)} \sum_{v=0}^{\infty} \frac{(n\theta)^v}{\Gamma(v+\varrho)} \frac{\Gamma(v+2)}{\Gamma(v+\varrho)}.$$

By using Lemma 1.1, we derive

$$\begin{aligned} \left|D_n^{(\varrho)}(y,\theta) - \theta\right| &\leq \left| \frac{1}{nMLF_{1,\varrho}(n\theta)} \sum_{v=0}^{\infty} \frac{(n\theta)^v}{\Gamma(v+\varrho)} (v+1) - \theta \right| \\ &\leq \left|L_n^{(\varrho)}(y,\theta) - \theta\right| + \frac{1}{n} \leq \frac{|1-\varrho|+1}{n}. \end{aligned}$$

To compute the second moment, we utilize identity (6) as follows:

$$\begin{aligned} D_n^{(\varrho)}(y^2;\theta) &= \frac{n\Gamma(\varrho)}{MLF_{1,\varrho}(n\theta)} \sum_{v=0}^{\infty} \frac{(n\theta)^v}{\Gamma(v+\varrho)} \int_0^\infty \frac{MLF_{1,\varrho}^{\varrho}(-nt)(nt)^v}{\Gamma(v+\varrho)} y^2 dt \\ &= \frac{1}{MLF_{1,\varrho}(n\theta)} \sum_{v=0}^{\infty} \frac{(n\theta)^v}{\Gamma(v+\varrho)} \frac{\Gamma(k+3)}{n^2\Gamma(v+\varrho)}, \end{aligned}$$

Using the property that the gamma function is monotonically increasing for positive real numbers, we obtain

$$\begin{aligned} \left|D_n^{(\varrho)}(y^2;\theta) - \theta^2\right| &\leq \left| \frac{1}{MLF_{1,\varrho}(n\theta)} \sum_{v=0}^{\infty} \frac{(n\theta)^v}{\Gamma(v+\varrho)} \frac{(v+1)(v+2)}{n^2} - \theta^2 \right| \\ &\leq \left|L_n^{(\varrho)}(y^2,\theta) - \theta^2\right| + \frac{3}{n}\left|L_n^{(\varrho)}(y,\theta)\right| + \frac{2}{n^2}\left|L_n^{(\varrho)}(1,\theta)\right|, \\ &\leq \frac{(2|1-\varrho|+1)}{n}\theta + \frac{2(1-\varrho)^2 + |1-\varrho| + |1-\varrho||\varrho-2|}{n^2} + \frac{3\theta}{n} + \frac{3(|1-\varrho|+1)}{n^2} + \frac{2}{n^2} \\ &= \frac{2(|1-\varrho|+2)}{n}\theta + \frac{2(1-\varrho)^2 + 4|1-\varrho| + |1-\varrho||\varrho-2| + 5}{n^2}. \end{aligned}$$

Finally,

$$\begin{aligned} D_n^{(\varrho)}(\phi_\theta^2;\theta) &\leq \left|L_n^{(\varrho)}(y^2,\theta)-\theta^2\right|+2\theta\left|L_n^{(\varrho)}(y,\theta)-\theta\right|+\theta^2\left|L_n^{(\varrho)}(1,\theta)-1\right| \\ &\leq \frac{2\left(|1-\varrho|+3\right)}{n}\theta+\frac{2(1-\varrho)^2+4|1-\varrho|+|1-\varrho||\varrho-2|+5}{n^2}, \end{aligned}$$

which completes the proof. □

2 Rate of Convergence

The rate of convergence of $D_n^{(\varrho)}$ is covered in this section. It can be seen from the following lemma that $D_n^{(\varrho)}$ maps E into itself.

Lemma 2.1. *Let $\varrho > 1$ remain constant. A constant $M(\varrho)$ exists such that for any $\theta \in [0,\infty)$ and $n \in \mathbb{N}$, where $\omega(\theta) = \frac{1}{1+\theta^2}$,*

$$\omega(\theta)D_n^{(\varrho)}\left(\frac{1}{\omega},\theta\right) \leq M(\varrho)$$

holds. Furthermore, we have

$$\|D_n^{(\varrho)}(\psi,\cdot)\|_2 \leq M(\varrho)\|\psi\|_2.$$

for every $\psi \in E$.

Proof. From Lemma 1.2, we conclude that

$$\begin{aligned} \omega(\theta)D_n^{(\varrho)}\left(\frac{1}{\omega},\theta\right) &= \frac{1}{1+\theta^2}\left[D_n^{(\varrho)}(1,\theta)+D_n^{(\varrho)}\left(y^2,\theta\right)\right] \\ &\leq \frac{1}{1+\theta^2}\left[1+\theta^2+\frac{2\left(|1-\varrho|+2\right)}{n}\theta+\frac{2(1-\varrho)^2+4|1-\varrho|+|1-\varrho||\varrho-2|+5}{n^2}\right] \\ &\leq M(\varrho). \end{aligned}$$

This follows from the following inequality:

$$\omega(\theta)\left|D_n^{(\varrho)}(\psi,\theta)\right| = \omega(\theta)\left|D_n^{(\varrho)}\left(\omega\frac{f}{\omega},\theta\right)\right| \leq \|\psi\|_2\omega(\theta)D_n^{(\varrho)}\left(\frac{1}{\omega},\theta\right) \leq M(\varrho)\|\psi\|_2.$$

In the inequality above, the answer may be obtained by taking the supremum over $\theta \in [0,\infty)$. □

Remember that

$$\omega_L(\psi,\delta) = \sup\{|\psi(t)-\psi(\theta)| : |t-\theta| \leq \delta,\ \theta,t \in [0,L]\}.$$

This is the typical modulus of continuity of ψ on the closed interval $[0,L]$. As is often known, we have

$$\lim_{\delta\to 0}\omega_L(\psi,\delta) = 0.$$

for a function $\psi \in E$. The rate of convergence of the operators $D_n^{(\varrho)}(\psi,\theta)$ for every $\psi \in E$ is given in the following theorem.

Theorem 2.2. *The modulus of continuity of ψ on the interval $[0,L+1]$ (where $L > 0$) may be represented as $\varrho > 1$, $\psi \in E$, and $\omega_{L+1}(\psi,\delta)$. Next, we have*

$$\|D_n^{(\varrho)}(\psi,\cdot)-\psi(\cdot)\|_{C[0,L]} \leq M_\psi(\varrho,L)\delta_n(\varrho,L)+2\omega_{L+1}\left(\psi,\delta_n^{1/2}(\varrho,L)\right),$$

where $\delta_n(\varrho,B) = \frac{L}{n\varrho}$ and $M_\psi(\varrho,L)$ is a constant that depends on ψ, ϱ, and L.

Proof. Let $\varrho > 0$ remain constant. The well-known inequality for $\theta \in [0,L]$ and $t \leq L+1$ is as follows:

$$|\psi(t)-\psi(\theta)| \leq \omega_{L+1}(\psi,|t-\theta|) \leq \left(1+\frac{|t-\theta|}{\delta}\right)\omega_{L+1}(\psi,\delta), \tag{12}$$

where $\delta > 0$. Now, applying the inequality $t-\theta > 1$ for $\theta \in [0,L]$ and $t > L+1$, we obtain

$$\begin{aligned} |\psi(t)-\psi(\theta)| &\leq A_f(1+\theta^2+y^2) \\ &\leq A_f(2+3\theta^2+2(t-\theta)^2) \\ &\leq 6A_f(1+L^2)(t-\theta)^2. \end{aligned} \tag{13}$$

Using equations (12) and (13), we can conclude that for all $\theta \in [0,L]$ and $t \geq 0$, we have

$$|\psi(t)-\psi(\theta)| \leq 6A_f(1+L^2)(t-\theta)^2+\left(1+\frac{|t-\theta|}{\delta}\right)\omega_{L+1}(\psi,\delta). \tag{14}$$

Therefore,

$$\left|D_n^{(\varrho)}(\psi,\theta)-\psi(\theta)\right| \leq 6A_f(1+L^2)D_n^{(\varrho)}\left(\phi_\theta^2,\theta\right)+\left(1+\frac{D_n^{(\varrho)}(|t-\theta|,\theta)}{\delta}\right)\omega_{L+1}(\psi,\delta).$$

Using Lemma 1.2 and the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned}
\left|D_n^{(\varrho)}(\psi,\theta)-\psi(\theta)\right| &\leq 6A_f(1+L^2)D_n^{(\varrho)}\left(\phi_\theta^2,\theta\right)+\left(1+\frac{\left[D_n^{(\varrho)}\left(\phi_\theta^2,\theta\right)\right]^{1/2}}{\delta}\right)\omega_{L+1}(\psi,\delta)\\
&\leq 6A_f(1+L^2)\left(\frac{2\,(|1-\varrho|+3)}{n}L+\frac{2(1-\varrho)^2+4|1-\varrho|+|1-\varrho||\varrho-2|+5}{n^2}\right)\\
&\quad+\left(1+\frac{\left[\frac{2\,(|1-\varrho|+3)}{n}L+\frac{2(1-\varrho)^2+4|1-\varrho|+|1-\varrho||\varrho-2|+5}{n^2}\right]^{1/2}}{\delta}\right)\omega_{L+1}(\psi,\delta)\\
&\leq M_\psi(\varrho,B)\delta_n(\varrho,L)+2\omega_{L+1}\left(\psi,(\delta_n(\varrho,L))^{1/2}\right),
\end{aligned}$$

where $M_\psi(\varrho,B)=6A_f(1+L^2)$ and $\delta_n(\varrho,L)=\frac{2\,(|1-\varrho|+3)}{n}L+\frac{2(1-\varrho)^2+4|1-\varrho|+|1-\varrho||\varrho-2|+5}{n^2}$. Whence the result follows. □

3 Examining Convergence Rates for Lipschitz-Type functions

The rate of convergence of the operators $D_n^{(\varrho)}$ for locally Lipschitz functions is examined in this section. Note that $C_L[0,\infty)$, which represents the space of bounded continuous functions on the interval $[0,\infty)$, is a subspace of the space of locally Lipschitz functions. Szász has addressed the order of approximation for modified Lipschitz class functions [11].

Theorem 3.1. *Let $0<\mu\leq 1$ and let S be any bounded subset of the interval $[0,\infty)$. If $\psi\in C_L[0,\infty)$ is locally Lipschitz with respect to the parameter Υ, it satisfies the condition*

$$|\psi(y)-\psi(\theta)|\leq D|y-\theta|^\mu,\quad y\in S \text{ and } \theta\in[0,\infty), \tag{15}$$

for some constant D. We therefore have

$$|D_n^{(\varrho)}(\psi,\theta)-\psi(\theta)|\leq D\left\{\left(\frac{2\,(|1-\varrho|+3)}{n}\theta+\frac{2(1-\varrho)^2+4|1-\varrho|+|1-\varrho||\varrho-2|+5}{n^2}\right)^{\mu/2}+2(d(\theta,S))^\mu\right\},$$

where $d(\theta,S)$ is the distance from θ to the set S, given as

$$d(\theta,S)=\inf\{|y-\theta|:y\in S\}.$$

and D is a constant that depends on μ and ψ.

Proof. The closure of S in $[0,\infty)$ is represented as $\overline{S}$. At $\theta_0\in\overline{S}$, there is a point such that $|\theta-\theta_0|=d(\theta,S)$. Using the triangle inequality, we arrive at

$$|\psi(y)-\psi(\theta)|\leq|\psi(y)-\psi(\theta_0)|+|\psi(\theta)-\psi(\theta_0)|.$$

Thus, using the condition given in (15), we have

$$|\psi(y)-\psi(\theta_0)|\leq D|y-\theta_0|^\mu.$$

Consequently, we can express the inequality as

$$|\psi(y)-\psi(\theta)|\leq D|y-\theta_0|^\mu+|\psi(\theta)-\psi(\theta_0)|,$$

which allows us to further analyze the convergence properties of $D_n^{(\varrho)}(\psi,\theta)$.

$$\begin{aligned}
\left|D_n^{(\varrho)}(\psi,\theta)-\psi(\theta)\right| &\leq D_n^{(\varrho)}(|\psi(y)-\psi(\theta)|,\theta)\\
&\leq n\sum_{v=0}^{\infty}p_{n,v}(\theta,\varrho)\int_0^\infty q_{n,v}(\theta,\varrho)|\psi(t)-\psi(\theta_0)|dt+|\psi(\theta)-\psi(\theta_0)|\\
&\leq Dn\sum_{v=0}^{\infty}p_{n,v}(\theta,\varrho)\int_0^\infty q_{n,v}(\theta,\varrho)|t-\theta_0|^\mu dt+D|\theta-\theta_0|^\mu\\
&\leq Dn\sum_{v=0}^{\infty}p_{n,v}(\theta,\varrho)\int_0^\infty q_{n,v}(\theta,\varrho)|t-\theta_0|^\mu dt+2D|\theta-\theta_0|^\mu.
\end{aligned}$$

Using $p=\frac{2}{\mu}$ and $q=\frac{2}{2-\mu}$ in Holder's inequality, we get:

$$\left|D_n^{(\varrho)}(\psi,\theta)-\psi(\theta)\right|\leq D\left\{D_n^{(\varrho)}((y-\theta)^2,\theta)^{\mu/2}+2(d(\theta,S))^{\mu}\right\}.$$

Ultimately, we obtain the outcome by using Lemma 1.2. □

We now provide the subsequent Lipschitz-type space:

$$Lip_D^*(\mu)=\left\{\psi\in C[0,\infty):|\psi(y)-\psi(\theta)|\leq D\frac{|y-\theta|^{\mu}}{(y+\theta+1)^{\mu/2}},\quad \theta,y\in[0,\infty)\right\},$$

where $0<\mu\leq 1$ and D are positive constants. For the space $Lip_D^*(\mu)$, a noteworthy result has been made, called the local approximation theorem. A earlier discussion of this theorem may be found in [11].

Theorem 3.2. *For any function $\psi\in Lip_D^*(\mu)$ with $\mu\in(0,1]$, and for a fixed $\varrho>0$ as well as for every $\theta\in[0,\infty)$ and $n\in\mathbb{N}$, we have*

$$\left|D_n^{(\varrho)}(\psi,\theta)-\psi(\theta)\right|\leq D\left\{\frac{2(|1-\varrho|+3)}{n}\theta+\frac{2(1-\varrho)^2+4|1-\varrho|+|1-\varrho||\varrho-2|+5}{n^2}\right\}^{\mu/2}.$$

Proof. Let $\mu=1$. Then, for any function $\psi\in Lip_D^*(1)$ and for every $\theta\in[0,\infty)$, we have

$$\begin{aligned}\left|D_n^{(\varrho)}(\psi,\theta)-\psi(\theta)\right| &\leq D_n^{(\varrho)}(|\psi(y)-\psi(\theta)|,\theta)\\ &\leq Dn\sum_{v=0}^{\infty}p_{n,v}(\theta,\varrho)\int_0^{\infty}q_{n,v}(\theta,\varrho)\frac{|t-\theta|}{(t+\theta+1)^{1/2}}dt\\ &\leq \frac{Dn}{(\theta+1)^{1/2}}\sum_{v=0}^{\infty}p_{n,v}(\theta,\varrho)\int_0^{\infty}q_{n,v}(\theta,\varrho)|t-\theta|dt.\end{aligned}$$

The Cauchy-Schwarz inequality is applied, and the result is

$$\begin{aligned}\left|D_n^{(\varrho)}(\psi,\theta)-\psi(\theta)\right| &\leq \frac{D}{(\theta+1)^{1/2}}\sqrt{D_n^{(\varrho)}(|t-\theta|^2,\theta)}\\ &\leq \frac{D}{(\theta+1)^{1/2}}\left\{\frac{2(|1-\varrho|+3)}{n}\theta+\frac{2(1-\varrho)^2+4|1-\varrho|+|1-\varrho||\varrho-2|+5}{n^2}\right\}^{1/2}\\ &\leq D\left\{\frac{2(|1-\varrho|+3)}{n}+\frac{2(1-\varrho)^2+4|1-\varrho|+|1-\varrho||\varrho-2|+5}{n^2}\right\}^{1/2}.\end{aligned}$$

For $0<\Upsilon<1$, $f\in Lip_D^*(\Upsilon)$ and $\theta\geq 0$, we have

$$\begin{aligned}\left|D_n^{(\varrho)}(\psi,\theta)-\psi(\theta)\right| &\leq D_n^{(\varrho)}(|\psi(y)-\psi(\theta)|,\theta)\\ &\leq Dn\sum_{v=0}^{\infty}p_{n,v}(\theta,\varrho)\int_0^{\infty}q_{n,v}(\theta,\varrho)\frac{|t-\theta|^{\mu}}{(t+\theta+1)^{\mu/2}}dt\\ &\leq \frac{Dn}{(\theta+1)^{\mu/2}}\sum_{v=0}^{\infty}p_{n,v}(\theta,\varrho)\int_0^{\infty}q_{n,v}(\theta,\varrho)|t-\theta|^{\mu}dt.\end{aligned}$$

Using Hölder's inequality and setting $p=\frac{2}{\mu}$ and $q=\frac{2}{2-\mu}$, we get:

$$\begin{aligned}\left|D_n^{(\varrho)}(\psi,\theta)-\psi(\theta)\right| &\leq \frac{D}{(\theta+1)^{\mu/2}}\left[D_n^{(\varrho)}((t-\theta)^2,\theta)\right]^{\mu/2}\\ &\leq \frac{D}{(\theta+1)^{\mu/2}}\left\{\frac{2(|1-\varrho|+3)}{n}\theta+\frac{2(1-\varrho)^2+4|1-\varrho|+|1-\varrho||\varrho-2|+5}{n^2}\right\}^{\mu/2}\\ &\leq D\left\{\frac{2(|1-\varrho|+3)}{n}+\frac{2(1-\varrho)^2+4|1-\varrho|+|1-\varrho||\varrho-2|+5}{n^2}\right\}^{\mu/2}.\end{aligned}$$

□

4 Examining *A*-Statistical Convergence for the operators

We start this part by going over a few concepts and notations associated with the notion of *A*-statistical convergence. Here for a non-negative regular summability matrix $A = (a_{jk})$, the definition of *A*-density of a subset K of $\mathbb{N}$ is as

$$\delta_A(K) = \lim_j \sum_{k \in K} a_{j,k},$$

where the limit exists. If a sequence $x = (x_n)$ convergent to l meaning A-statistically converges symbolized by $st_A - \lim x = l$ where every $\epsilon > 0$; $\delta_A\{n \in \mathbb{N} : |x - l| \geq \epsilon\} = 0$. *A*-statistical convergence simplifies to statistical convergence when it A is substituted with C_1, the Cesáro matrix of order one. In a similar vein, *A*-statistical convergence and ordinary convergence coincide when considering $A = I$, the identity matrix. The statistical convergence of different kinds of operators has been studied by several scholars in ([26, 27, 28, 29]). We shall use *A*-statistical convergence to demonstrate the weighted Korovkin theorem in the next sections. We remember that Duman and Orhan established the weighted Korovkin-type approximation theorem for *A*-statistical convergence in [30].

In Section 2 and 3, we have demonstrated that the operators $D_n^{(\varrho)}$ converge uniformly to the function f according to the classical Korovkin's theorem. Since these operators exhibit uniform convergence, it follows that they also converge statistically to f. Hence, there is no need to establish the statistical convergence separately. We will now alter the specified operators by

$$\hat{D}_n^{(\varrho)}(f, \theta) = (1 + p\alpha_n) D_n^{(\varrho)}(f, \theta),$$

where $p \in \mathbb{R} \setminus \{0\}$ and (α_n) be any unbounded sequence, which is *A*-statistically converging to 0. Now, we are going to prove the following statistical convergence theorem for $\left(\hat{D}_n^{(\varrho)}\right)$.

Theorem 4.1. *Given a non-negative regular summability matrix $A = (a_{nk})$, a fixed $\varrho > 1$, and $\theta \in [0, \infty)$, for every $f \in E$, we obtain*

$$st_A - \lim_n \|\hat{D}_n^{(\varrho)}(\psi, \cdot) - \psi\|_{C[0,B]} = 0.$$

Proof. If we apply result given in [30, p. 191, Th. 3], It is enough to demonstrate that $st_A - \lim_n \|\hat{D}_n^{(\varrho)}\left(y^i, \cdot\right) - y^i\|_{C[0,B]} = 0$, where $i = 0, 1, 2$.
In view of Lemma 1.2, it follows that

$$\|\hat{D}_n^{(\varrho)}(1, \cdot) - 1\|_{C[0,B]} = \|(1 + p\alpha_n)\hat{D}_n^{(\varrho)}(1, \cdot) - 1\|_{C[0,B]} \leq |p|\|\alpha_n\|_{C[0,B]}.$$

Therefore,

$$st_A - \lim_n \|\hat{D}_n^{(\varrho)}(1, \cdot) - 1\|_{C[0,B]} = 0.$$

We note that

$$\|\hat{D}_n^{(\varrho)}(y, \cdot) - y\|_{C[0,B]} \leq \|D_n^{(\varrho)}(y, \cdot) - y\|_{C[0,B]} + |p|\|\alpha_n\|_{C[0,B]}\|D_n^{(\varrho)}(y, \cdot)\|_{C[0,B]}.$$

Therefore

$$st_A - \lim_n \|\hat{D}_n^{(\varrho)}(y, \cdot) - y\|_{C[0,B]} = 0.$$

Finally,

$$\|\hat{D}_n^{(\varrho)}\left(y^2, \cdot\right) - y^2\|_{C[0,B]} \leq \|D_n^{(\varrho)}\left(y^2, \cdot\right) - y^2\|_{C[0,B]} + |p|\|\alpha_n\|_{C[0,B]}\|D_n^{(\varrho)}\left(y^2, \cdot\right)\|_{C[0,B]}.$$

Hence,

$$\lim_{n \to \infty} st_A - \|\hat{D}_n^{(\varrho)}\left(y^2, \cdot\right) - y^2\|_{C[0,B]} = 0.$$

Therefore, we get the desired result. □

On the other hand, $D_n^{(\varrho)}(f, 0) = f(0)$ and therefore $\hat{D}_n^{(\varrho)}(f, 0) = (1 + p\alpha_n) f(0)$, then we have

$$\|\hat{D}_n^{(\varrho)}(f, \cdot) - f(\cdot)\|_{C[0,B]} \geq |\hat{D}_n^{(\varrho)}(f, 0) - f(0)| = |p||\alpha_n||f(0)|.$$

Combining the above equation with the fact that $\lim sup_{n \to \infty} \alpha_n = \infty$, we say that $\hat{D}_n^{(\varrho)}$ does not satisfy the classical Korovkin's theorem.

5 Conclusion

In conclusion, we have presented a Durrmeyer version of the Mittag-Leffler operators, highlighting their key characteristics, including the computation of moments. By employing traditional modulus of continuity techniques, we have provided convergence estimates for these operators. Our study also extends to *A*-statistical convergence and the convergence rate within Lipschitz spaces. These results offer valuable insights into the behavior and performance of Durmmeyer version Mittag-Leffler operators, contributing to the broader understanding of their convergence properties and their applications. One can study the Stancu version of these Durmmeyer variants.

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