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Multivariate φ-Variational Approximation of Mellin-Type Nonlinear Integral Operators via Summability Methods

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Abstract

In this paper, we construct the nonlinear form of multidimensional Mellin-type integral operators and improve them using summability methods. We utilize Tonelli sense convergence in φ-variation, incorporating the Haar measure into our framework. Additionally, we study the rate of approximation and provide a characterization theorem for functions that are absolutely continuous in the Tonelli sense. Finally, we present illustrations of our approximations to support the theoretical findings.

Keywords: convergence in φ-variation, Mellin operators, summability methods, rate of convergence, characterization of absolute continuity

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1 Introduction

Inspired by the works [15] and [6], which establish the one-dimensional nonlinear form and the multidimensional linear form, respectively, our goal is to construct a multidimensional nonlinear form of Mellin-type integral operators in (Tonelli sense) φ-variation [5, 31]. It is well known that such operators have significant applications in optical physics, signal analysis, and engineering [18, 19, 20, 24, 32]. Additionally, by employing summability methods, we aim to develop a general form of these operators. For insights into the effects of summability methods in one-dimensional nonlinear case, we refer to [12]. We also refer to the recent papers in [21, 22, 23] for the effects of summability methods on approximation theory.

In the present paper, we consider Bell-type summability method. For a given family of infinite matrices with real or complex entries $\mathcal{A} = \{A^{\upsilon}\} = \left\{\left[a^{\upsilon}_{nk}\right]\right\}_{\upsilon\in\mathbb{N}}$ $(k, n \in \mathbb{N})$ and for a given sequence $u = (u_k)_{k\in\mathbb{N}}$, we call that u is $\mathcal{A}$-summable to a number L, if $\mathcal{A}$-transform of u (that is, $\sum_{k=1}^{\infty} a^{\upsilon}_{nk} u_k$) is finite for all $n, \upsilon \in \mathbb{N}$ and

$$\mathcal{A} - \lim u := \lim_{n\to\infty} \sum_{k=1}^{\infty} a^{\upsilon}_{nk} u_k = L \text{ (uniformly in } \upsilon \in \mathbb{N}) \text{ [16, 17]}$$

holds. The summability method $\mathcal{A}$ is called regular, if for any $u_k \to L$ implies $\sum_{k=1}^{\infty} a^{\upsilon}_{nk} u_k \to L$ (uniformly in $\upsilon \in \mathbb{N}$). Regular summability methods can be characterized by the following [17]:

$$\mathcal{A} \text{ is regular } \Leftrightarrow \begin{array}{l} 1)\ \lim\limits_{n\to\infty} a_{nk} = 0 \text{ for each } k \in \mathbb{N} \text{ (uniformly in } \upsilon \in \mathbb{N}) \\ 2)\ \lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} a^{\upsilon}_{nk} = 1 \text{ (uniformly in } \upsilon \in \mathbb{N}) \\ 3)\ a_{n,\upsilon} := \sum\limits_{k=1}^{\infty} \left|a^{\upsilon}_{nk}\right| < \infty \text{ for all } n, \upsilon \in \mathbb{N} \text{ and there} \\ \quad \text{exists } N, M \in \mathbb{N} \text{ such that } \sup_{n\geq N, \upsilon\in\mathbb{N}} a_{n,\upsilon} \leq M. \end{array} \tag{1}$$

Throughout the paper, we assume that $\mathcal{A}$ is regular summability method with nonnegative real entries. Here, we note that beyond classical convergence, it is also possible to achieve different types of summability methods, such as Cesàro means and convergence, by utilizing various types of regular matrices [16, 17, 27, 29].

Due to its suitability in the Mellin setting (see [6, 12, 15]), we use the Haar measure on the space $\mathbb{R}^N_+$ given by $\mu(A) := \int_A 1/\langle \mathbf{t} \rangle \, d\mathbf{t}$ together with $L^1_\mu\left(\mathbb{R}^N_+\right)$ space given below

$$L^1_\mu\left(\mathbb{R}^N_+\right) := \left\{ f : \mathbb{R}^N_+ \to \mathbb{R} \mid \|f\|_{L^1_\mu} := \int_{\mathbb{R}^N_+} |f| \, d\mu = \int_{\mathbb{R}^N_+} |f(\mathbf{t})| \frac{d\mathbf{t}}{\langle \mathbf{t} \rangle} < \infty \right\}.$$

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By $\langle \mathbf{t} \rangle$ and $\left\langle \mathbf{t}'_j \right\rangle$, we mean the multiplication $\Pi_{i=1}^{N} t_i$ and $\Pi_{i=1,i\neq j}^{N} t_i$ respectively. We denote by Φ the class of all convex φ-functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ (φ- function means that φ is continuous, nondecreasing on $\mathbb{R}_0^+$, $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$) satisfying $\lim_{u\to 0^+} \varphi(u)/u = 0$. From now on, we assume that $\varphi \in \Phi$.

Let I be an N-dimensional interval such that $I = \Pi_{i=1}^{N} [a_i, b_i] \subset \mathbb{R}_+^N$. Then by the notation $[a'_j, b'_j]$, we mean $N-1$ dimensional interval $\Pi_{i=1,i\neq j}^{N} [a_i, b_i] \subset \mathbb{R}_+^{N-1}$. Moreover, we use the following notation $\mathbf{x} = (x_1, x_2, \cdots, x_N)$ and $\mathbf{x}'_j := \left(x_1, x_2, \cdots, x_{j-1}, x_{j+1}, \cdots, x_N\right)$. Using the above notation, if we are interested in j-th coordinate of the interval I, vector $\mathbf{x}$ and function $f(\mathbf{x})$, we write $I = [a'_j, b'_j] \times \left[a_j, b_j\right]$, $\mathbf{x} = (\mathbf{x}'_j, x_j)$ and $f(\mathbf{x}'_j, x_j)$, respectively. We also use the symbol $|\cdot|$ for Euclidean norm of multidimensional vectors.

We remind that there are different type of variations and hence their corresponding variational approximations [7, 9, 10, 25, 26, 28, 30, 33, 34]. In our study we consider Tonelli sense φ-variation which is defined as follows:
For a given function $f : \mathbb{R}_+ \to \mathbb{R}$, φ-variation of f on $[a, b] \subset \mathbb{R}_+$ is defined as follows:

$$V_{[a,b]}^{\varphi}[f] := \sup_{P} \sum_{i=1}^{n} \varphi\left(|f(s_i) - f(s_{i-1})|\right) \text{ (see [35])},$$

where the supremum is taken over all the partitions $P = \{a = s_0, s_1, \cdots, s_n = b\}$ of $[a, b]$. Moreover, φ-variation of a given function f on $\mathbb{R}_+$ is defined as

$$V^{\varphi}[f] := \sup_{[a,b]\subset\mathbb{R}_+} V_{[a,b]}^{\varphi}[f] \text{ (see [6])}.$$

Considering these definitions, a function $f : \mathbb{R}_+ \to \mathbb{R}$ is called bounded φ-variation, if there exists a $\lambda > 0$ such that $V^{\varphi}[\lambda f]$ is finite. Throughout the paper, the space of all functions of bounded φ-variation will be denoted by $BV^{\varphi}(\mathbb{R}_+)$.

Tonelli sense φ-variation in N-dimension is defined as follows [5, 6]:

Let $f : \mathbb{R}_+^N \to \mathbb{R}$ be given. Then for a given N-dimensional interval $I \subset \mathbb{R}_+^N$, define $\Phi^{\varphi}(f, I)$ as follows

$$\Phi^{\varphi}(f, I) := \left(\sum_{j=1}^{N} [\Phi_j^{\varphi}(f, I)]^2\right)^{\frac{1}{2}},$$

where $\Phi_j^{\varphi}(f, I) := \int_{a'_j}^{b'_j} V_{[a_j,b_j]}^{\varphi}[f(\mathbf{x}'_j \cdot)] d\mathbf{x}'_j / \left\langle \mathbf{x}'_j \right\rangle$. Here $V_{[a_j,b_j]}^{\varphi}[f(\mathbf{x}'_j \cdot)]$ denotes the one dimensional φ-variation of f on the interval $\left[a_j, b_j\right]$ for each fixed $\mathbf{x}'_j \in \mathbb{R}_+^{N-1}$. Taking these definitions into account, N-dimensional φ-variation of f on the interval $I \subset \mathbb{R}_+^N$ is given by

$$V_I^{\varphi}[f] := \sup \sum_{r=1}^{m} \Phi^{\varphi}(f, J_r),$$

where $\{J_1, J_2, \cdots, J_m\}$ is a partition of I and the supremum is taken over all the partitions of I. By the above definition, multidimensional φ-variation of f on $\mathbb{R}_+^N$ is defined as

$$V^{\varphi}[f] = \sup_{I\subset\mathbb{R}_+^N} V_I^{\varphi}[f].$$

Analogously to one dimensional case, f is called bounded φ-variation (in Tonelli sense), if there exists a $\lambda > 0$ such that $V^{\varphi}[\lambda f]$ is finite. The space of such functions will be denoted by $BV^{\varphi}\left(\mathbb{R}_+^N\right)$.

On the other hand, we also need the following definitions of absolute continuity. A function $f : [a, b] \subset \mathbb{R}_+ \to \mathbb{R}$ is called φ-absolutely continuous on $[a, b]$, if one can find a $\lambda > 0$ satisfying that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{m} \varphi\left(\lambda |f(\beta_i) - f(\alpha_i)|\right) < \varepsilon$$

for all finite collections of non-overlapping intervals $[\alpha_i, \beta_i] \subset [a, b]$, $i = 1, 2, \ldots, m$, whenever

$$\sum_{i=1}^{m} \varphi(\beta_i - \alpha_i) < \delta.$$

A multidimensional counterpart of this definition is given by the following:

$f : \mathbb{R}_+^N \to \mathbb{R}$ is called locally φ-absolutely continuous if for any interval $I = \Pi_{i=1}^{N} [a_i, b_i] \subset \mathbb{R}_+^N$ and for all $j = 1, 2, \ldots, N$, $f(\mathbf{x}'_j \cdot) : \left[a_j, b_j\right] \to \mathbb{R}$ are absolutely continuous for (uniformly) almost every $\mathbf{x}'_j \in \mathbb{R}_+^{N-1}$.

Now, $f : \mathbb{R}_+^N \to \mathbb{R}$ is called φ-absolutely continuous, if $f \in BV^{\varphi}\left(\mathbb{R}_+^N\right)$ and f is locally φ-absolutely continuous. The space of all φ-absolutely continuous functions will be denoted by $AC^{\varphi}\left(\mathbb{R}_+^N\right)$ (see [5, 6]).

Let $\varphi, \eta \in \Phi$ and, ψ be φ-function. Then (φ, η, ψ) is called properly directed (see [2]), if for all $\mu \in (0, 1)$, there can be found a $C_\mu > 0$ such that

$$\varphi\left(C_\mu \psi(|g|)\right) \leq \eta(\mu |g|)$$

for all (Haar) measurable function $g : \mathbb{R}_+ \to \mathbb{R}$. From now on, we assume that (φ, η, ψ) is properly directed.

Some important properties of φ-variation are given below:

Let $f_1,\ldots,f_n \in L_\mu^N(\mathbb{R}_+)$ be given. Then if $N=1$, there holds

$$V^\varphi\left[\sum_{i=1}^n f_i\right] \le \frac{1}{n}\sum_{i=1}^n V^\varphi[nf_i], \tag{2}$$

if $N>1$, then

$$V^\varphi\left[\sum_{i=1}^n f_i\right] \le \frac{1}{\sqrt{n}}\sum_{i=1}^n V^\varphi[nf_i] \tag{3}$$

holds.

2 Approximation Theorem

Generalized nonlinear Mellin operator is defined as follows:

$$\mathcal{T}_{n,\upsilon}(f;\mathbf{r}) = \sum_{k=1}^\infty a_{nk}^\upsilon \int_{\mathbb{R}_+^N} L_k(\mathbf{t}) H_k(f(\mathbf{rt}))\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle},\ \ (\mathbf{r}\in\mathbb{R}_+^N,\ n,\upsilon\in\mathbb{N}),$$

where $\langle\mathbf{t}\rangle := \Pi_{i=1}^N t_i$ and $\mathbf{rt} := (r_1t_1, r_2t_2,\cdots,r_Nt_N)$ for $\mathbf{r},\mathbf{t}\in\mathbb{R}_+^N$. In the definition above, $L_k:\mathbb{R}_+^N\to\mathbb{R}$ is assumed to be $L_k\in L_\mu^1\left(\mathbb{R}_+^N\right)$ and $H_k:\mathbb{R}\to\mathbb{R}$ being $H_k(0)=0$. We also assume that H_k is ψ-Lipschitz, that is,
there exists a number $L>0$ such that

$$|H_k(x)-H_k(y)| \le L\psi(|x-y|) \tag{4}$$

for all $x,y\in\mathbb{R}$ and $k\in\mathbb{N}$, where ψ is a φ-function (see also [1]).

In order to prove our approximations, we need the following general form of the approximate identities:

- **(i)** $\sup_{k\in\mathbb{N}}\|L_k\|_{L_\mu^1} = A < \infty$,
- **(ii)** for any fixed $0<\delta<1$
$$\mathcal{A}-\lim \int_{|\mathbf{1}-\mathbf{t}|\ge\delta} L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle} = 0,$$
where $\mathbf{1}=(1,1,\cdots,1)\in\mathbb{R}_+^N$,
- **(iii)**
$$\mathcal{A}-\lim \int_{\mathbb{R}_+^N} L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle} = 1,$$

and

- **(iv)** denoting $G_k(u) := H_k(u)-u$ for all $u\in\mathbb{R}$,
for all $\gamma>0$, there exists a $\lambda>0$ such that
$$\lim_{k\to\infty}\frac{V^\varphi[\lambda G_k]}{\varphi(\gamma m(J))} = 0\ \ \text{(uniformly in every proper bounded interval } J\subset\mathbb{R})$$
where $m(J)$ denotes the length of the interval J.

Lemma 2.1. *Let $\psi\circ|f|\in L_\mu^\infty\left(\mathbb{R}_+^N\right)$ and* **(i)** *holds. Then we have $\mathcal{T}_{n,\upsilon}(f;\mathbf{r})<\infty$ for all $r\in\mathbb{R}_+^N$. In addition, if* **(i)** *is satisfied and $\psi\circ|f|\in L_\mu^1\left(\mathbb{R}_+^N\right)$, then $\mathcal{T}_{n,\upsilon}(f)\in L_\mu^1\left(\mathbb{R}_+^N\right)$.*

Proof. For any $\mathbf{r}\in\mathbb{R}_+^N$,

$$\begin{aligned}\left|\mathcal{T}_{n,\upsilon}(f;\mathbf{r})\right| &\le \sum_{k=1}^\infty a_{nk}^\upsilon\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,|H_k(f(\mathbf{rt}))|\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\\ &\le L\sum_{k=1}^\infty a_{nk}^\upsilon\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,\psi(|f(\mathbf{rt})|)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\end{aligned}$$

holds. Now since $\psi\circ|f|\in L_\mu^\infty\left(\mathbb{R}_+^N\right)$, from Hölder's inequality and **(i)** there holds

$$\left|\mathcal{T}_{n,\upsilon}(f;\mathbf{r})\right| \le LA\|\psi\circ|f|\|_{L_\mu^\infty}\sum_{k=1}^\infty a_{nk}^\upsilon$$

and finally from (1), for every $n,\upsilon\in\mathbb{N}$ we get

$$\left|\mathcal{T}_{n,\upsilon}(f;\mathbf{s})\right| < \infty.$$

For the second part of the proof, if $\psi \circ |f| \in L^1_\mu\left(\mathbb{R}^N_+\right)$ then from (4), (**i**) and the Fubini-Tonelli theorem

$$\begin{aligned}\int_{\mathbb{R}^N_+} \left|\mathcal{T}_{n,\upsilon}(f;\mathbf{r})\right| \frac{d\mathbf{r}}{\langle\mathbf{r}\rangle} &\le L\int_{\mathbb{R}^N_+}\left\{\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+}|L_k(\mathbf{t})|\,\psi(|f(\mathbf{rt})|)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right\}\frac{d\mathbf{r}}{\langle\mathbf{r}\rangle}\\ &= L\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+}|L_k(\mathbf{t})|\left\{\int_{\mathbb{R}^N_+}\psi(|f(\mathbf{rt})|)\frac{d\mathbf{r}}{\langle\mathbf{r}\rangle}\right\}\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\\ &\le LA\|\psi\circ|f|\|_{L^1_\mu}\sum_{k=1}^{\infty} a^{\upsilon}_{nk} < \infty\end{aligned}$$

holds. □

Lemma 2.2. *Let $f \in BV^{\eta}\left(\mathbb{R}^N_+\right)$ and (**i**) holds. Then there exists a $\gamma > 0$ such that*

$$V^{\varphi}\left[\gamma\mathcal{T}_{n,\upsilon}(f)\right] \le V^{\eta}[\mu f]$$

for which $V^{\eta}[\mu f] < \infty$, which means $\mathcal{T}_{n,\upsilon}$ maps from $BV^{\eta}\left(\mathbb{R}^N_+\right)$ to $BV^{\varphi}\left(\mathbb{R}^N_+\right)$.

Proof. Let $\{J_1, J_2, \cdots, J_m\}$ be a partition of the N-dimensional interval $I = \Pi_{j=1}^N\left[a_j, b_j\right] \subset \mathbb{R}^N_+$, where $J_q = \Pi_{j=1}^N[{}^q a_j, {}^q b_j]$ for $q = 1, \cdots, m$. Furthermore, let $\left\{r^0_j = {}^q a_j, r^1_j, \cdots, r^{\omega}_j = {}^q b_j\right\}$ be a partition of the interval $[{}^q a_j, {}^q b_j]$ for $q = 1, \cdots, m$ and $j = 1, \cdots, N$. Now, for all $\gamma > 0$ and $\mathbf{r}'_j \in I'_j$, from (4) we have

$$\begin{aligned}U_j &= \sum_{\tau=1}^{\omega}\varphi\left(\gamma\left|\mathcal{T}_{n,\upsilon}(f;\mathbf{r}'_j, r^{\tau}_j) - \mathcal{T}_{n,\upsilon}(f;\mathbf{r}'_j, r^{\tau-1}_j)\right|\right)\\ &= \sum_{\tau=1}^{\omega}\varphi\left(\gamma\left|\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+} L_k(\mathbf{t})H_k(f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau}_j t_j))\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle} - \sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+} L_k(\mathbf{t})H_k(f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau-1}_j t_j))\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right|\right)\\ &\le \sum_{\tau=1}^{\omega}\varphi\left(\gamma\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+}|L_k(\mathbf{t})|\left|H_k(f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau}_j t_j)) - H_k(f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau-1}_j t_j))\right|\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right)\\ &\le \sum_{\tau=1}^{\omega}\varphi\left(\gamma L\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+}|L_k(\mathbf{t})|\,\psi\left(\left|f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau}_j t_j) - f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau-1}_j t_j)\right|\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right).\end{aligned}$$

Considering Jensen's inequality together with the regularity of $\mathcal{A}$, from assumption (**i**)

$$\begin{aligned}U_j &\le \frac{1}{a_{n,\upsilon}}\sum_{\tau=1}^{\omega}\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\varphi\left(\gamma L a_{n,\upsilon}\int_{\mathbb{R}^N_+}|L_k(\mathbf{t})|\,\psi\left(\left|f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau}_j t_j) - f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau-1}_j t_j)\right|\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right)\\ &\le \frac{1}{a_{n,\upsilon}A}\sum_{\tau=1}^{\omega}\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+}|L_k(\mathbf{t})|\,\varphi\left(\gamma A L a_{n,\upsilon}\psi\left(\left|f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau}_j t_j) - f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau-1}_j t_j)\right|\right)\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\end{aligned}$$

yields, where $a_{n,\upsilon}$ (defined in (1)) is finite. On the other hand, since (φ, η, ψ) properly directed, for all $\mu \in (0, 1)$ we can find a $C_\mu > 0$ such that

$$U_j \le \frac{1}{a_{n,\upsilon}A}\sum_{\tau=1}^{\omega}\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+}|L_k(\mathbf{t})|\,\eta\left(\mu\left|f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau}_j t_j) - f(\mathbf{r}'_j\mathbf{t}'_j, r^{\tau-1}_j t_j)\right|\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$

and hence

$$U_j \le \frac{1}{a_{n,\upsilon}A}\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+}|L_k(\mathbf{t})|\,V^{\eta}_{[{}^q a_j, {}^q b_j]}[\mu f(\mathbf{r}'_j\mathbf{t}'_j, \cdot t_j)]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}, \tag{5}$$

where $\gamma \le C_\mu / \left(A L a_{n,\upsilon}\right)$. Then from the Fubini-Tonelli theorem and (5), we obtain

$$\begin{aligned}\Phi^{\varphi}_j\left(\gamma\mathcal{T}_{n,\upsilon}(f), J_q\right) &:= \int_{a'_j}^{b'_j} V^{\varphi}_{[{}^q a_j, {}^q b_j]}[\gamma\mathcal{T}_{n,\upsilon}(f;\mathbf{r}'_j, \cdot)]\frac{d\mathbf{r}'_j}{\left\langle\mathbf{r}'_j\right\rangle}\\ &\le \int_{a'_j}^{b'_j}\left(\frac{1}{a_{n,\upsilon}A}\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+}|L_k(\mathbf{t})|\,V^{\eta}_{[{}^q a_j, {}^q b_j]}[\mu f(\mathbf{r}'_j\mathbf{t}'_j, \cdot t_j)]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right)\frac{d\mathbf{r}'_j}{\left\langle\mathbf{r}'_j\right\rangle}\\ &= \frac{1}{a_{n,\upsilon}A}\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+}|L_k(\mathbf{t})|\left(\int_{{}^q a'_j}^{{}^q b'_j} V^{\eta}_{[{}^q a_j, {}^q b_j]}[\mu f(\mathbf{r}'_j\mathbf{t}'_j, \cdot t_j)]\frac{d\mathbf{r}'_j}{\left\langle\mathbf{r}'_j\right\rangle}\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\\ &= \frac{1}{a_{n,\upsilon}A}\sum_{k=1}^{\infty} a^{\upsilon}_{nk}\int_{\mathbb{R}^N_+}|L_k(\mathbf{t})|\,\Phi^{\eta}_j\left(\mu f(\cdot\mathbf{t}), J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}.\end{aligned}$$

Then applying two times generalized Minkowski-inequality, one may easily get

$$\begin{aligned}\Phi^{\varphi}\left(\gamma\mathcal{T}_{n,\upsilon}(f),J_q\right) &:= \left(\sum_{j=1}^{n}\left[\Phi_j^{\varphi}\left(\gamma\mathcal{T}_{n,\upsilon}(f),J_q\right)\right]^2\right)^{\frac{1}{2}}\\ &\leq \left(\sum_{j=1}^{n}\left[\frac{1}{a_{n,\upsilon}A}\sum_{k=1}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,\Phi_j^{\eta}\left(\mu f(\cdot\mathbf{t}),J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right]^2\right)^{\frac{1}{2}}\\ &\leq \frac{1}{a_{n,\upsilon}A}\sum_{k=1}^{\infty}a_{nk}^{\upsilon}\left(\sum_{j=1}^{n}\left[\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,\Phi_j^{\eta}\left(\mu f(\cdot\mathbf{t}),J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right]^2\right)^{\frac{1}{2}}\\ &\leq \frac{1}{a_{n,\upsilon}A}\sum_{k=1}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\left(\sum_{j=1}^{n}\left[\Phi_j^{\eta}\left(\mu f(\cdot\mathbf{t}),J_q\right)\right]^2\right)^{\frac{1}{2}}\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\\ &= \frac{1}{a_{n,\upsilon}A}\sum_{k=1}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,\Phi^{\eta}\left(\mu f(\cdot\mathbf{t}),J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}.\end{aligned}$$

Then there holds

$$\begin{aligned}V_I^{\varphi}\left[\gamma\mathcal{T}_{n,\upsilon}(f)\right] &:= \sup_{\{J_1,J_2,\cdots,J_m\}}\sum_{q=1}^{m}\Phi^{\varphi}\left(\gamma\mathcal{T}_{n,\upsilon}(f),J_q\right)\\ &\leq \frac{1}{a_{n,\upsilon}A}\sum_{k=1}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\left(\sup_{\{J_1,J_2,\cdots,J_m\}}\sum_{k=1}^{m}\Phi^{\eta}\left(\mu f(\cdot\mathbf{t}),J_q\right)\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\\ &= \frac{1}{a_{n,\upsilon}A}\sum_{k=1}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,V_I^{\varphi}\left[\mu f(\cdot\mathbf{t})\right]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}.\end{aligned}$$

Considering (**i**) and (1)

$$\begin{aligned}V^{\varphi}\left[\gamma\mathcal{T}_{n,\upsilon}(f)\right] &:= \sup_{I\subset\mathbb{R}_+^N}V_I^{\varphi}\left[\gamma\mathcal{T}_{n,\upsilon}(f)\right]\\ &\leq \frac{1}{a_{n,\upsilon}A}\sum_{k=1}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\sup_{I\subset\mathbb{R}_+^N}V_I^{\varphi}\left[\mu f(\cdot\mathbf{t})\right]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\end{aligned}$$

holds. Here, notice that

$$\sup_{I\subset\mathbb{R}_+^N}V_I^{\varphi}\left[\mu f(\cdot\mathbf{t})\right] = V^{\varphi}\left[\mu f(\cdot\mathbf{t})\right] = V^{\varphi}\left[\mu f\right]$$

and hence we finally conclude that

$$\begin{aligned}V^{\varphi}\left[\gamma\mathcal{T}_{n,\upsilon}(f)\right] &\leq \frac{V^{\varphi}\left[\mu f\right]}{a_{n,\upsilon}A}\sum_{k=1}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\\ &\leq V^{\varphi}\left[\mu f\right]\end{aligned}$$

holds, which completes the proof. □

Main approximation theorem is given below.

Theorem 2.3. *Assume that $f\in AC^{\varphi}\left(\mathbb{R}_+^N\right)\cap BV^{\eta}\left(\mathbb{R}_+^N\right)$. If* (**i**)−(**iv**) *are satisfied, then there exists a $\lambda>0$ satisfying that*

$$\lim_{n\to\infty}V^{\varphi}\left[\lambda\left(\mathcal{T}_{n,\upsilon}(f)-f\right)\right]=0 \text{ uniformly in } \upsilon\in\mathbb{N}.$$

Proof. Considering the similar notations in the previous lemma, it is possible to write for every $\lambda>0$ that

$$\begin{aligned}U &:= \sum_{\tau=1}^{\omega}\varphi\left(\lambda\left|\mathcal{T}_{n,\upsilon}(f;\mathbf{r}_j',r_j^{\tau})-f(\mathbf{r}_j',r_j^{\tau})-\mathcal{T}_{n,\upsilon}(f;\mathbf{r}_j',r_j^{\tau-1})-f(\mathbf{r}_j',r_j^{\tau-1})\right|\right)\\ &= \sum_{\tau=1}^{\omega}\varphi\left(\lambda\left|\sum_{k=1}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}L_k(\mathbf{t})\left[H_k(f(\mathbf{r}_j'\mathbf{t}_j',r_j^{\tau}t_j))-f(\mathbf{r}_j'\mathbf{t}_j',r_j^{\tau}t_j)\ -H_k(f(\mathbf{r}_j'\mathbf{t}_j',r_j^{\tau-1}t_j))+f(\mathbf{r}_j'\mathbf{t}_j',r_j^{\tau-1}t_j)\right]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right.\right.\\ &\quad+\sum_{k=1}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}L_k(\mathbf{t})\left[f(\mathbf{r}_j'\mathbf{t}_j',r_j^{\tau}t_j)-f(\mathbf{r}_j',r_j^{\tau})-f(\mathbf{r}_j'\mathbf{t}_j',r_j^{\tau-1}t_j)+f(\mathbf{r}_j',r_j^{\tau})\right]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\\ &\quad\left.\left.+\left(\sum_{k=1}^{\infty}a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}-1\right)\left[f(\mathbf{r}_j',r_j^{\tau})-f(\mathbf{r}_j',r_j^{\tau-1})\right]\right|\right).\end{aligned}$$

Using the convexity of φ, we can easily see that

$$U \le \frac{1}{3}\sum_{\tau=1}^{\omega}\varphi\left(3\lambda\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\left|H_k(f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau}t_j))-f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau}t_j)\ \ -H_k(f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau-1}t_j))+f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau-1}t_j)\right|\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right)$$
$$+\frac{1}{3}\sum_{\tau=1}^{\omega}\varphi\left(3\lambda\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\left|f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau}t_j)-f(\mathbf{r}_j', r_j^{\tau})\ \ -f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau-1}t_j)+f(\mathbf{r}_j', r_j^{\tau})\right|\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right)$$
$$+\frac{1}{3}\sum_{\tau=1}^{\omega}\varphi\left(3\lambda\left|\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}-1\right|\left|f(\mathbf{r}_j', r_j^{\tau})-f(\mathbf{r}_j', r_j^{\tau-1})\right|\right)$$

holds. Taking Jensen's inequality (discrete and continuous forms) into account in the first two expression, from the convexity of φ, (**i**) and (1) we obtain

$$U \le \frac{1}{3MA}\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\sum_{\tau=1}^{\omega}\varphi\left(3MA\lambda\left|H_k(f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau}t_j))-f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau}t_j)\ \ -H_k(f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau-1}t_j))+f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau-1}t_j)\right|\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$
$$+\frac{1}{3MA}\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\sum_{\tau=1}^{\omega}\varphi\left(3MA\lambda\left|f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau}t_j)-f(\mathbf{r}_j', r_j^{\tau})\ \ -f(\mathbf{r}_j'\mathbf{t}_j', r_j^{\tau-1}t_j)+f(\mathbf{r}_j', r_j^{\tau})\right|\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$
$$+\frac{1}{3}\left|\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}-1\right|\sum_{\tau=1}^{\omega}\varphi\left(3\lambda\left|f(\mathbf{r}_j', r_j^{\tau})-f(\mathbf{r}_j', r_j^{\tau-1})\right|\right)$$

for sufficiently large $n\in\mathbb{N}$, where M comes from the regularity of $\mathcal{A}$. Notice that from (**i**), it is possible to find a number n_0 such that for all $n\ge n_0$, $\left|\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}-1\right|<1$ and $\sup_{n\ge N, v\in\mathbb{N}}\sum_{k=1}^{\infty}a_{nk}^{v}\le M$. Now, if we take supremum over all the partitions of the interval $[{}^{q}a_j, {}^{q}b_j]$, we get

$$U \le \frac{1}{3MA}\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,V^{\varphi}_{[{}^{q}a_j,{}^{q}b_j]}[3MA\lambda(H_k\circ f)(\mathbf{r}_j'\mathbf{t}_j',\cdot t_j)-f(\mathbf{r}_j'\mathbf{t}_j',\cdot t_j)]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$
$$+\frac{1}{3MA}\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,V^{\varphi}_{[{}^{q}a_j,{}^{q}b_j]}[3MA\lambda(f(\mathbf{r}_j'\mathbf{t}_j',\cdot t_j)-f(\mathbf{r}_j',\cdot))]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$
$$+\frac{V^{\varphi}_{[{}^{q}a_j,{}^{q}b_j]}\left[3\lambda f(\mathbf{r}_j',\cdot)\right]}{3}\left|\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}-1\right|$$

for sufficiently large $n\in\mathbb{N}$. Then using the Fubini-Tonelli theorem, for every $j=1,\cdots,N$

$$\Phi_j^{\varphi}\left(\lambda\left(\mathcal{T}_{n,v}(f)-f\right),J_q\right):=\int_{a_j'}^{b_j'}V^{\varphi}_{[{}^{q}a_j,{}^{q}b_j]}[\lambda(\mathcal{T}_{n,v}(f;\mathbf{r}_j',\cdot)-f(\mathbf{r}_j',\cdot))]\frac{d\mathbf{r}_j'}{\left\langle\mathbf{r}_j'\right\rangle}$$
$$\le\frac{1}{3MA}\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,\Phi_j^{\varphi}\left(3MA\lambda(H_k\circ f(\cdot\mathbf{t})-f(\cdot\mathbf{t})),J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$
$$+\frac{1}{3MA}\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,\Phi_j^{\varphi}\left(3MA\lambda(f(\cdot\mathbf{t})-f),J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}+\frac{\Phi_j^{\varphi}(3\lambda f,J_q)}{3}\left|\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}-1\right|$$

holds. Subsequently, by the Minkowski inequality, for every $q=1,\cdots,m$ there holds

$$\Phi^{\varphi}\left(\lambda\left(\mathcal{T}_{n,v}(f)-f\right),J_q\right):=\left\{\sum_{j=1}^{N}\left[\Phi_j^{\varphi}\left(\lambda\left(\mathcal{T}_{n,v}(f)-f\right),J_q\right)\right]\right\}^{\frac{1}{2}}$$
$$\le\frac{1}{3MA}\left\{\sum_{j=1}^{N}\left(\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,\Phi_j^{\varphi}\left(3MA\lambda(H_k\circ f(\cdot\mathbf{t})-f(\cdot\mathbf{t})),J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right)^2\right\}^{\frac{1}{2}}$$
$$+\frac{1}{3MA}\left\{\sum_{j=1}^{N}\left(\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,\Phi_j^{\varphi}\left(3MA\lambda(f(\cdot\mathbf{t})-f),J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right)^2\right\}^{\frac{1}{2}}$$
$$+\frac{1}{3}\left|\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}_+^N}L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}-1\right|\left\{\sum_{j=1}^{N}[\Phi_j^{\varphi}\left(3\lambda f,J_q\right)]^2\right\}^{\frac{1}{2}}$$
$$=: U_1+U_2+U_3.$$

Using generalized Minkowski inequality in U_1 and U_2 twice in each, we see that

$$U_1 \leq \frac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\left\{\sum_{j=1}^{N}\left(\int_{\mathbb{R}_+^N} |L_k(\mathbf{t})|\,\Phi_j^{\varphi}\left(3MA\lambda(H_k\circ f(\cdot\mathbf{t})-f(\cdot\mathbf{t})),J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right)^2\right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N} |L_k(\mathbf{t})|\left\{\sum_{j=1}^{N}[\Phi_j^{\varphi}\left(3MA\lambda(H_k\circ f(\cdot\mathbf{t})-f(\cdot\mathbf{t})),J_q\right)]^2\right\}^{\frac{1}{2}}\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$

$$= \frac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N} |L_k(\mathbf{t})|\,\Phi^{\varphi}\left(3MA\lambda(H_k\circ f(\cdot\mathbf{t})-f(\cdot\mathbf{t})),J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$

and

$$U_2 \leq \frac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\left\{\sum_{j=1}^{N}\left(\int_{\mathbb{R}_+^N} |L_k(\mathbf{t})|\,\Phi_j^{\varphi}\left(3MA\lambda(f(\cdot\mathbf{t})-f),J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}\right)^2\right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N} |L_k(\mathbf{t})|\left\{\sum_{j=1}^{N}[\Phi_j^{\varphi}\left(3MA\lambda(f(\cdot\mathbf{t})-f),J_q\right)]^2\right\}^{\frac{1}{2}}\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$

$$= \frac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N} |L_k(\mathbf{t})|\,\Phi^{\varphi}\left(3MA\lambda(f(\cdot\mathbf{t})-f),J_q\right)\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}.$$

Now, if we sum over $q=1,\cdots,m$ and take the supremum over all possible partitions of $\{J_1,J_2,\cdots,J_m\}$, we have the followings

$$V_I^{\varphi}\left[\lambda\left(\mathcal{T}_{n,\upsilon}(f)-f\right)\right] := \sup_{\{J_1,J_2,\cdots,J_m\}}\sum_{q=1}^{m}\Phi^{\varphi}\left(\lambda\left(\mathcal{T}_{n,\upsilon}(f)-f\right),J_q\right)$$

$$\leq \frac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N} |L_k(\mathbf{t})|\,V_I^{\varphi}\left[3MA\lambda(H_k\circ f(\cdot\mathbf{t})-f(\cdot\mathbf{t}))\right]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$

$$+\frac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N} |L_k(\mathbf{t})|\,V_I^{\varphi}\left[3MA\lambda(f(\cdot\mathbf{t})-f)\right]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}+\frac{1}{3}\left|\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N} L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}-1\right|V_I^{\varphi}\left[3\lambda f\right]$$

and since $I\subset\mathbb{R}_+^N$ is arbitrary, we get

$$V^{\varphi}\left[\lambda\left(\mathcal{T}_{n,\upsilon}(f)-f\right)\right] \leq \frac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N} |L_k(\mathbf{t})|\,V^{\varphi}\left[3MA\lambda(H_k\circ f(\cdot\mathbf{t})-f(\cdot\mathbf{t}))\right]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}+\frac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N} |L_k(\mathbf{t})|\,V^{\varphi}\left[3MA\lambda(f(\cdot\mathbf{t})-f)\right]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$

$$+\frac{1}{3}\left|\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N} L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}-1\right|V^{\varphi}\left[3\lambda f\right]$$

$$=: V_1+V_2+V_3.$$

Notice that $V^{\varphi}\left[3MA\lambda(H_k\circ f(\cdot\mathbf{t})-f(\cdot\mathbf{t}))\right]=V^{\varphi}\left[3MA\lambda(H_k\circ f-f)\right]$ for all $\mathbf{t}\in\mathbb{R}_+^N$. Considering **(iv)** in Lemma 1 of [4], then there exists a $\gamma>0$ such that there exists a number $k_0\in\mathbb{N}$ satisfying

$$V^{\varphi}\left[3MA\lambda(H_k\circ f-f)\right]<\varepsilon \tag{6}$$

for all $k>k_0$ where λ is sufficiently small such that $\lambda\leq\gamma/(3MA)$. Therefore, if we divide V_1 as follows

$$V_1=\frac{1}{3MA}\sum_{k=1}^{k_0} a_{nk}^{\upsilon}V^{\varphi}\left[3MA\lambda(H_k\circ f-f)\right]\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}+\frac{1}{3MA}\sum_{k=k_0+1}^{\infty} a_{nk}^{\upsilon}V^{\varphi}\left[3MA\lambda(H_k\circ f-f)\right]\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$

$$=: V_1^1+V_1^2,$$

from (6), **(i)** and (1)

$$V_1^2\leq\frac{\varepsilon}{3MA}\sum_{k_0+1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$

$$\leq\frac{\varepsilon}{3}.$$

On the other hand, it is not hard to see from (1) that

$$V_1^1\leq\frac{1}{3M}\sum_{k=1}^{k_0} a_{nk}^{\upsilon}V^{\varphi}\left[3MA\lambda(H_k\circ f-f)\right]$$

$$\leq\frac{K}{3M}\sum_{k=1}^{k_0} a_{nk}^{\upsilon}<\frac{Kk_0}{3M}\varepsilon$$

where $K = \max_{k\in\{1,\cdots,k_0\}} V^{\varphi}\left[3MA\lambda(H_k \circ f - f)\right]$.

In V_2, one can obtain from [5] that there exists a $\bar{\lambda} > 0$ such that for any $\varepsilon > 0$

$$V^{\varphi}\left[\bar{\lambda}(f(\cdot\mathbf{t}) - f)\right] < \varepsilon \tag{7}$$

whenever

$$|\mathbf{1}-\mathbf{t}| < \delta.$$

Now, dividing the integral in V_2 as follows

$$\begin{aligned} V_2 = & \tfrac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon} \int\limits_{|\mathbf{1}-\mathbf{t}|<\delta} |L_k(\mathbf{t})|\, V^{\varphi}\left[3MA\lambda(f(\cdot\mathbf{t}) - f)\right] \frac{d\mathbf{t}}{\langle \mathbf{t}\rangle} \\ & + \tfrac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon} \int\limits_{|\mathbf{1}-\mathbf{t}|\geq\delta} |L_k(\mathbf{t})|\, V^{\varphi}\left[3MA\lambda(f(\cdot\mathbf{t}) - f)\right] \frac{d\mathbf{t}}{\langle \mathbf{t}\rangle} \\ & =: V_2^1 + V_2^2 \end{aligned}$$

from (7) there holds

$$\begin{aligned} V_2^1 & < \tfrac{\varepsilon}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon} \int\limits_{|\mathbf{1}-\mathbf{t}|<\delta} |L_k(\mathbf{t})| \frac{d\mathbf{t}}{\langle \mathbf{t}\rangle} \\ & \leq \tfrac{\varepsilon}{3} \end{aligned}$$

for $\lambda \leq \bar{\lambda}/3MA$. For V_2, since from (3) $V^{\varphi}\left[3MA\lambda(f(\cdot\mathbf{t}) - f)\right] \leq \sqrt{2} V^{\varphi}\left[6MA\lambda f\right]$, by the assumption **(ii)**

$$\begin{aligned} V_2^2 & \leq \tfrac{V^{\varphi}[6MA\lambda f]\sqrt{2}}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon} \int\limits_{|\mathbf{1}-\mathbf{t}|\geq\delta} |L_k(\mathbf{t})| \frac{d\mathbf{t}}{\langle \mathbf{t}\rangle} \\ & < \tfrac{V^{\varphi}[6MA\lambda f]\sqrt{2}}{3MA}\varepsilon \end{aligned}$$

for sufficiently large $n \in \mathbb{N}$.

Finally, directly from **(iii)**, we get

$$V_3 < \tfrac{V^{\varphi}[3\lambda f]}{3}\varepsilon$$

for sufficiently large $n \in \mathbb{N}$, which completes the proof. □

3 Rate of Approximation

Now, we investigate the rate of approximation of our operators. To this end, we need certain assumptions:

Let $\alpha \in (0,1]$. For any fixed $0 < \delta < 1$,

(I)

$$\sum_{k=1}^{\infty} a_{nk}^{\upsilon} \int\limits_{|\mathbf{1}-\mathbf{t}|\geq\delta} |L_k(\mathbf{t})| \frac{d\mathbf{t}}{\langle \mathbf{t}\rangle} = O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \upsilon)$$

(II)

$$\sum_{k=1}^{\infty} a_{nk}^{\upsilon} \int\limits_{|\mathbf{1}-\mathbf{t}|<\delta} |L_k(\mathbf{t})|\, |\log \mathbf{t}|^{\alpha} \frac{d\mathbf{t}}{\langle \mathbf{t}\rangle} = O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \upsilon)$$

(III)

$$\sum_{k=1}^{\infty} a_{nk}^{\upsilon} \int\limits_{\mathbb{R}_+^N} L_k(\mathbf{t}) \frac{d\mathbf{t}}{\langle \mathbf{t}\rangle} - 1 = O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \upsilon)$$

(IV) for all $\gamma > 0$, there exists a $\lambda > 0$ such that

$$\sum_{k=1}^{\infty} a_{nk}^{\upsilon} \frac{V^{\varphi}\left[\lambda G_k\right]}{\varphi(\gamma m(J))} = O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \upsilon \text{ and every proper bounded interval } J\subset\mathbb{R}).$$

We also need the following class of φ-absolutely continuous functions.

$$V_N^{\varphi} Lip(\alpha) := \left\{ f \in AC^{\varphi}\left(\mathbb{R}_+^N\right) : \exists\bar{\lambda} > 0 \text{ s.t. } V^{\varphi}\left[\bar{\lambda}(f(\cdot\mathbf{t}) - f)\right] = O\left(|\log \mathbf{t}|^{\alpha}\right) \text{ as } |\mathbf{1}-\mathbf{t}| \to 0 \right\}.$$

Theorem 3.1. *Suppose that $\alpha \in (0,1]$ and* **(I)**–**(IV)** *and* **(i)** *hold. If $f \in V_N^{\varphi} Lip(\alpha) \cap BV^{\eta}\left(\mathbb{R}_+^N\right)$, then there exists a $\lambda > 0$ such that*

$$V^{\varphi}\left[\lambda\left(\mathcal{T}_{n,\upsilon}(f) - f\right)\right] = O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \upsilon)$$

holds.

Proof. Considering **(i)**, from (3) we have

$$V^{\varphi}\left[\lambda\left(\mathcal{T}_{n,\upsilon}(f)-f\right)\right] \leq \tfrac{1}{3M}\sum_{k=1}^{\infty} a_{nk}^{\upsilon} V^{\varphi}\left[3MA\lambda(H_k\circ f-f)\right]+\tfrac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N}|L_k(\mathbf{t})|\,V^{\varphi}\left[3MA\lambda(f(\cdot\mathbf{t})-f)\right]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$
$$+\tfrac{1}{3}\left|\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{\mathbb{R}_+^N} L_k(\mathbf{t})\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}-1\right| V^{\varphi}[3\lambda f]$$
$$=: W_1+W_2+W_3$$

for sufficiently large $n\in\mathbb{N}$. In W_1, from assumption **(IV)** and [4], for all $\gamma>0$ there exists a $\bar{\lambda}>0$ such that

$$\sum_{k=1}^{\infty} a_{nk}^{\upsilon} V^{\varphi}\left[3MA\lambda(H_k\circ f-f)\right]\leq RV^{\varphi}[\gamma f]\,n^{-\alpha}$$
$$=O\left(n^{-\alpha}\right)\text{ as } n\to\infty$$

for some $R>0$, for which $\lambda\leq\bar{\lambda}/(3MA)$ and $V^{\varphi}[\gamma f]<\infty$. Therefore, we have

$$W_1=O\left(n^{-\alpha}\right)\text{ as } n\to\infty.$$

Now, in W_2, since $f\in V_N^{\varphi}Lip(\alpha)$, $\exists\tilde{\lambda}>0$ scuh that there exists a number $S,\delta_0>0$ satisfying

$$V^{\varphi}\left[\tilde{\lambda}\left(f(\cdot\mathbf{t})-f\right)\right]\leq S\,|\log\mathbf{t}|^{\alpha}$$

whenever $|\mathbf{1}-\mathbf{t}|<\delta_0$. Taking this expression into account, if we divide the integral in W_2 as follows

$$W_2=\tfrac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{|\mathbf{1}-\mathbf{t}|<\delta_0}|L_k(\mathbf{t})|\,V^{\varphi}\left[3MA\lambda(f(\cdot\mathbf{t})-f)\right]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}+\tfrac{1}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{|\mathbf{1}-\mathbf{t}|\geq\delta_0}|L_k(\mathbf{t})|\,V^{\varphi}\left[3MA\lambda(f(\cdot\mathbf{t})-f)\right]\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$
$$=: W_2^1+W_2^2$$

we may obtain from **(II)** that

$$W_2^1\leq\tfrac{S}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{|\mathbf{1}-\mathbf{t}|<\delta_0}|L_k(\mathbf{t})|\,|\log\mathbf{t}|^{\alpha}\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$
$$=O\left(n^{-\alpha}\right)\text{ as } n\to\infty$$

for $\lambda\leq\tilde{\lambda}/3MA$. On the other hand, we know from (3) that

$$V^{\varphi}\left[3MA\lambda(f(\cdot\mathbf{t})-f)\right]\leq\sqrt{2}V^{\varphi}[6MA\lambda f]$$

holds. Therefore, from **(I)**

$$W_2^2\leq\tfrac{V^{\varphi}[6MA\lambda f]\sqrt{2}}{3MA}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\int_{|\mathbf{1}-\mathbf{t}|\geq\delta_0}|L_k(\mathbf{t})|\frac{d\mathbf{t}}{\langle\mathbf{t}\rangle}$$
$$=O\left(n^{-\alpha}\right)\text{ as } n\to\infty.$$

Finally, from **(III)**

$$W_3=O\left(n^{-\alpha}\right)\text{ as } n\to\infty.$$

□

4 Characterization of Absolute Continuity

We need the following assumption.

(V) For any N-dimensional interval $I=\Pi_{i=1}^N[a_i,b_i]\subset\mathbb{R}_+^N$ and for all nonoverlapping intervals $\left\{\left[{}^q\alpha_j,{}^q\beta_j\right]\right\}_{q=1,\cdots,m}$ of the one dimensional interval $\left[a_j,b_j\right]$, there can be found a $\lambda>0$ such that: for all $\varepsilon>0$, $\exists\delta>0$ such that

$$\sum_{q=1}^{m}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\varphi(\lambda\left|L_k(\mathbf{r}'_j,{}^q\beta_j)-L_k(\mathbf{r}'_j,{}^q\alpha_j)\right|)<\varepsilon$$

whenever $\sum_{q=1}^m\varphi({}^q\beta_j-{}^q\alpha_j)<\delta$ for every $j=1,\cdots,N$.

(V′) For any N-dimensional interval $I=\Pi_{i=1}^N[a_i,b_i]\subset\mathbb{R}_+^N$ and for all nonoverlapping intervals $\left\{\left[{}^q\alpha_j,{}^q\beta_j\right]\right\}_{q=1,\cdots,m}$ of the one dimensional interval $\left[a_j,b_j\right]$, there can be found a $\lambda>0$ such that: for all $\varepsilon>0$, $\exists\delta>0$ such that

$$\sum_{q=1}^{m}\sum_{k=1}^{\infty} a_{nk}^{\upsilon}\varphi(\lambda\left|L_k(\mathbf{r}'_j,{}^q\beta_j)-L_k(\mathbf{r}'_j,{}^q\alpha_j)\right|)<\varepsilon$$

whenever $\sum_{q=1}^m\varphi(\log({}^q\beta_j)-\log({}^q\alpha_j))<\delta$ for every $j=1,\cdots,N$.

Remark 1. We remind that, from Proposition 3.3 in [8], one can clearly understand that conditions (**V**) and (**V**′) are equivalent. We also remind that if $\mathcal{A}$ is row finite, that is, for all $\upsilon \in \mathbb{N}$, each row of A^{υ} contains finite number of nonzero terms, then the conditions (**V**) and (**V**′) reduces to $L_k \in AC^{\varphi}_{loc}\left(\mathbb{R}^N_+\right)$ or equivalently $L_k \in \log\text{-}AC^{\varphi}_{loc}\left(\mathbb{R}^N_+\right)$ defined in [8, 14].

Lemma 4.1. *Assume that* (**i**) *holds and* L_k *satisfies* (**V**) *(or* (**V**′)*). If* $\psi \circ |f| \in L^1_{\mu}\left(\mathbb{R}^N_+\right)$ *and* $f \in BV^{\eta}\left(\mathbb{R}^N_+\right) \cap AC^{\varphi}\left(\mathbb{R}^N_+\right)$*, then* $\mathcal{T}_{n,\upsilon}(f) \in AC^{\varphi}\left(\mathbb{R}^N_+\right)$.

Proof. Using the substitution $\mathbf{rt} = \mathbf{x}$ in our operator, we immediately obtain

$$\mathcal{T}_{n,\upsilon}(f;\mathbf{r}) = \sum_{k=1}^{\infty} a^{\upsilon}_{nk} \int_{\mathbb{R}^N_+} L_k\left(\frac{\mathbf{x}}{\mathbf{r}}\right) H_k(f(\mathbf{x})) \frac{d\mathbf{x}}{\langle \mathbf{x} \rangle}$$

where $\frac{\mathbf{x}}{\mathbf{r}} = \left(\frac{x_1}{r_1}, \cdots, \frac{x_N}{r_N}\right)$. Let $\left\{\left[{}^q\alpha_j, {}^q\beta_j\right]\right\}_{q=1,\cdots,m}$ be a partition of the j-th section $\left[a_j, b_j\right]$ of $\Pi^N_{i=1}[a_i, b_i] \subset \mathbb{R}^N_+$ such that

$$\sum_{q=1}^{m} \varphi(\log({}^q\beta_j) - \log({}^q\alpha_j)) < \delta \text{ for every } j = 1, \cdots, N.$$

Then since

$$\sum_{q=1}^{m} \varphi\left(\log\left(\frac{x_j}{{}^q\beta_j}\right) - \log\left(\frac{x_j}{{}^q\alpha_j}\right)\right) < \delta,$$

from (1), (4), (**V**′), Jensen's inequality and Fubini-Tonelli theorem, one can easily observe that

$$\begin{aligned}
\sum_{q=1}^{m} \varphi\left(\lambda \left|\mathcal{T}_{n,\upsilon}(f;\mathbf{r}'_j, {}^q\beta_j) - \mathcal{T}_{n,\upsilon}(f;\mathbf{r}'_j, {}^q\alpha_j)\right|\right) &\leq \sum_{q=1}^{m} \varphi\left(\lambda L \sum_{k=1}^{\infty} a^{\upsilon}_{nk} \int_{\mathbb{R}^N_+} \left|L_k\left(\frac{\mathbf{x}'_j}{\mathbf{r}'_j}, \frac{x_j}{{}^q\beta_j}\right) - L_k\left(\frac{\mathbf{x}'_j}{\mathbf{r}'_j}, \frac{x_j}{{}^q\alpha_j}\right)\right| \psi(|f(\mathbf{x})|) \frac{d\mathbf{x}}{\langle \mathbf{x} \rangle}\right) \\
&\leq \frac{1}{M} \sum_{q=1}^{m} \sum_{k=1}^{\infty} a^{\upsilon}_{nk} \varphi\left(\lambda M L \int_{\mathbb{R}^N_+} \left|L_k\left(\frac{\mathbf{x}'_j}{\mathbf{r}'_j}, \frac{x_j}{{}^q\beta_j}\right) - L_k\left(\frac{\mathbf{x}'_j}{\mathbf{r}'_j}, \frac{x_j}{{}^q\alpha_j}\right)\right| \psi(|f(\mathbf{x})|) \frac{d\mathbf{x}}{\langle \mathbf{x} \rangle}\right) \\
&\leq \frac{1}{M \|\psi \circ |f|\|_{L^1_\mu}} \sum_{q=1}^{m} \sum_{k=1}^{\infty} a^{\upsilon}_{nk} \int_{\mathbb{R}^N_+} \psi(|f(\mathbf{x})|) \varphi\left(\lambda M L \|\psi \circ |f|\|_{L^1_\mu} \left|L_k\left(\frac{\mathbf{x}'_j}{\mathbf{r}'_j}, \frac{x_j}{{}^q\beta_j}\right)\right.\right. \\
&\quad \left.\left. - L_k\left(\frac{\mathbf{x}'_j}{\mathbf{r}'_j}, \frac{x_j}{{}^q\alpha_j}\right)\right|\right) \frac{d\mathbf{x}}{\langle \mathbf{x} \rangle} \\
&< \frac{\varepsilon}{M},
\end{aligned}$$

for sufficiently small $\lambda > 0$ and for every $j = 1, \cdots, N$. On the other hand, we know from Lemma 2.2 that $\mathcal{T}_{n,\upsilon}(f) \in BV^{\varphi}\left(\mathbb{R}^N_+\right)$, which means $\mathcal{T}_{n,\upsilon}(f) \in AC^{\varphi}\left(\mathbb{R}^N_+\right)$. □

Theorem 4.2. *Suppose that* (**I**)−(**V**) *hold. Suppose further that* $(\psi \circ |f|) \in L^1_{\mu}\left(\mathbb{R}^N_+\right)$ *and* $f \in BV^{\eta}\left(\mathbb{R}^N_+\right)$*. Then there holds*

$$f \in AC^{\varphi}\left(\mathbb{R}^N_+\right) \iff \exists \mu > 0, \ \lim_{n\to\infty} V^{\varphi}\left[\mu\left(\mathcal{T}_{n,\upsilon}(f) - f\right)\right] = 0 \textit{ uniformly in } \upsilon.$$

Proof. If $f \in AC^{\varphi}\left(\mathbb{R}^N_+\right)$, then from Theorem 2.3, we obtain the sufficiency part. For the necessity part, it is known from Proposition 4.3 in [8] that $AC^{\varphi}\left(\mathbb{R}^N_+\right)$ is a closed subspace of $BV^{\varphi}\left(\mathbb{R}^N_+\right)$ under the topology generated by convergence in φ-variation. Since $\mathcal{T}_{n,\upsilon}(f) \in AC^{\varphi}\left(\mathbb{R}^N_+\right)$ by the previous lemma, we finally have $f \in AC^{\varphi}\left(\mathbb{R}^N_+\right)$. □

5 Applications

Now, we will demonstrate our kernels that satisfy Theorem 4.2 and present our main approximation theorem.

Let $L_k : \mathbb{R}^2_+ \to \mathbb{R}$ be given by

$$L_k(t_1, t_2) := \tilde{L}_k(t_1)\tilde{L}_k(t_2) \tag{8}$$

where $\tilde{L}_k : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\tilde{L}_k(t) = \begin{cases} \frac{kt}{2} - \frac{k^2 t}{2}|t-1|; & |t-1| \leq \frac{1}{k} \\ 0; & \text{otherwise.} \end{cases}$$

The conditions of Theorem 4.2 are easily satisfied for $\mathcal{A} = \{I\}$, the identity matrix. See Figure 1 for the characterization kernel in (8).

Another example involves our main approximation theorem for $N = 2$. To this end, we consider the following kernel $L_k : \mathbb{R}^2_+ \to \mathbb{R}$, defined as:

$$L_k(t_1, t_2) := \begin{cases} k^2 t_1 t_2; & |t_1 - 1| \leq \frac{1}{2k} \text{ and } |t_2 - 1| \leq \frac{1}{2k} \\ 0; & \text{otherwise,} \end{cases} \tag{9}$$

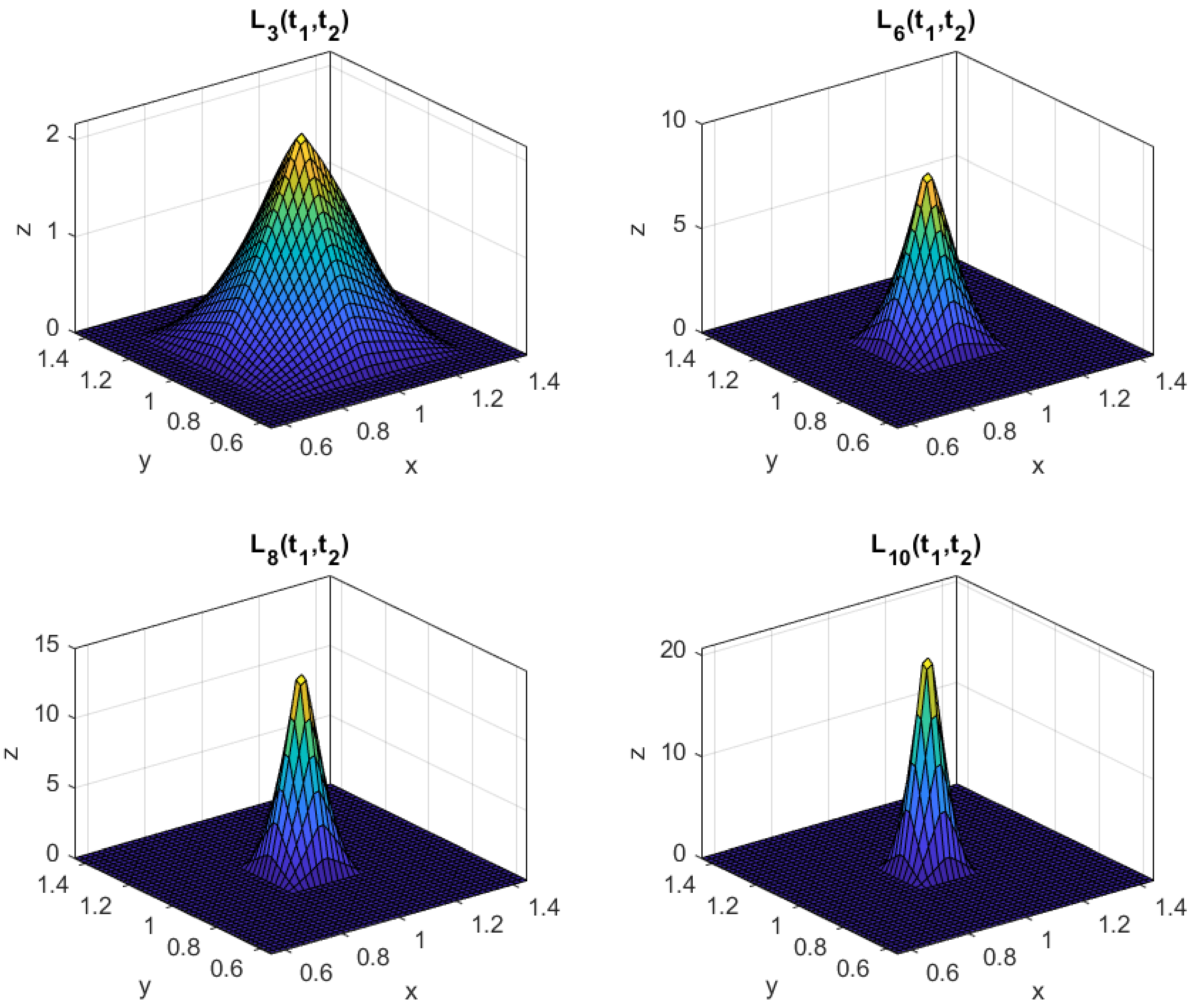

Figure 1: The two-dimensional kernel function for characterization in (8).

and $H_k : \mathbb{R} \to \mathbb{R}$ such that, H_k is given by

$$H_k(u) = \begin{cases} \log\left(1 + \frac{u}{k}\right); & 0 \le u < 1 \\ \log\left(1 + \frac{1}{ku}\right); & u \ge 1 \end{cases}$$

on $[0, \infty)$ and extended it in the odd way ([4]). Then it is not hard to see that L_k satisfies **(i)**-**(iii)** for $\mathcal{A} = \{I\}$ and H_k satisfies **(iv)** (see [4]). If we also assume that $\psi(|u|) = |u|$, then condition (4) is satisfied. The graph of the kernel in (9) is shown in Figure 2. Now, using the above definitions and considering $f(x, y) : \mathbb{R}_+^2 \to \mathbb{R}$ such that $f(x, y) = \arctan x \arctan y$, we obtain the following approximations by means of Mellin-type nonlinear integral operator assuming $\mathcal{A} = \{I\}$, the identity matrix, in Figure 3. We should also note that, since classical nonlinear multidimensional approximation has not been previously investigated, we illustrate our examples under the assumption $\mathcal{A} = \{I\}$. However, it is not difficult to find non-trivial examples where classical convergence fails, yet summability methods are effective (see, for instance, kernel (4.4) in [12]).

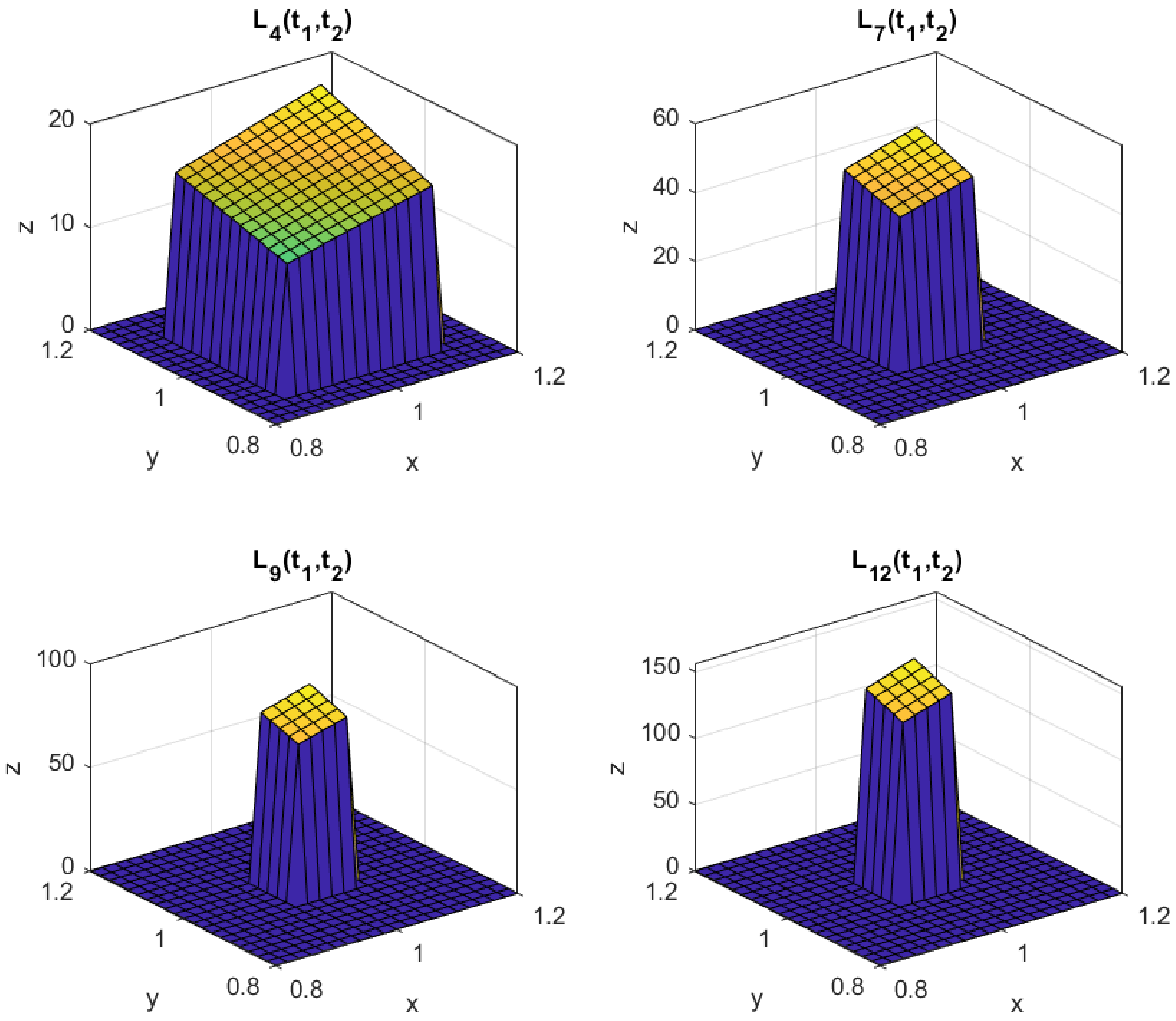

Figure 2: The two-dimensional kernel function for approximation in (9).

6 Conclusion

In this paper, we prove approximation theorems for N-dimensional Mellin-type nonlinear integral operators under convergence in φ-variation and generalize it with Bell-type summability method. Then we also give a general characterization theorem for the space of (Tonelli sense) φ-absolutely continuous functions. If we select specific families of matrices, such as the Cesàro matrix ([27]), the almost convergence matrix ([27, 29]), or the identity matrix (see [13]), in place of $\mathcal{A}$, all these results reduce to Cesàro mean convergence, almost convergence, and classical convergence, respectively. On the other hand, it is also possible to extend φ-variation theory to the $\mathcal{F}^{\varphi}$-variation, which is inspired by $\mathcal{F}$-variation (see [3, 11]).

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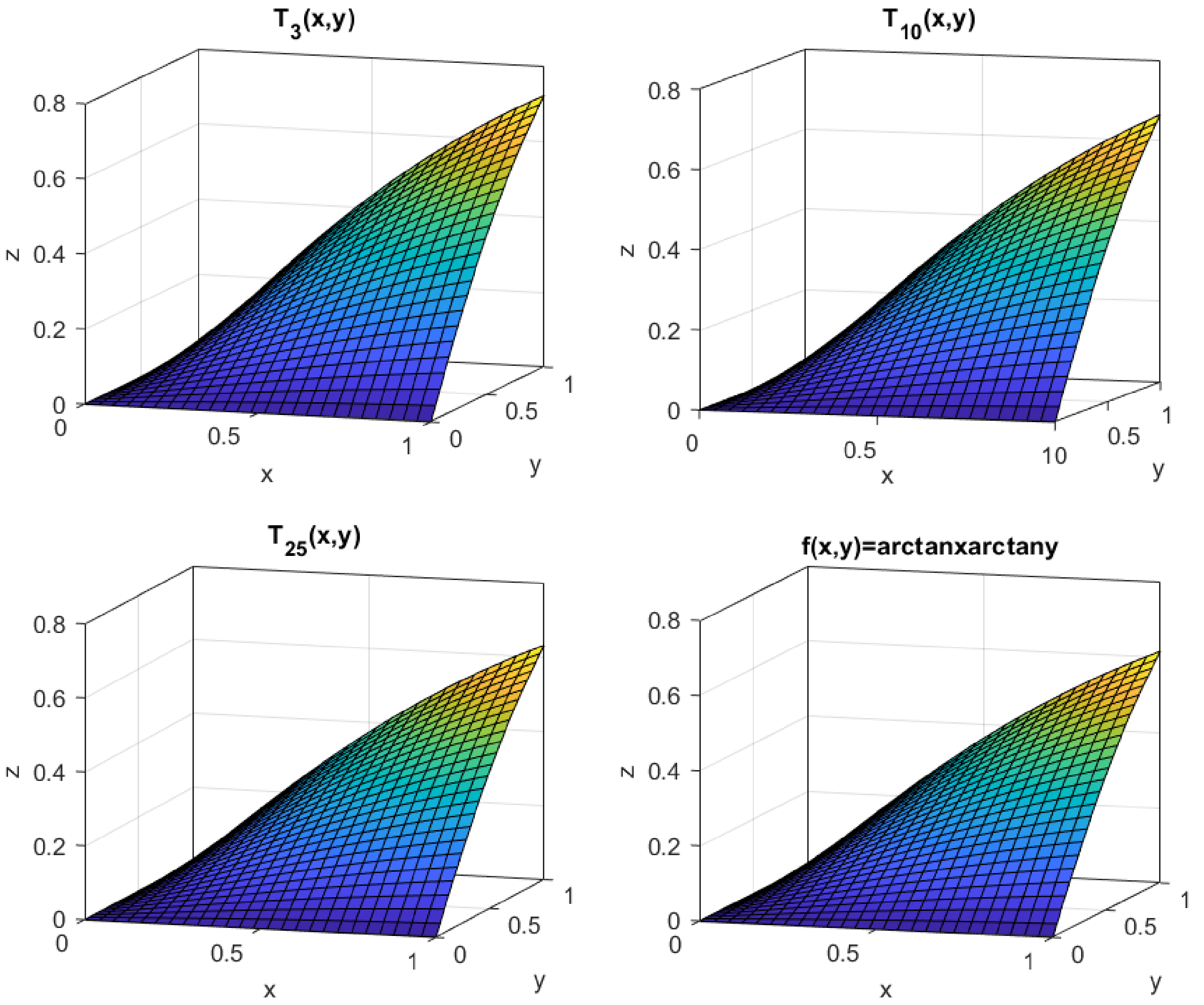

Figure 3: Approximation to f by nonlinear Mellin operator

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