PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 18 · 2025 · Pages 8–16

Asymptotic Expansion of Wavelet Type Generalized Bézier Operators

Huseyin Erhan Altin[a] · Harun Karsli[b]

Abstract

In this paper we attain certain asymptotic properties of the recently introduced [4] wavelet type generalized Bézier operators by means of the compacted Daubechies of ξ. The basis used in constructing these types of operators is the wavelet expansion of ξ, rather than its sampled values $\xi\left(\frac{k}{n}\right)$. Clearly, our wavelet operators are more flexile than the former ones, encompassing at least the classical version, as well as the Kantorovich forms of the generalized Bézier operators. As a result, our findings extend several previous results on generalized Bézier operators.

Keywords: Bézier basis, wavelets, asymptotic approximation.

AMS Subject Classification 41A35, 42C40, 47A58

1 Introduction

For a function $\xi \in B[0,1]$, a well-known operators $B_n\xi$, Bernstein operators [11], are described for $n \geq 1$ by

$$(B_n\xi)(z) = \sum_{k=0}^{n} \xi\left(\frac{k}{n}\right) p_{n,k}(z) \quad , \quad z \in [0,1], \tag{1}$$

with the Bernstein basis, $p_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}$.

In 1972, [9], Bézier introduced a basis functions, $J_{n,k}(z) = \sum_{j=k}^{n} p_{n,j}(z)$ where $0 \leq z \leq 1$. Then for $\alpha > 0$ and a function $\xi \in B[0,1]$, the operators $B_{n,\alpha}\xi$, Bézier modification of operators (1), are described by

$$\left(B_{n,\alpha}\xi\right)(z) = \sum_{k=0}^{n} \xi\left(\frac{k}{n}\right) Q_{n,k}^{(\alpha)}(z) \quad , \quad z \in [0,1], \tag{2}$$

where $Q_{n,k}^{(\alpha)}(z) = J_{n,k}^{\alpha}(z) - J_{n,k+1}^{\alpha}(z)$, with $J_{n,l}(z) \equiv 0$ if $l > n$. If $\alpha = 1$, then $B_{n,\alpha}\xi$ reduce to the operators (1).

These operators, (1) and (2), and their certain modifications are famous in theory of approximation. Furthermore, since the discrete type positive linear operators, say Bernstein polynomials (1) and its Bézier variants (2), cannot used in L^p approximation $(1 \leq p < \infty)$, their other modifications, such as Durrmeyer and Kantorovich, are regarded to get some constructive results for functions in L^p spaces.

For a function $\xi \in L^1[0,1]$, the wide class of integrable functions on $[0,1]$, an operators K_n, Kantorovich modification of (1), are described by

$$(K_n\xi)(z) = \sum_{k=0}^{n} p_{n,k}(z) \int_0^1 (n+1)\chi_{nk}(u)\xi(u)\,du,$$

in [8], where χ_{nk} is the characteristic function of the interval $\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$ and $p_{n,k}(z)$ is, again, the Bernstein basis.

If Υ specifies an indefinite integral of ξ, then it can be easily seen that the operators K_n are described by $(K_n\xi)(z) = \frac{d}{dz}(B_{n+1}\Upsilon)(z)$. Actually, approximation properties of those operators, the classical Kantorovich, have been studied by several authors. And a result

$$\lim_{n\to\infty} (K_n\xi)(z) = \xi(z) \text{ almost everywhere on } [0,1]$$

is well-known when ξ is Lebesgue integrable.

[a]Department of Mathematics, Bolu Abant Izzet Baysal University, Turkey
[b]Department of Mathematics, Bolu Abant Izzet Baysal University, Turkey

In 2004, [13], Gupta considered an operators $K_{n,c}\xi$, the generalized Kantorovich type, as

$$\left(K_{n,c}\xi\right)(z) = n\sum_{k=0}^{\infty} p_{n,k,c}(z) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \xi(u)\,du. \tag{3}$$

In fact, particular cases of (3) allows us to achieve well-known operators, Baskakov-Kantorovich and Szász-Kantorovich, by describing

$$p_{n,k,c}(z) = (-1)^k \frac{x^k}{k!}\phi_{n,c}^{(k)}(z).$$

Indeed, by specifying $\phi_{n,c}(z) = (1+cz)^{-n/c}$ for $c=1$, and $\phi_{n,c}(z) = \exp(-nz)$ for $c=0$, the operators (3) turns out to the operators, Baskakov&Szász-Kantorovich, respectively.

In [13], Gupta also considered an operators $K_{n,\alpha,c}\xi$, the Bézier variant of (3), as

$$\left(K_{n,\alpha,c}\xi\right)(z) = n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \xi(u)\,du, \tag{4}$$

where $Q_{n,k,c}^{(\alpha)}(z) = J_{n,k,c}^{\alpha}(z) - J_{n,k+1,c}^{\alpha}(z)$ $(\alpha \geq 1)$ with the basis functions $J_{n,k,c}(z) = \sum_{j=k}^{\infty} p_{n,j,c}(z)$, which turns out to the basis, Baskakov and Szász, for $c=1$ and $c=0$, respectively. Clearly, if $\alpha = 1$ then $K_{n,\alpha,c}\xi$ in (4) reduces to the operators in (3) for locally integrable ξ on $[0,\infty)$.

In recent years, within the studies about wavelet type operators, Karsli, [3] and [4], introduced the operators $WB_{n,\alpha}\xi$, wavelet type generalized Bézier, as an extension and also a generalization of (4), by using the compacted Daubechies. With a (father) wavelet, $\eta \in L_\infty(\mathbb{R})$, which provides some suitable conditions, $WB_{n,\alpha} : B[0,\infty) \to C[0,\infty)$ are defined as

$$\left(WB_{n,\alpha}\xi\right)(z) = n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z) \int_{\mathbb{R}} \xi(u)\,\eta(nu-k)\,du \quad , \quad (z \in \mathbb{R}). \tag{5}$$

This study fundamentally aims to derive asymptotic properties of the operators $WB_{n,\alpha}$ (5), wavelet type generalized Bézier, by means of the compactly supported Daubechies wavelets, which will be used exclusively as the compacted Daubechies throughout the rest of the paper.

The basis used in constructing these types of operators is the wavelet expansion of the function ξ, rather than its sampled values $\xi\left(\frac{k}{n}\right)$. Clearly, our wavelet operators are more flexile than the former ones, encompassing at least the classical version, as well as the Kantorovich forms of the generalized Bézier operators. As a result, our findings extend several previous results on generalized Bézier operators. For further details and also for advantages of the wavelets, please see [1].

2 Preliminaries

This part introduces some preliminary notations and needed definitions that will be used in the construction of the compacted Daubechies (see [5]- [6]), and the operators specifically, the wavelet type generalized Bézier, that are central to this study.

$C[0,\infty)$ denotes the Banach space of the functions $\xi : [0,\infty) \to \mathbb{R}$, bounded and continuous, with norm

$$\|\xi\| = \sup\{|\xi(z)| : z \in [0,\infty)\},$$

and for $1 \leq p \leq \infty$, $L^p[0,\infty)$ denotes the Lebesgue measurable function space of the functions ξ, satisfy certain conditions connected with the power p, with norms

$$\|\xi\|_p = \left(\int_0^{\infty} |\xi(z)|^p\,dz\right)^{1/p} < \infty,$$

and

$$\|\xi\|_\infty = \operatorname{esssup}\{|\xi(z)| : z \in [0,\infty)\}.$$

Moreover, the notation $[a]$ express the greatest integer of a.

Definition 2.1. (MRA) A multiresolution analysis is a sequence of closed subspaces of $L^2(\mathbb{R})$, say $(\Lambda_i)_{i\in\mathbb{Z}}$, such that

i) Λ_i is a set of all $\xi \in L^2(\mathbb{R})$ which are constant on intervals, 2^{-i} length, and

$$\ldots \subset \Lambda_{-2} \subset \Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_i \subset \Lambda_{i+1} \subset \ldots \subset L^2(\mathbb{R}),$$

with

$$\overline{\bigcup_i \Lambda_i} = L^2(\mathbb{R}),$$

ii)

$$\begin{aligned}\forall i &\in \mathbb{Z} \quad , \quad \xi(z) \in \Lambda_i \Longleftrightarrow \xi(2z) \in \Lambda_{i+1},\\ \forall j &\in \mathbb{Z} \quad , \quad \xi(z) \in \Lambda_0 \Longleftrightarrow \xi(z-j) \in \Lambda_0,\\ \forall i,j &\in \mathbb{Z} \quad , \quad \xi(z) \in \Lambda_i \Longleftrightarrow \xi\left(z-2^{-i}j\right) \in \Lambda_i,\end{aligned}$$

and

iii)

$$\bigcap_i \Lambda_i = \{0\}$$

hold.

Definition 2.2. (Wavelet) A wavelet is a localized wave that oscillates and gradually decays in the time domain. A father wavelet $\eta(z)$, scaling function, and a mother wavelet $\vartheta(z)$, wavelet function, are both orthogonal functions which generate whole wavelet basis set by scaling and translation. These two functions, scaling (father) and wavelet (mother), have finite energy, that is to say $\eta, \vartheta \in L^2(\mathbb{R})$ and orthogonal, and satisfy

$$\int_{-\infty}^{\infty} \eta(z)\,dz = 1 \quad , \quad \int_{-\infty}^{\infty} \vartheta(z)\,dz = 0$$

respectively. The wavelets, generally, imply the family of orthonormal functions set with the form

$$\vartheta_{a,b}(z) = \frac{1}{\sqrt{a}}\vartheta\left(\frac{z-b}{a}\right) \tag{6}$$

where $a > 0,\ b \in \mathbb{R}$ and ϑ be the (mother) wavelet.

Haar Wavelet, is an example of the simplest one, is described by

$$\vartheta(z) = \begin{cases} 1 & , \quad 0 \le z < 1/2 \\ -1 & , \quad 1/2 \le z < 1 \\ 0 & , \quad e.w. \end{cases}$$

with the scaling function

$$\eta(z) = \begin{cases} 1 & , \quad 0 \le z < 1 \\ 0 & , \quad e.w. \end{cases}$$

Haar wavelet, clearly, form an orthonormal basis for the space of functions, square integrable, on the whole real line. On the other hand, this wavelet is only appropriate to describe discrete signals not to describe smooth signals or functions just because it is not continuous and therefore not differentiable.

Now, consider an orthonormal basis of wavelets in $L^2(\mathbb{R})$, and suppose there is a father wavelet $\eta(z)$, whose translates $\{\eta(z-n)\}$ are orthogonal, and a mother wavelet $\vartheta(z)$, generates the orthonormal basis $\vartheta_{i,j}(z)$ of $L^2(\mathbb{R})$, with

$$\vartheta_{i,j}(z) = 2^{i/2}\vartheta\left(2^i z - j\right).$$

Hence, each $\xi \in L^2(\mathbb{R})$ can be represented by

$$\xi(z) = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} b_{i,j}\vartheta_{i,j}(z) \tag{7}$$

by using a multiresolution analysis, where

$$b_{i,j} = \left\langle \xi(z), \varphi_{i,j}(z)\right\rangle = 2^{i/2}\int_{\mathbb{R}} \xi(z)\,\overline{\varphi(2^i z - j)}\,dz \tag{8}$$

are wavelet coefficients. Here, (7) is named wavelet expansion of $\xi \in L^2(\mathbb{R})$. Some approximation results about wavelet expansion can be found in [2], [10] and [14]. In addition to this, using same special a and b in (6) and (8), one can obtain certain type of wavelets, such as Haar, Strömber, Meyer, Daubechies etc...

Now we are ready to introduce compacted Daubechies. First, suppose that $\eta \in L_\infty(\mathbb{R})$, scaling function, satisfies:

a) η is compactly supported, that is to say $\operatorname{supp}\eta \subset [0,\lambda]$ for a real constant $\lambda > 0$,

b)

$$\int_{-\infty}^{\infty} \eta(z)\,dz = 1,$$

c) the first N moments of η satisfy

$$m_i(\eta) := \int_{-\infty}^{\infty} x^i \eta(z)\,dz = 0 \quad , \quad i = 1, ..., N.$$

Hence, the absolute moments of η, obviously, satisfy

$$M_i(\eta) := \int_{-\infty}^{\infty} |x|^i |\eta(z)|\,dz < +\infty$$

for every $i \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

Wavelets that satisfy the former conditions are referred to as compacted Daubechies. These wavelets exhibit strong relationships with qualities such as continuity and differentiability. This implies that when using Daubechies wavelets to regenerate a function, it is more practical to select or regulate wavelets depending upon the continuity or differentiability qualities of the function.

By the help of the former definitions, we can define the operators, wavelet type generalized Bézier, by means of the compacted Daubechies as follows:

Definition 2.3. [4] Let $\xi \in B[0,\infty)$ and $\eta \in L_\infty(\mathbb{R})$, a (scaling) wavelet, satisfying (a)-(c). In this case, the wavelet type generalized Bézier operators, by means of the compacted Daubechies are defined by:

$$\begin{aligned} \left(WB_{n,\alpha}\xi\right)(z) &= n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z) \int_{\mathbb{R}} \xi(u)\,\eta(nu-k)\,du \quad , \quad (z \in \mathbb{R}) \\ &= \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z) \int_{\mathbb{R}} \xi\left(\frac{u+k}{n}\right)\eta(u)\,du \quad , \quad (z \in \mathbb{R}) \\ &= \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z) \int_{0}^{\lambda} \xi\left(\frac{u+k}{n}\right)\eta(u)\,du \quad , \quad (z \in \mathbb{R}). \end{aligned}$$

Remark 1. If we let $\eta(u) = \chi_{[0,1]}(u)$, in other words if we choose father wavelet η as Haar scaling function, in this case the operators $WB_{n,\alpha}$ (5), wavelet type generalized Bézier, by means of the compacted Daubechies reduce to the operators (4), generalized Bézier Kantorovich, which considered by Abel and Gupta in [12] and [13]. Indeed;

$$\begin{aligned} \left(WB_{n,\alpha}\xi\right)(z) &= n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z) \int_{\mathbb{R}} \xi(u)\,\eta(nu-k)\,du \\ &= \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z) \int_{\mathbb{R}} \xi\left(\frac{u+k}{n}\right)\eta(u)\,du \\ &= \sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z) \int_{0}^{1} \xi\left(\frac{u+k}{n}\right)\eta(u)\,du. \end{aligned}$$

In that case the operators $WB_{n,\alpha}$ (5), wavelet type generalized Bézier, by means of the compacted Daubechies are a natural extension of the operators (4), generalized Bézier Kantorovich.

3 Auxiliary Results

This section presents some preliminary results that will be utilized in the proof of the main findings of our work. Specifically, our first result demonstrates a strong connection between the previously established operator (2), Bézier, and the newly defined one (5), generalized Bézier, which is constructed by wavelets.

Theorem 3.1. *[4] Let $\xi \in C[0,\infty)$ and $\eta \in L_\infty(\mathbb{R})$,a (scaling) wavelet, satisfying (a)-(c). In this case, the moments of (5), wavelet type generalized Bézier, by means of the compacted Daubechies and (2), generalized Bézier, are identical, that is to say*

$$\left(WB_{n,\alpha}x^\beta\right)(z) = \sum_{k=0}^{\infty} \frac{k^\beta}{n^\beta} Q_{n,k,c}^{(\alpha)}(z) \quad , \quad \beta = 0, 1, ..., K$$

holds true.

Remark 2. [4] Furthermore, by the properties (b) and (c), the central moments of (5) and (2) are also identical, that is

$$\begin{aligned}\left(WB_{n,\alpha}(x-z)^{\beta}\right)(z) &= \frac{1}{n^{\beta}}\sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(z)(k-nz)^{\beta}\\ &= \left(B_{n,\alpha}(x-z)^{\beta}\right)(z)\end{aligned}$$

holds true. Suppose, moreover, that the central moments of (5), wavelet type generalized Bézier, by means of the compacted Daubechies, satisfy

$$\mu_0(z) \; : \; = \left(WB_{n,\alpha}1\right)(z) = 1,$$

$$\mu_1(z) \; : \; = \left(WB_{n,\alpha}(x-z)\right)(z) = 0, \tag{9}$$

$$\mu_2(z) \; : \; = \left(WB_{n,\alpha}(x-z)^2\right)(z) = \frac{\alpha z(1+cz)}{n}, \tag{10}$$

for $c=1$ and $c=0$. In addition, by [13] Remark 1, one has

$$\mu_m(z) := \left(WB_{n,\alpha}(x-z)^m\right)(z) = O\left(n^{-[(m+1)/2]}\right), \quad n\to\infty$$

for each $z\in[0,\infty)$ while $n\to\infty$ and [13] Lemma 3, for all $x>0$ and $n,k\in\mathbb{N}$, we have

$$Q_{n,k,c}^{(\alpha)}(z) \le \alpha p_{n,k}(z) < \begin{cases} \frac{\alpha C_1}{\sqrt{nz(1+z)}} & , \quad c=1 \\ \frac{\alpha}{\sqrt{\pi nz}} & , \quad c=0 \end{cases}$$

where $C_1=1$ if $n=1$, $C_1=2\sqrt{2}/3\sqrt{2}$ if $n\ge 2$, $k=0$, and C_1 depends on n with

$$(3/2)^{3/2}n^{3/2}\frac{(n-1)^{n-1}}{(n+1/2)^{n+1/2}}$$

if $n\ge 2$, $k\ge 1$. On the other hand, by [7] Theorem 1, with gamma function $\Gamma(.)$, one also has

$$\left(B_n|x-z|^{\beta}\right)(z) \le 2\Gamma\left(\frac{\beta}{2}+1\right)\frac{1}{n^{\beta/2}}$$

for any $n\in\mathbb{N}$, $z\in[0,1]$ and $\beta>0$. Therefore, the discrete absolute moments of order β satisfy

$$\widetilde{\mu}_{\beta}(z) := \left(WB_{n,\alpha}|x-z|^{\beta}\right)(z) < \infty.$$

In [3], Karsli also proved the following substantial theorem.

Theorem 3.2. *Let $\xi\in L_1[0,\infty)$ and $\eta\in L_\infty(\mathbb{R})$,a (scaling) wavelet, satisfying (a)-(c). Then*

$$\lim_{n\to\infty}\left(WB_{n,\alpha}\xi\right)(z_0) = \xi(z_0)$$

holds true at each point z_0 of continuity of f.

Corollary 3.3. *Let $\xi\in C[0,\infty)\cap L_\infty(\mathbb{R})$ and $\eta\in L_\infty(\mathbb{R})$,a (scaling) wavelet, satisfying (a)-(c). In this case, the convergence is uniform, that is*

$$\lim_{n\to\infty}\left\|\left(WB_{n,\alpha}\xi\right)-\xi\right\|_\infty = 0$$

with respect to $z\in[0,\infty)$.

4 Main Results

We are now prepared to present the key results of our work.

Theorem 4.1. *Let $\xi\in B[0,\infty)$ and $\eta\in L_\infty(\mathbb{R})$,a (scaling) wavelet, satisfies (a)-(c). Suppose, moreover, that $\xi'(z)$ exists at a fixed point z. In this case, the asymptotic formula*

$$\left(WB_{n,\alpha}\xi\right)(z) = \xi(z) + o\left(n^{-1/2}\right)$$

holds while $n\to\infty$.

Proof. In consideretion of the local Taylor's formula, at a point z where $\xi'(z)$ exists, we have

$$\xi(u) = \xi(z) + \xi'(z)(u-z) + h(u-z)(u-z), \tag{11}$$

with a bounded function h, satisfies $\lim_{t\to 0} h(t) = 0$. Using (11) in (5) we can write

$$\begin{aligned}
\left(WB_{n,\alpha}\xi\right)(z) &= n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} \xi(u)\,\eta(nu-k)\,du \\
&= n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} \left[\xi(z)+\xi'(z)(u-z)+h(u-z)(u-z)\right]\eta(nu-k)\,du \\
&= n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} \left[\xi(z)+\xi'(z)(u-z)\right]\eta(nu-k)\,du \\
&\quad + n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} h(u-z)(u-z)\,\eta(nu-k)\,du \\
&= : I_1 + R_W.
\end{aligned}$$

Now we analyse I_1 and R_W respectively. Begin with the term I_1.

$$\begin{aligned}
I_1 &= n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} \left[\xi(z)+\xi'(z)(u-z)\right]\eta(nu-k)\,du \\
&= \xi(z)+\xi'(z)\,n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} (u-z)\,\eta(nu-k)\,du.
\end{aligned}$$

Note that

$$u-z=\frac{nu-nz}{n}=\frac{nu-k-nz+k}{n}=\frac{nu-k}{n}-\frac{nz-k}{n} \tag{12}$$

and hence

$$|u-z|\le\left|\frac{nu-k}{n}\right|+\left|\frac{nz-k}{n}\right| \tag{13}$$

holds true. So, one has from (12)

$$\begin{aligned}
I_1 &= \xi(z)+\xi'(z)\,n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} (u-z)\,\eta(nu-k)\,du \\
&= \xi(z)+\xi'(z)\,n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} \left(\frac{nu-k}{n}-\frac{nz-k}{n}\right)\eta(nu-k)\,du \\
&= \xi(z)+\xi'(z)\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} (nu-k)\,\eta(nu-k)\,du \\
&\quad -\xi'(z)\sum_{k=0}^{\infty} (nz-k)\,Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} \eta(nu-k)\,du.
\end{aligned}$$

Using (b), (c) and (9),the first central moment, one has

$$\begin{aligned}
I_1 &= \xi(z)+\xi'(z)\frac{m_1(\eta)}{n}+\xi'(z)\mu_1(z) \\
&= \xi(z).
\end{aligned}$$

Next we take into account the remainder term R_W. Since $\lim_{t\to 0} h(t) = 0$ for a bounded function h, for fixed $\epsilon > 0$, we have $\delta > 0$, such that $|h(t)| \le \epsilon$ whenever $|t| \le \delta$, and hence

$$\begin{aligned}
R_W &= n\sum_{k=0}^{\infty} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} h(u-z)(u-z)\,\eta(nu-k)\,du \\
&= n\sum_{\left|\frac{k}{n}-z\right|\ge\delta} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} h(u-z)(u-z)\,\eta(nu-k)\,du \\
&\quad + n\sum_{\left|\frac{k}{n}-z\right|<\delta} Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}} h(u-z)(u-z)\,\eta(nu-k)\,du \\
&: = R_{W_1}+R_{W_2}.
\end{aligned}$$

By (13), we get

$$\begin{aligned}
\left|R_{W_2}\right| &= \left| n \sum_{\left|\frac{k}{n}-z\right|<\delta} Q^{(\alpha)}_{n,k,c}(z) \int_{\mathbb{R}} h(u-z)(u-z)\eta(nu-k)\,du \right| \\
&\leq \epsilon \sum_{\left|\frac{k}{n}-z\right|<\delta} Q^{(\alpha)}_{n,k,c}(z) \int_{\mathbb{R}} \left[|nu-k|+|nz-k|\right]|\eta(nu-k)|\,du \\
&= \epsilon \sum_{\left|\frac{k}{n}-z\right|<\delta} Q^{(\alpha)}_{n,k,c}(z) \int_{\mathbb{R}} |nu-k|\,|\eta(nu-k)|\,du \\
&\quad +\epsilon \sum_{\left|\frac{k}{n}-z\right|<\delta} Q^{(\alpha)}_{n,k,c}(z)\,|nz-k| \int_{\mathbb{R}} |\eta(nu-k)|\,du \\
&\leq \epsilon\left(\frac{M_1(\eta)}{n} + \widetilde{\mu}_1(z) M_0(\eta)\right) = o\left(n^{-1/2}\right)
\end{aligned}$$

as $n \to \infty$. In addition to that, for a constant $B > 0$ we have $|h(t)| \leq B$ and hence

$$\left|R_{W_1}\right| \leq B\left(\frac{M_1(\eta)}{n} + \widetilde{\mu}_1(z) M_0(\eta)\right) = o\left(n^{-1/2}\right)$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty} |R_W| = 0,$$

holds, which completes the assertion. □

Theorem 4.2. *Let $\xi \in B[0,\infty)$ and $\eta \in L_\infty(\mathbb{R})$,a (scaling) wavelet, satisfies (a)-(c). Suppose, moreover, that for a fixed $z \in [0,\infty)$ and for certain $r \in \mathbb{N}$, $\xi \in C^r$ locally at that point z. In this case, the asymptotic formula*

$$\left(WB_{n,\alpha}\xi\right)(z) = \xi(z) + \sum_{i=1}^{r} \frac{\xi^{(i)}(z)}{i!}\mu_i(z) + o\left(n^{-r/2}\right) \tag{14}$$

holds as $n \to \infty$, and μ_i is the (algebraic) moment of i−th order.

Proof. In consideretion of the local Taylor's formula, at a point z where $\xi'(z)$ exists, we have

$$\xi(u) = \sum_{i=0}^{r} \frac{\xi^{(i)}(z)}{i!}(u-z)^i + h(u-z)(u-z)^r \tag{15}$$

with a bounded function h, satisfies $\lim_{t\to 0} h(t) = 0$. Using (15) in (5) we can write

$$\begin{aligned}
\left(WB_{n,\alpha}\xi\right)(z) &= n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(z) \int_{\mathbb{R}} \xi(u)\eta(nu-k)\,du \\
&= n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(z) \int_{\mathbb{R}} \left(\sum_{i=0}^{r} \frac{\xi^{(i)}(z)}{i!}(u-z)^i + h(u-z)(u-z)^r\right)\eta(nu-k)\,du \\
&= n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(z) \int_{\mathbb{R}} \left(\sum_{i=0}^{r} \frac{\xi^{(i)}(z)}{i!}(u-z)^i\right)\eta(nu-k)\,du \\
&\quad +n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(z) \int_{\mathbb{R}} h(u-z)(u-z)^r\,\eta(nu-k)\,du \\
&= : I_1 + R_W.
\end{aligned}$$

Now we analyse I_1 and R_W respectively. Begin with the term I_1.

$$\begin{aligned}
I_1 &= n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(z) \int_{\mathbb{R}} \left(\sum_{i=0}^{r} \frac{\xi^{(i)}(z)}{i!}(u-z)^i\right)\eta(nu-k)\,du \\
&= \xi(z) + n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(z) \int_{\mathbb{R}} \left(\sum_{i=1}^{r} \frac{\xi^{(i)}(z)}{i!}(u-z)^i\right)\eta(nu-k)\,du \\
&= \xi(z) + n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k,c}(z) \left(\sum_{i=1}^{r} \frac{\xi^{(i)}(z)}{i!}\right) \int_{\mathbb{R}} (u-z)^i\,\eta(nu-k)\,du.
\end{aligned}$$

Likewise (12), note that

$$(u-z)^i=\left(\frac{nu-nz}{n}\right)^i=\left(\frac{nu-k-nz+k}{n}\right)^i,$$

and by binomial expansion,

$$(u-z)^i=\sum_{v=0}^{i}\binom{i}{v}\left(\frac{nu-k}{n}\right)^v\left(-\frac{nz-k}{n}\right)^{i-v} \tag{16}$$

holds true. So, one has from (16)

$$\begin{aligned}I_1 &= \xi(z)+n\sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(z)\left(\sum_{i=1}^{r}\frac{\xi^{(i)}(z)}{i!}\right)\int_{\mathbb{R}}\left(\sum_{v=0}^{i}\binom{i}{v}\left(\frac{nu-k}{n}\right)^v\left(-\frac{nz-k}{n}\right)^{i-v}\right)\eta(nu-k)\,du\\ &= \xi(z)+n\sum_{i=1}^{r}\frac{\xi^{(i)}(z)}{i!n^i}\sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(z)\sum_{v=0}^{i}\binom{i}{v}(k-nz)^{i-v}\int_{\mathbb{R}}(nu-k)^v\,\eta(nu-k)\,du\\ &= \xi(z)+n\sum_{i=1}^{r}\frac{\xi^{(i)}(z)}{i!n^i}\sum_{v=0}^{i}\binom{i}{v}\sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(z)(k-nz)^{i-v}\int_{\mathbb{R}}(nu-k)^v\,\eta(nu-k)\,du.\end{aligned}$$

Using (b), (c) when $N=r$ and Remark 6, one has

$$I_1=\xi(z)+\sum_{i=1}^{r}\frac{\xi^{(i)}(z)}{i!}\mu_i(z).$$

Next we consider the remainder term R_W. Since $\lim_{t\to 0}h(t)=0$ for a bounded function h, for fixed $\epsilon>0$, we have $\delta>0$, such that $|h(t)|\le\epsilon$ whenever $|t|\le\delta$, and hence

$$\begin{aligned}|R_W| &\le n\sum_{k=0}^{\infty}Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}}|h(u-z)|\,|u-z|^r\,|\eta(nu-k)|\,du\\ &= n\left(\sum_{\left|\frac{k}{n}-z\right|\ge\delta}+\sum_{\left|\frac{k}{n}-z\right|<\delta}\right)Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}}|h(u-z)|\,|u-z|^r\,|\eta(nu-k)|\,du\\ &=: R_{W_1}+R_{W_2}\end{aligned}$$

say. The latter term

$$\begin{aligned}R_{W_2} &\le \epsilon n\sum_{\left|\frac{k}{n}-z\right|<\delta}Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}}|u-z|^r\,|\eta(nu-k)|\,du\\ &= \epsilon n\sum_{\left|\frac{k}{n}-z\right|<\delta}Q_{n,k,c}^{(\alpha)}(z)\int_{\mathbb{R}}\sum_{v=0}^{r}\binom{r}{v}\left|\frac{nu-k}{n}\right|^v\left|-\frac{nz-k}{n}\right|^{r-v}|\eta(nu-k)|\,du\\ &= \epsilon n\sum_{v=0}^{r}\binom{r}{v}\sum_{\left|\frac{k}{n}-z\right|<\delta}Q_{n,k,c}^{(\alpha)}(z)\left|-\frac{nz-k}{n}\right|^{r-v}\int_{\mathbb{R}}\left|\frac{nu-k}{n}\right|^v|\eta(nu-k)|\,du\\ &= \frac{\epsilon}{n^r}\sum_{v=0}^{r}\binom{r}{v}\sum_{\left|\frac{k}{n}-z\right|<\delta}Q_{n,k,c}^{(\alpha)}(z)\,|k-nz|^{r-v}M_v(\eta)\\ &\le \epsilon\sum_{v=0}^{r}\binom{r}{v}\frac{M_v(\eta)\,\widetilde{\mu}_{r-v}(z)}{n^v}.\end{aligned}$$

In addition to that, for a constant $B>0$ we have $|h(t)|\le B$ and hence

$$R_{W_1}\le B\sum_{v=0}^{r}\binom{r}{v}\frac{M_v(\eta)\,\widetilde{\mu}_{r-v}(z)}{n^v}.$$

Thus

$$\lim_{n\to\infty}|R_W|=0,$$

holds, which completes the assertion. □

As an obvious outcome of the above theorems, one can state the following Voronovskaya type theorems, first and second order, respectively.

Theorem 4.3. *Let $\xi \in B[0,\infty)$ and $\eta \in L_{\infty}(\mathbb{R})$,a (scaling) wavelet, satisfies (a)-(c). Suppose, moreover, that for a fixed $z \in [0,\infty)$ and for certain $r \in \mathbb{N}$, $\xi \in C^1$ locally at that point z. In this case*

$$\lim_{n\to\infty} n^{1/2}\left[WB_{n,\alpha}\xi(z) - \xi(z)\right] = 0$$

holds.

Proof. In view of (14) for $r = 1$, we have

$$\left(WB_{n,\alpha}\xi\right)(z) = \xi(z) + \xi'(z)\mu_1(z) + o\left(n^{-1/2}\right) \quad , \quad (n \to \infty).$$

Then using (9) and taking to the limit for $n \to \infty$, we get the required result. □

Theorem 4.4. *Let $\xi \in B[0,\infty)$ and $\eta \in L_{\infty}(\mathbb{R})$,a (scaling) wavelet, satisfies (a)-(c). Suppose, moreover, that for a fixed $z \in [0,\infty)$ and for certain $r \in \mathbb{N}$, $\xi \in C^2$ locally at that point z. In this case*

$$\begin{aligned}\lim_{n\to\infty} n\left[\left(WB_{n,\alpha}\xi\right)(z) - \xi(z)\right] &= \frac{1}{2}\xi''(z)\mu_2(z)\\ &= \frac{\alpha z(1+cz)}{2}\xi''(z).\end{aligned}$$

holds.

Proof. In view of (14) for $r = 2$, we have

$$\left(WB_{n,\alpha}\xi\right)(z) = \xi(z) + \sum_{i=1}^{2}\frac{\xi^{(i)}(z)}{i!}\mu_i(z) + o\left(n^{-2/2}\right) \quad , \quad (n \to \infty).$$

Then using (9) and (10) with taking to the limit for $n \to \infty$, we get the required result. □

5 Conclusion

In contrast to Fourier analysis, wavelets reveal both the frequencies within a signal and the specific times at which these frequencies appear. This makes wavelets a more effective tool for analyzing dynamic signals, offering enhanced resolution in both the time and frequency domains.

Furthermore, based on the definitions and properties of wavelet bases, wavelet-based operators can be utilized for approximation problems in L^p spaces. Given the numerous advantages of wavelets in approximating within L^p spaces, as well as their potential applications in machine learning and neural networks, future work will focus on adapting the advancements in wavelet theory to these spaces and frameworks.

6 Acknowledgment

We would like to thank the editor and referees for their careful reading, valuable suggestions and corrections that improved the quality and presentation of the paper.

References

[1] C. S. Burrus, R. A. Gopinath, & H. Guo, Introduction to wavelets and wavelet transforms, A primer. Prentice Hall, 1998.

[2] G. G. Walter, Pointwise convergence of wavelet expansions, J. Approx. Theory, 80: 108-118, 1995.

[3] H. Karsli, Extension of the generalized Bézier operators by wavelets, General Mathematics, 30(2): 3-15, 2022.

[4] H. Karsli, On wavelet type generalized Bézier operators, Math. Found. of Comp., 6(3): 439-452, 2023.

[5] I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math.,41: 909-996, 1988.

[6] I. Daubechies, Ten Lectures on Wavelets, CBMS-NSF Series in Appl. Math. 61, SIAM Publ. Philadelphia, 1992.

[7] J. A. Adell, J. Bustamante, J.M. Quesada, Estimates for the momentsof Bernstein polynomials, J. Math. Anal. Appl., 432:114-128, 2015.

[8] L. Kantorovich, Sur certains développements suivant les polynomes de la forme de S. Bernstein. I. II., C. R. Acad. Sci. URSS, 563-568, 595-600, 1930.

[9] P. Bézier, Numerical Control Mathematics and Applications, Wiley, London, 1972.

[10] S. E. Kelly, M. A. Kon and L. A. Raphael, Pointwise convergence of wavelet expansions, Bull. Amer. Math. Soc., 30:87-94, 1994.

[11] S. N. Bernstein, Demonstration du Thĕoreme de Weierstrass fondĕe sur le calcul des probabilitĕs, Comm. Soc. Math. Kharkow 13: 1-2, 1912/13.

[12] U. Abel and V. Gupta, An estimate on the rate of convergence of Bézier variant of Baskakov Kantorovich operators, Demonstratio Math., 36: 123-136, 2003.

[13] V. Gupta, The Bézier variant of Kantorovich operators, Comput. Math. Appl. 47: 227-232, 2004.

[14] W. Lenski and B. Szal, Approximation of integrable functions by wavelet expansion, Results Math., 72: 1203-1211, 2017.