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Approximation Results on an Infinite Interval Based on Power Series Statistical Sense

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Abstract

This paper introduces a new approximation theorem, type of Korovkin, for positive linear operators (pLO) defined on the Banach space $C_*[0,\infty)$ comprising all real-valued continuous functions on $[0,\infty)$ that converge to a finite limit as their argument approaches infinity. By applying statistical convergence with respect to power series methods and employing the test functions 1, $\exp(-u)$ and $\exp(-2u)$, we derive a novel approximation result. Our findings demonstrate that the proposed method outperforms classical and statistical approaches, as illustrated by a concrete example. Furthermore, we explore the rate of convergence associated with this new approximation theorem.

Keywords Statistical convergence, power series method, Korovkin type theorem, rate of convergence

2020 Mathematics Subject Classification 40A35, 40G10, 41A25, 41A36

1 Introduction and Preliminaries

Approximation theory forms a fundamental aspect of mathematical analysis, aiming to represent complex functions through simpler, more manageable ones. The well-known Korovkin theorem [6, 16] provides the necessary conditions under which a sequence of positive linear operators (pLO) converges to the target function. In the mid-20th century, Korovkin [16] pioneered the study of these conditions and identified the test functions 1, v and v^2 as sufficient for uniform convergence. Subsequent research, exemplified by the works of [1] and [14], has extended these investigations to various operator classes and function spaces. The introduction of statistical convergence into approximation theory has brought significant benefits. This elegant and straightforward approach, which differs from traditional matrix summability methods, provides a powerful tool for addressing convergence problems in operators such as those of the Hermite-Fejér type [3]. These operators, known for their non-convergence at points of simple discontinuity, can be effectively corrected by statistical convergence depends on the natural density. Recent advances, as exemplified by the studies of [13] and [11], have used the concept of statistical convergence to develop novel Korovkin-type theorems. These generalized theorems extend the classical results and provide a more comprehensive framework for approximation theory. The concept of replacing natural density with density via the power series method (referred to as P-density) was recently introduced in [24]. This approach has since been applied to define several related notions, including equi-P-statistical convergence [7], P-statistical relative uniform convergence of function sequences at a point [8] and P-statistical relative modular convergence [10].

In the present work, we give a novel Korovkin-type approximation theorem for pLO defined on the Banach space $C_*[0,\infty)$ comprising all real-valued continuous functions on $[0,\infty)$ that converge to a finite limit as their argument approaches infinity. By employing statistical convergence with respect to power series methods and utilizing the test functions 1, $\exp(-v)$ and $\exp(-2v)$, we establish a new approximation result. Our findings demonstrate that the proposed method outperforms classical and statistical approaches, as illustrated by a concrete example. Moreover, we investigate the rate of convergence for the new approximation theorem.

The requisite definitions and notational conventions employed throughout this work are now delineated.

Let $\mathbb{N}$ represent the set of natural numbers, and consider any subset $F \subseteq \mathbb{N}$. The natural density of F, denoted by $\delta(F)$, is defined as:

$$\delta(F) := \lim_{i} \frac{\sharp\{j \le i \ : j \in F\}}{i}$$

whenever the limit exists, where $\sharp\{\cdot\}$ indicates the cardinality of the set [18].

A sequence of numbers $v = (v_j)$ is called statistically (st) convergent to $\mathcal{L}$ if, for every $\xi > 0$,

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$$\lim_{i} \frac{\sharp\left\{j \le i : \left|v_j - \mathcal{L}\right| \ge \xi\right\}}{i} = 0$$

meaning that the set,

$$F_k(\xi) := \left\{j \le i : \left|v_j - \mathcal{L}\right| \ge \xi\right\}$$

has natural density zero. This type of convergence is denoted by $st - \lim_j v_j = \mathcal{L}$ ([12], [20]). As per the definition, a sequence that converges in the classical sense is also statistically convergent to the same limit. However, the converse does not hold; a statistically convergent sequence may not necessarily converge in the classical sense.

A more recent form of statistical convergence, known as P-statistical convergence, was introduced by Ünver and Orhan [24] (see also [23]). This method, which is based on power series, offers a distinct approach to convergence analysis. The power series methods' two notable examples are the Abel and Borel methods, which often provide superior results compared to ordinary convergence.

Let $\left(p_j\right)$ be a positive sequence of real numbers such that $p_0 > 0$ and the associated power series

$$p(t) := \sum_{j=0}^{\infty} p_j t^j$$

has radius of convergence $\mathcal{R}$ with $0 < \mathcal{R} \le \infty$. A sequence $v = \left(v_j\right)$ is *convergent in the sense of power series method* provided that $\lim_{0<t\to\mathcal{R}^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j v_j = \mathcal{L}$ ([17], [19]).

Keep in mind that the method is regular iff $\lim\limits_{0<t\to\mathcal{R}^-} \dfrac{p_j t^j}{p(t)} = 0$ for every j (see,e.g. [4]).

Remark 1. It is important to highlight specific cases of power series methods. When $\mathcal{R} = 1$, if $p_j = 1$, the method reduces to the Abel method, and if $p_j = \frac{1}{j+1}$, it corresponds to the logarithmic method. Furthermore, if $\mathcal{R} = \infty$ and $p_j = \frac{1}{j!}$, the power series method becomes the Borel method.

It is henceforth assumed that the power series method adheres to the regularity condition, which is a fundamental assumption in this analysis.

Now, let us reconsider the following definitions within the framework of power series methods:

Definition 1.1. [24] Let $F \subset \mathbb{N}_0$. The limit

$$\delta_P(F) := \lim_{0<t\to\mathcal{R}^-} \frac{1}{p(t)} \sum_{j\in F} p_j t^j$$

is referred to as the P-density of F. According to the definition of the power series method and P-density, we have $0 \le \delta_P(F) \le 1$.

Lemma 1.1. *(Elementary Properties)*

(i) $\delta_P(\mathbb{N}_0) = 1$,

(ii) *if* $F \subset G$ *then* $\delta_P(F) \le \delta_P(G)$,

(iii) *if* F *has* P*-density then* $\delta_P(\mathbb{N}_0 \backslash F) = 1 - \delta_P(F)$.

Definition 1.2. [24] Let $v = \left(v_j\right)$ be a sequence. Then v is called P-statistically convergent, i.e., statistically convergent with respect to power series method to $\mathcal{L}$ if, for any $\xi > 0$

$$\lim_{0<t\to\mathcal{R}^-} \frac{1}{p(t)} \sum_{j\in F_\xi} p_j t^j = 0$$

which means that $\delta_P\left(F_\xi\right) = 0$ for any $\xi > 0$ where $F_\xi = \left\{j \in \mathbb{N}_0 : \left|v_j - \mathcal{L}\right| \ge \xi\right\}$. This convergence is denoted by $st_P - \lim v_j = \mathcal{L}$.

Definition 1.3. A sequence of real numbers $v = \left(v_j\right)$ is called P-statistically bounded if for some $H > 0$ such that $\delta_P\left(\left\{j : \left|v_j\right| > H\right\}\right) = 0$.

Here, c denotes the set of all convergent sequences, st represents the set of statistically convergent sequences and st_P corresponds to the set of P-statistically convergent sequences.

Remark 2. i) Statistical convergence and P-statistical convergence are distinct concepts that do not necessarily coincide; i.e., $st \nsubseteq st_P$ and $st_P \nsubseteq st$.

ii) While classical convergence guarantees statistical and P-statistical convergence to the same limit, the converse is not necessarily true. There are sequences that converge statistically or P-statistically but not in the ordinary sense, i.e., $st \nsubseteq c$ and $st_P \nsubseteq c$.

In recent years, there has been a notable expansion in the study of approximation theorems of the Korovkin type, particularly with the incorporation of statistical convergence methods. A noteworthy approach is the application of the power series method to establish theorems of Korovkin-type on unbounded intervals. This has previously presented a challenge due to the complexity of controlling the behaviour of approximating functions at infinity. The concept of statistical convergence, which generalizes classical convergence by relaxing the uniformity of convergence conditions, provides a flexible framework for addressing these challenges. Previous research in this field has predominantly concentrated on bounded intervals. However, recent developments indicate that utilising the power series method can facilitate a deeper understanding of the rate of approximation and its behaviour over unbounded domains. This study aims to extend current knowledge by investigating approximation processes that not only satisfy Korovkin-type conditions statistically but also provide explicit rates of convergence under these expanded frameworks.

2 Approximation in the Statistical Sense Via Power Series Methods

Power series methods serve as a fundamental tool for approximating complex functions in various fields of mathematical analysis and applied sciences. The space $C_*[0,\infty)$ served as the backdrop for Boyanov and Veselinov's (1970) innovative Korovkin-type theorem, equipped with the standard supremum norm $\|g\| = \sup_{v\in[0,\infty)} |g(v)|$ ([5]). Boyanov and Veselinov utilized the test functions 1, $\exp(-v)$ and $\exp(-2v)$ instead of the classical 1, x and x^2. In this section, we introduce a Korovkin-type approximation theorem by utilizing the concept of P-statistical convergence, along with the test functions 1, $\exp(-v)$ and $\exp(-2v)$.

In the framework of power series statistical convergence, we establish a Korovkin-type result. To begin, we revisit the classical Korovkin theorem for sequences of pLO, as formulated by Boyanov and Veselinov [5] and later extended by Demirci and Karakuş [9]. Building on these foundational results, we proceed to present an analogous theorem within the context of power series statistical convergence, highlighting its relevance and applicability in this new setting.

Let T be a linear operator from $C_*[0,\infty)$ into itself. We define T as a positive linear operator if it preserves non-negativity, meaning that $T(g)\geq 0$ for all non-negative functions g. We denote the value of $T(g)$ at a point $v\in[0,\infty)$ by $T(g;v)$.

Theorem 2.1. *[5] Let (T_j) be a sequence of pLO, acting from $C_*[0,\infty)$ into itself, satisfies the conditions*

a) $\lim_j \left\|T_j(1)-1\right\| = 0,$

b) $\lim_j \left\|T_j(\exp(-s))-\exp(-v)\right\| = 0,$

c) $\lim_j \left\|T_j(\exp(-2s))-\exp(-2v)\right\| = 0,$

then, for all $g\in C_*[0,\infty)$

$$\lim_j \left\|T_j(g)-g\right\| = 0.$$

Theorem 2.2. *[9] Let (T_j) be a sequence of pLO acting from $C_*[0,\infty)$ into itself. Then, for all $g\in C_*[0,\infty)$*

$$st-\lim_j \left\|T_j(g)-g\right\| = 0$$

if and only if the following statements hold:

a) $st-\lim_j \left\|T_j(1)-1\right\| = 0,$

b) $st-\lim_j \left\|T_j(\exp(-s))-\exp(-v)\right\| = 0,$

c) $st-\lim_j \left\|T_j(\exp(-2s))-\exp(-2v)\right\| = 0.$

We now prove the following lemma:

Lemma 2.3. *Consider a sequence (T_j) of pLO mapping $C_*[0,\infty)$ into itself. Then, for every $\xi>0$, there exists a $\delta>0$ and $M>0$ such that for every $T_j: C_*[0,\infty)\to C_*[0,\infty)$, $g\in C_*[0,\infty)$ and $v\in[0,\infty)$ the following inequality holds*

$$\begin{aligned} & \left|T_j(g;v)-g(v)\right| \\ \leq\ & \xi+\left(\xi+M+\frac{2M}{\delta^2}\right)\left|T_j(1;v)-1\right| \\ & +\frac{4M}{\delta^2}\left|T_j(\exp(-s);v)-\exp(-v)\right|+\frac{2M}{\delta^2}\left|T_j(\exp(-2s);v)-\exp(-2v)\right|. \end{aligned}$$

Proof. Let $g\in C_*[0,\infty)$. There exists a constant M such that $|g(v)|\leq M$ for each $v\in[0,\infty)$. Because $|\exp(-s)-\exp(-v)|\leq |s-v|$ $(s,v\geq 0)$, for a given $\xi>0$, there exists a $\delta>0$ such that $|g(s)-g(v)|<\xi$ whenever $|\exp(-s)-\exp(-v)|<\delta$, for every $s,v\in[0,\infty)$. Then it is seen that the following inequality holds

$$|g(s)-g(v)| < \xi+\frac{2M}{\delta^2}(\exp(-s)-\exp(-v))^2$$

for all $s,v\in[0,\infty)$, namely,

$$-\xi-\frac{2M}{\delta^2}(\exp(-s)-\exp(-v))^2 < g(s)-g(v) < \xi+\frac{2M}{\delta^2}(\exp(-s)-\exp(-v))^2.$$

Operating $T_j(1;v)$ to this inequality and in the light of the linearity and the positivity of the operators T_j, we have that

$$\begin{aligned}&\left|T_j(g;v)-g(v)\right|\\ \leq\ & \xi+(\xi+M)\left|T_j(1;v)-1\right|+\frac{2M}{\delta^2}\left|\exp(-2v)\right|\left|T_j(1;v)-1\right|\\ &+\frac{2M}{\delta^2}\left|T_j(\exp(-2s);v)-\exp(-2v)\right|+\frac{4M}{\delta^2}\left|e^{-v}\right|\left|T_j(\exp(-s);v)-\exp(-v)\right|\\ \leq\ & \xi+\left(\xi+M+\frac{2M}{\delta^2}\right)\left|T_j(1;v)-1\right|\\ &+\frac{4M}{\delta^2}\left|T_j(\exp(-s);v)-\exp(-v)\right|+\frac{2M}{\delta^2}\left|T_j(\exp(-2s);v)-\exp(-2v)\right|,\end{aligned}$$

where $|\exp(-kv)|\leq 1$ for all $v\in[0,\infty)$ and $k\in\mathbb{N}$. Thus, we reach to our required result. □

The primary result of the relevant section is presented in the following theorem:

Theorem 2.4. *Let (T_j) be a sequence of pLO acting from $C_*[0,\infty)$ into itself. Then, for all $g\in C_*[0,\infty)$*

$$st_P-\lim\left\|T_j(g)-g\right\|=0$$

iff the following statements hold:

a) $st_P-\lim\left\|T_j(1)-1\right\|=0,$

b) $st_P-\lim\left\|T_j(\exp(-s))-\exp(-v)\right\|=0,$

c) $st_P-\lim\left\|T_j(\exp(-2s))-\exp(-2v)\right\|=0.$

Proof. It is sufficient to demonstrate the condition of sufficiency. Assume that conditions (a), (b) and (c) are satisfied. Let $g\in C_*[0,\infty)$. Then, thanks to Lemma 2.3, for every $\xi>0$, there exists a $\delta>0$ and $M>0$ such that for every $v\in[0,\infty)$, the following inequality holds

$$\begin{aligned}&\left|T_j(g;v)-g(v)\right|\\ \leq\ & \xi+\left(\xi+M+\frac{2M}{\delta^2}\right)\left|T_j(1;v)-1\right|\\ &+\frac{4M}{\delta^2}\left|T_j(\exp(-s);v)-\exp(-v)\right|+\frac{2M}{\delta^2}\left|T_j(\exp(-2s);v)-\exp(-2v)\right|.\end{aligned}$$

Then taking the supremum over $v\in[0,\infty)$, we get

$$\begin{aligned}&\left\|T_j(g)-g\right\|\\ \leq\ & \xi+K\left\{\left\|T_j(1)-1\right\|+\left\|T_j(\exp(-s))-\exp(-v)\right\|+\left\|T_j(\exp(-2s))-\exp(-2v)\right\|\right\},\end{aligned}\tag{1}$$

where $K:=\max\left\{\xi+M+\frac{2M}{\delta^2},\frac{4M}{\delta^2},\frac{2M}{\delta^2}\right\}$. For a given $\eta>0$ choose $\xi>0$ such that $\xi<\eta$ and let

$$\begin{aligned}S &:= \left\{j:\ \left\|T_j(g)-g\right\|\geq\eta\right\},\\ S^1 &:= \left\{j:\ \left\|T_j(1)-1\right\|\geq\frac{\eta-\xi}{3K}\right\},\\ S^2 &:= \left\{j:\ \left\|T_j(\exp(-s))-\exp(-v)\right\|\geq\frac{\eta-\xi}{3K}\right\},\\ S^3 &:= \left\{j:\ \left\|T_j(\exp(-2s))-\exp(-2v)\right\|\geq\frac{\eta-\xi}{3K}\right\}.\end{aligned}$$

In the view of (1), it can be shown easily that $S\subset S^1\cup S^2\cup S^3$. Thus, we may write

$$\delta_P(S)\leq\delta_P\left(S^1\right)+\delta_P\left(S^2\right)+\delta_P\left(S^3\right).\tag{2}$$

From (2), using (a), (b) and (c), we get that

$$\delta_P(S)=0$$

whence the result. □

Remark 3. Since convergence implies P-statistical convergence, any sequence that adheres to the conditions of Boyanov and Veselinov's classical theorem (Theorem 2.1) will necessarily fulfill the conditions of its P-statistical analogue (Theorem 2.4).

Now, we present the following example of sequences of pLO. This example demonstrates that there may exist a sequence of pLO that satisfies the conditions of Theorem 2.4, but fails to meet the conditions outlined in Theorem 2.1 and Theorem 2.2.

Example 2.1. Consider the following Szasz-Mirakyan operators (see [21, 22])

$$M_j(g;v)=e^{-jv}\sum_{k=0}^{\infty}g\left(\frac{k}{j}\right)\frac{(jv)^k}{k!},\tag{3}$$

where $v\in[0,\infty)$, $g\in C_*[0,\infty)$. Also, observe that

$$\begin{aligned}M_j(1;v)&=1,\\M_j(\exp(-s);v)&=\exp(-jv\left(1-\exp\left(-\frac{1}{j}\right)\right)),\\M_j(\exp(-2s);v)&=\exp(-jv\left(1-\exp\left(-\frac{2}{j}\right)\right)).\end{aligned}$$

Now we take $v=(v_j)$ by

$$v_j=\begin{cases}j, & j=2k,\\0, & \text{otherwise},\end{cases}\quad k=1,2,...,\tag{4}$$

and the power series method is provided with

$$p_j=\begin{cases}0, & j=2k,\\1, & \text{otherwise},\end{cases}\quad k=1,2,....\tag{5}$$

It is clear that

$$st_P-\lim v_j=0.\tag{6}$$

Now in the view of (3), (4) and (5), we define the following pLO on $C_*[0,\infty)$ as follows:

$$T_j(g;v)=(1+v_j)M_j(g;v).\tag{7}$$

So, by the Theorem 2.4 and (6), we see that

$$st_P-\lim\left\|T_j(g)-g\right\|=0.$$

Also, since (v_j) is not usual convergent and statistically convergent, we can say that classical and statistical versions of theorems of Korovkin type, i.e., Theorem 2.1 and Theorem 2.2 do not work for our operators defined by (7).

3 The rate of convergence for new theorem

Our focus in this section is on the convergence speed of a sequence of pLO mapping $C_*[0,\infty)$ into itself. In order to achieve this, we are using the modulus of continuity. Building upon the foundational work presented in [10] (see also [2]), we commence by revisiting the following:

Definition 3.1. [10] Let (α_j) be a non-increasing sequence of positive real numbers. A sequence $v=(v_j)$ is P-statistically convergent to a number $\mathcal{L}$ with the rate of $o(\alpha_j)$ if, for every $\xi>0$,

$$\lim_{0<t\to\mathcal{R}^-}\frac{1}{p(t)}\sum_{j\in F_\xi}p_jt^j=0,$$

where $F_\xi=\left\{j\in\mathbb{N}_0:\left|v_j-\mathcal{L}\right|\geq\xi\alpha_j\right\}$. In this case, we write

$$v_j-\mathcal{L}=st_P-o(\alpha_j).$$

Definition 3.2. [10] Let (α_j) be a positive non-increasing sequence. A sequence $v=(v_j)$ is P-statistically bounded with the rate of $O(\alpha_j)$ if there is an $B>0$ with

$$\lim_{0<t\to\mathcal{R}^-}\frac{1}{p(t)}\sum_{j\in F_\xi}p_jt^j=0,$$

where $F_\xi=\left\{j\in\mathbb{N}_0:\left|v_j\right|\geq B\alpha_j\right\}$. In this case, we write

$$v_j-\mathcal{L}=st_P-O(\alpha_j).$$

Lemma 3.1. *[10] Let (u_j) and (v_j) be sequences. Assume that (α_j) and (β_j) be non-increasing sequence of positive real numbers. If $u_j-\mathcal{L}_1=st_P-o(\alpha_j)$ and $v_j-\mathcal{L}_2=st_P-o(\beta_j)$, then we have*

(i) $(u_j-\mathcal{L}_1)\mp(v_j-\mathcal{L}_2)=st_P-o(\gamma_j)$, where $\gamma_j:=\max\{\alpha_j,\beta_j\}$ for each $j\in\mathbb{N}_0$,

(ii) $\lambda(u_j-\mathcal{L}_1)=st_P-o(\alpha_j)$ for any real number λ.

Furthermore, equivalent findings are observed when the "o" notation is replaced by "O" notation.

Recall that for a continuous function $g \in C(I)$ defined on an interval $I \subset \mathbb{R}$, the function $\omega : C(I) \times [0, \infty) \to \mathbb{R} \cup \{\infty\}$, in the following way:

$$\omega(g;\delta) := \sup\{|g(s) - g(v)| : s, v \in [0, \infty), |s - v| \leq \delta\}$$

is called its usual modulus of continuity.

To estimate the rate of convergence for a sequence of pLO mapping from $C_*[0, \infty)$ into itself, we utilize the following modulus of continuity, as introduced in [15]:

$$\omega^*(g;\delta) := \sup\{|g(s) - g(v)| : s, v \in [0, \infty), |\exp(-s) - \exp(-v)| \leq \delta\},$$

where $g \in C_*[0, \infty)$ and $\delta > 0$. This modulus can be expressed in terms of the usual modulus of continuity as:

$$\omega^*(g;\delta) = \omega(g^*;\delta),$$

where g is the continuous function defined on $[0, 1]$ by

$$g^*(v) = \begin{cases} g(-\ln v), & v \in (0, 1], \\ \lim_{s\to\infty} g(s), & v = 0. \end{cases}$$

A straightforward computation reveals that for any $\delta > 0$

$$|g(s) - g(v)| \leq \left(1 + \frac{(\exp(-s) - \exp(-v))^2}{\delta^2}\right)\omega^*(g;\delta)$$

for all $g \in C_*[0, \infty)$ (see also, [15]).

The main result of this section is given in the following theorem:

Theorem 3.2. *Let (T_j) be a sequence of pLO from $C_*[0, \infty)$ into itself. Assume that (α_j) and (β_j) be positive non-increasing sequences. Suppose that the following conditions hold:*

(i) $\|T_j(1) - 1\| = st_P - o(\alpha_j)$,

(ii) $\omega^*(g;\delta_j) = st_P - o(\beta_j)$, *where* $\delta_j := \sqrt{\|T_j(\varphi_{(v)})\|}$ *with* $\varphi_{(v)}(s) = (\exp(-s) - \exp(-v))^2$. *Then, for any* $g \in C_*[0, \infty)$

$$\|T_j(g) - g\| = st_P - o(\gamma_j),$$

where $\gamma_j := \max\{\alpha_j, \beta_j\}$. *Replacing "o" with "O" yields comparable findings.*

Proof. Let $g \in C_*[0, \infty)$ and $v \in [0, \infty)$ be fixed. Using linearity and positivity of the T_j, we have,

$$\begin{aligned}
& \left|T_j(g;v) - g(v)\right| \\
= & \left|T_j(g(s) - g(v);v) - g(v)\left(T_j(1;v) - 1\right)\right| \\
\leq & T_j(|g(s) - g(v)|;v) + M\left|T_j(1;v) - 1\right| \\
\leq & T_j\left(\left(1 + \frac{(\exp(-s) - \exp(-v))^2}{\delta^2}\right)\omega^*(g;\delta);v\right) + M\left|T_j(1;v) - 1\right| \\
\leq & \omega^*(g;\delta)T_j\left(\left(1 + \frac{(\exp(-s) - \exp(-v))^2}{\delta^2}\right);v\right) + M\left|T_j(1;v) - 1\right| \\
\leq & \omega^*(g;\delta)\left|T_j(1;v) - 1\right| + \frac{\omega^*(g;\delta)}{\delta^2}T_j(\varphi_{(v)};v) + \omega^*(g;\delta) \\
& +M\left|T_j(1;v) - 1\right|,
\end{aligned}$$

where $M := \|g\|$. Thus, taking supremum over $v \in [0, \infty)$ on the both-sides of the above inequality, we get, for any $\delta > 0$,

$$\begin{aligned}
\left\|T_j(g) - g\right\| \leq & \ \omega^*(g;\delta)\left\|T_j(1) - 1\right\| + \frac{\omega^*(g;\delta)}{\delta^2}\left\|T_j\left(\varphi_{(v)}\right)\right\| \\
& + \omega^*(g;\delta) + M\left\|T_j(1) - 1\right\|.
\end{aligned}$$

Now if $\delta := \delta_j := \sqrt{\|T_j(\varphi_{(v)})\|}$, then we can write

$$\left\|T_j(g) - g\right\| \leq \omega^*(g;\delta)\left\|T_j(1) - 1\right\| + 2\omega^*(g;\delta) + M\left\|T_j(1) - 1\right\|$$

and hence

$$\left\|T_j(g) - g\right\| \leq N\left\{\omega^*(g;\delta)\left\|T_j(1) - 1\right\| + \omega^*(g;\delta) + \left\|T_j(1) - 1\right\|\right\} \tag{8}$$

where $N = \max\{2, M\}$. Now, for a given $\eta > 0$, we define the following sets:

$$U \quad : \quad = \left\{j : \left\|T_j(g) - g\right\| \geq \eta\gamma_j\right\},$$
$$U^1 \quad : \quad = \left\{j : \omega^*(g;\delta)\left\|T_j(1) - 1\right\| \geq \frac{\eta}{3N}\beta_j\alpha_j\right\},$$
$$U^2 \quad : \quad = \left\{j : \omega^*(g;\delta) \geq \frac{\eta}{3N}\beta_j\right\},$$
$$U^3 \quad : \quad = \left\{j :, \left\|T_j(1) - 1\right\| \geq \frac{\eta}{3N}\alpha_j\right\}.$$

In the view of (8) and since $\gamma_j := \max\{\alpha_j, \beta_j\}$, we have

$$U \subset U^1 \cup U^2 \cup U^3.$$

Also, define the sets:

$$U^4 \quad : \quad = \left\{j : \omega^*(g;\delta) \geq \sqrt{\frac{\eta}{3N}}\beta_j\right\},$$
$$U^5 \quad : \quad = \left\{j :, \left\|T_j(1) - 1\right\| \geq \sqrt{\frac{\eta}{3N}}\alpha_j\right\}.$$

Then, observe that $U^1 \subset U^4 \cup U^5$. So, we have $U \subset U^2 \cup U^3 \cup U^4 \cup U^5$ which gives,

$$\delta_P(U) \leq \delta_P\left(U^2\right) + \delta_P\left(U^3\right) + \delta_P\left(U^4\right) + \delta_P\left(U^5\right).$$

Under the hypothesis (i) and (ii), we conclude that

$$\delta_P(U) = 0.$$

The theorem's proof is now complete. □

Remark 4. What if we were to replace conditions (i) and (ii) in Theorem 3.2 with the following?

(i) $\left\|T_j(1) - 1\right\| = st_P - o(\alpha_j^1)$,

(ii) $\left\|T_j(\exp(-s)) - \exp(-v)\right\| = st_P - o(\alpha_j^2)$,

(iii) $\left\|T_j(\exp(-2s)) - \exp(-2v)\right\| = st_P - o(\alpha_j^3)$.

Then, since $T_j(\varphi_{(v)}; v) = \exp(-2v)\left|T_j(1; v) - 1\right| - 2\exp(-v)\left|T_j(\exp(-s); v) - \exp(-v)\right| + \left|T_j(\exp(-2s); v) - \exp(-2v)\right|$, we can write

$$\left\|T_j(\varphi_{(v)})\right\| \leq \left\|T_j(1) - 1\right\| + 2\left\|T_j(\exp(-s)) - \exp(-v)\right\| + \left\|T_j(\exp(-2s)) - \exp(-2v)\right\|. \tag{9}$$

Now it follows (i), (ii), (iii), (9) and Lemma 3.1 that

$$\delta_j = \sqrt{\left\|T_j(\varphi_{(v)})\right\|} = st_P - o(\alpha_j),$$

where $\alpha_j := \max\left\{\alpha_j^1, \alpha_j^2, \alpha_j^3\right\}$. This implies,

$$\omega^*\left(g;\delta_j\right) = st_P - o(\alpha_j)$$

and using this in Theorem 3.2, we immediately see that,

$$\left\|T_j(g) - g\right\| = st_P - o(\alpha_j).$$

Consequently, incorporating condition (iii) into Theorem 3.2, alongside conditions (i) and (ii), enables us to derive the rates of P-statistical convergence for the sequence of pLO as established in Theorem 2.4.

4 Concluding Remarks

This work presents a fresh perspective on Korovkin-type approximation within the Banach space $C_*[0,\infty)$, focusing specifically on real-valued continuous functions that converge to a finite value as their argument grows unbounded. Through the integration of statistical convergence in the context of the power series method, and by utilizing the test functions 1, $\exp(-v)$ and $\exp(-2v)$, we derive a novel approximation theorem. This approach demonstrates a marked improvement over conventional methods, particularly in its ability to efficiently manage approximations on unbounded domains. This theoretical development is further substantiated by an explicit example, highlighting the practical significance of the framework introduced. Our focus in third section is on the convergence speed. In order to achieve this, we are using the modulus of continuity. Our results suggest that this method offers a new direction for approximation theory, potentially broadening the scope of Korovkin-type theorems and enabling their application to more complex operators and diverse contexts. Future studies could expand on these findings by exploring the method's applicability to a wider class of function spaces and examining its implications within other areas of analysis.

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