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An application of the Euler-MacLaurin summation formula for estimating the order of approximation of sampling-type series

Marco Cantarini[a] · Danilo Costarelli[b]

Abstract

In this paper we establish a quantitative estimate for the order of approximation for the generalized and the Kantorovich sampling series based upon kernels with asymptotic decay as the function u^{-2}, that are characterized by an infinite first order discrete absolute moment. The key point of the above proof is provided by the application of a special case of the Euler-MacLaurin summation formula. Concrete examples are discussed, such as the critical case of the Fejér kernel.

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1 Introduction

In recent years, much attention from the scientific community has been given to the study of the sampling theory (see, e.g., [24, 25, 26, 27, 28, 29, 30, 23, 20, 21, 31]), with particular emphasis towards the sampling-type series, in view of their interests from both theoretical and practical points of view.

It is well-known that such a research topic arose thanks to the effort of P.L. Butzer and his school (see, e.g., [10, 9]), with the main aim to provide an approximate version of the celebrated Whittaker-Kotel'nikov-Shannon sampling theorem. Among the several instances of sampling-type series we can find the generalized sampling operators, their Kantorovich and Durrmeyer generalizations ([2, 18, 11]), and the very recent Steklov-type ones ([16]).

In particular, the Kantorovich sampling (KS) series have been also investigated in the setting of image resizing and the related applications (see, e.g., [19]), as well as from the point of view of Approximation Theory ([5, 12, 13, 3, 4]). We recall that, this family of linear operators are defined as follows:

$$(K_w^\chi f)(x) \;:=\; \sum_{k\in\mathbb{Z}} \chi(wx-k)\left[w\int_{k/w}^{(k+1)/w} f(u)\,du\right], \quad x\in\mathbb{R}, \quad w>0$$

where $f:\mathbb{R}\to\mathbb{R}$ is locally integrable, and $\chi:\mathbb{R}\to\mathbb{R}$ is a *kernel* function, satisfying the following conditions:

(χ1) $\chi \in L^1(\mathbb{R})$ is bounded on $\mathbb{R}$;

(χ2) χ satisfies the "partition of the unit property", that is:

$$\sum_{k\in\mathbb{Z}} \chi(u-k) \;=\; 1, \qquad u\in\mathbb{R};$$

(χ3) there exists $\beta>0$ such that the *discrete absolute moment of order* β is finite, i.e.:

$$M_\beta(\chi) \;:=\; \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |k-u|^\beta\,|\chi(u-k)| \;<\; +\infty.$$

Concerning the above topic, an interesting considered problem is that of the "order of approximation". A first work on this subject is by Bardaro and Mantellini ([6]), in which a Jackson-type estimate for the aliasing error $\|K_w^\chi f - f\|_\infty$ has been established, under the fundamental assumption that the kernel χ satisfies condition (χ3) with $\beta\geq 1$, and for $f\in C(\mathbb{R})$ (namely, the space of all uniformly continuous and bounded functions endowed with the usual sup-norm $\|\cdot\|_\infty$).

[a]Department of Mathematics and Computer Science, University of Perugia, Via Vanvitelli 1, 06123 Perugia, Italy, email: marco.cantarini@unipg.it
[b]Department of Mathematics and Computer Science, University of Perugia, Via Vanvitelli 1, 06123 Perugia, Italy, email: danilo.costarelli@unipg.it

Later, in the paper [17], the order of approximation has been estimated in the case when $(\chi 3)$ is satisfied by χ for $0<\beta<1$; in this case we have:

$$\|K_w^\chi f - f\|_\infty \leq C_1\, \omega(f, w^{-\beta}) + C_2\, w^{-\beta}, \quad as \quad w \to +\infty, \tag{1}$$

where C_1, $C_2>0$ are suitable absolute constants and $\omega(f,\delta)$ denotes the classical modulus of continuity of f.

It is well-known that, if χ is any continuous kernel, such that:

$$\chi(u) \;=\; \mathcal{O}(|u|^{-2}), \quad as \quad |u| \to +\infty,$$

it turns out that $M_\beta(\chi) < +\infty$ for any $0<\beta<1$, hence the inequality (1) holds for any $0<\beta<1$. At the same time, if $M_1(\chi)=+\infty$ the above estimate can not be improved, in fact avoiding to exactly determine what is the better exponent of the parameter w in the right-hand side of (1). This is exactly the situation that occurs when we consider the well-known Fejér kernel as the function χ (see [8]).

In the present paper, we deal with the above discussed open problem, that is to provide a definite quantitative estimate for the aliasing error when $M_\beta(\chi)<+\infty$ for any $0<\beta<1$ and $M_1(\chi)=+\infty$ in term of the modulus of continuity.

On the bases of the estimate established in this paper there is an application of a special form of the Euler-MacLaurin summation (EMS) formula. More precisely, we propose the use of a version of the EMS formula that can be deduced when differentiable functions are expanded to the first order by Bernoulli polynomials, with the remainder term expressed in the integral form.

This result completes, in fact, all the possible situations in which we can study the problem of the order of approximation for the KS series. With the proposed strategy, also the corresponding estimation for the generalized sampling (GS) series can be deduced. We recall that the definition of the GS operators can be given by replacing the first-order integral means of $K_w^\chi f$ with the sample values $f(k/w)$, $k\in\mathbb{Z}$, $w>0$. At the end of the paper, the critical case of the Fejér kernel is discussed in detail.

2 The Euler-MacLaurin summation formula and some preliminary notions

We need to recall the following special form of the Euler-MacLaurin summation formula.

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a given function with continuous derivative in $[a,b]$. Then*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)\,dt + \int_a^b \{t\}\, f'(t)\,dt - \{b\}\, f(b) + \{a\}\, f(a),$$

where the symbol $\{\cdot\}$ denotes the fractional part function of a given number, i.e.,

$$\{x\} := x - \lfloor x \rfloor \in [0,1), \quad x\in\mathbb{R},$$

with $\lfloor x\rfloor$ be the integer part of x.

This result can be found, for instance, in formula 2.1.2 of [32]. The above theorem will be crucial in the proof of the main result of this paper.

Furthermore, we recall the notion of the modulus of continuity of a given $f\in C(\mathbb{R})$, that is the following:

$$\omega(f,\delta) := \sup_{|t|\leq\delta} \|f(\cdot+t) - f(\cdot)\|_\infty, \quad \delta>0.$$

For a complete list of the properties of $\omega(f,\delta)$ we refer to [22]; here we recall only the following two useful facts:

$$(i) \quad \omega(f,\delta)\to 0, \quad as \quad \delta\to 0^+;$$

$$(ii) \quad \omega(f,\lambda\,\delta) \;\leq\; (1+\lambda)\,\omega(f,\delta), \quad \lambda,\,\delta>0.$$

Finally, we also recall the main condition that we assume from now on of the kernel functions χ, that is the following.

We always consider continuous χ such that:

$$\chi(u) \;=\; \mathcal{O}(|u|^{-2}), \quad as \quad |u|\to+\infty, \quad and\ with \quad M_1(\chi)=+\infty. \tag{2}$$

3 Main result for the KS operators

In this section we prove the following theorem.

Theorem 3.1. *Let χ be a given kernel satisfying (2). Then, for every $f\in C(\mathbb{R})$, we have:*

$$\left\|K_w^\chi f - f\right\|_\infty \;\leq\; C\left[\,\omega\!\left(f, \frac{\log(w)}{w}\right) + \frac{\|f\|_\infty}{w}\right],$$

for every sufficiently large $w>0$, where $C>0$ is a suitable absolute constant depending only on χ.

Proof. Since χ satisfies (2), there exists $x_0 > 0$ (we may suppose $x_0 > 1$ without any loss of generality) and a constant $C_1 > 0$ such that:

$$|\chi(u)| \le C_1 |u|^{-2},$$

for all $|u| > x_0$. Let us fix $w > x_0 > 0$. Then, using the partition of the unit property ($\chi 2$), we have for every $x \in \mathbb{R}$, that

$$\left|K_w^\chi f(x) - f(x)\right| \le \sum_{|wx-k|\le w} |\chi(wx-k)|\, w \int_{k/w}^{(k+1)/w} |f(u) - f(x)|\, du$$

$$+ \sum_{|wx-k|>w} |\chi(wx-k)|\, w \int_{k/w}^{(k+1)/w} |f(u) - f(x)|\, du =: I_1 + I_2.$$

Let us focus on I_1. By the monotonicity of the modulus of continuity, using the property (ii) with $\lambda = \frac{w}{\log(w)} |u-x|$, and recalling the definition of the discrete absolute moments, we obtain:

$$I_1 \le \sum_{|wx-k|\le w} |\chi(wx-k)|\, w \int_{k/w}^{(k+1)/w} \omega(f, |u-x|)\, du$$

$$\le \omega\left(f, \frac{\log(w)}{w}\right) \left[\sum_{|wx-k|\le w} |\chi(wx-k)|\, w \int_{k/w}^{(k+1)/w} \left(1 + \frac{w}{\log(w)} |u-x|\right) du \right]$$

$$\le \omega\left(f, \frac{\log(w)}{w}\right) \left[M_0(\chi) + \frac{w}{\log(w)} \sum_{|wx-k|\le w} |\chi(wx-k)|\, w \int_{k/w}^{(k+1)/w} |u-x|\, du \right].$$

Now, adding and subtracting k/w in the functions of the above integrals, we get:

$$I_1 \le \omega\left(f, \frac{\log(w)}{w}\right) M_0(\chi)$$

$$+ \frac{\omega\left(f, \frac{\log(w)}{w}\right)}{\log(w)} \left(\frac{1}{2} \sum_{|wx-k|\le w} |\chi(wx-k)| + \sum_{|wx-k|\le w} |\chi(wx-k)|\, |wx-k| \right)$$

$$\le \omega\left(f, \frac{\log(w)}{w}\right) \left[M_0(\chi) + \frac{M_0(\chi)}{\log(w)} \left(\frac{1}{2} + x_0\right) \right]$$

$$+ \frac{\omega\left(f, \frac{\log(w)}{w}\right)}{\log(w)} \sum_{x_0 < |wx-k| \le w} |\chi(wx-k)|\, |k - wx|.$$

Now, using (2) we can observe that

$$\sum_{x_0<|wx-k|\le w} |\chi(wx-k)|\, |wx-k| \le C_1 \sum_{x_0<|wx-k|\le w} |wx-k|^{-1}$$

$$= C_1 \sum_{-w+wx\le k<-x_0+wx} (wx-k)^{-1} + C_1 \sum_{x_0+wx<k\le w+wx} (k-wx)^{-1}$$

$$= C_1 \sum_{x_0-wx<\tilde{k}\le w-wx} \left(wx+\tilde{k}\right)^{-1} + C_1 \sum_{x_0+wx<k\le w+wx} (k-wx)^{-1}$$

Note that, the function $F_1(t) := 1/(wx+t)$, is differentiable with continuous derivative if $t \in J_1 := [x_0 - wx, w - wx]$, and the same results for $F_2(t) := 1/(t - wx)$, if $t \in J_2 := [x_0 + wx, w + wx]$, hence, by the Euler-MacLaurin summation formula applied to F_1 on J_1 we get:

$$\sum_{x_0-wx<\tilde{k}\le w-wx} (wx+\tilde{k})^{-1} = \int_{x_0-wx}^{w-wx} \frac{dt}{wx+t} - \int_{x_0-wx}^{w-wx} \frac{\{t\}\, dt}{(wx+t)^2}$$

$$-\frac{\{w-wx\}}{w} + \frac{\{x_0-wx\}}{x_0}$$

$$\le \log w + \frac{1}{w} + \frac{1}{x_0} + \log x_0 + \frac{\{w-wx\}}{w} + \frac{\{x_0-wx\}}{x_0} \le \widetilde{C_1} \log w,$$

for every sufficiently large $w > 0$, where the constant $\widetilde{C_1}$ is absolute. In a similar manner, using the Euler-MacLaurin summation formula for F_2 on J_2, we have:

$$\sum_{x_0+wx<k\le w+wx} (k-wx)^{-1} = \int_{x_0+wx}^{w+wx} \frac{dt}{t-wx} - \int_{x_0+wx}^{w+wx} \frac{\{t\}\, dt}{(t-wx)^2}$$

$$-\frac{\{w+wx\}}{w}+\frac{\{x_0+wx\}}{x_0} \leq \widetilde{C_2}\log w,$$

for every sufficiently large $w>0$, where also the constant $\widetilde{C_2}$ is absolute. Then we can conclude that there exists a constant $K_1>0$, which depends on χ (and that is independent on w and x), such that

$$I_1 \leq K_1\,\omega\left(f, \frac{\log(w)}{w}\right),$$

for $w>0$ sufficiently large. It remains to consider I_2. Since $w>x_0$, using again (2) we have

$$I_2 \leq 2C_1\,\|f\|_\infty \sum_{|wx-k|>w} |wx-k|^{-2}$$

$$= 2C_1\,\|f\|_\infty \sum_{k<wx-w}(k-wx)^{-2} + 2C_1\,\|f\|_\infty \sum_{k>wx+w}(k-wx)^{-2}.$$

Now, observing that the following inequalities hold:

$$\sum_{k<wx-w}(k-wx)^{-2} \leq \int_{-\infty}^{wx-w+1}(t-wx)^{-2}\,dt = \frac{1}{w-1} \leq \frac{2}{w},$$

and

$$\sum_{k>wx+w}(k-wx)^{-2} \leq \int_{wx+w-1}^{+\infty}(t-wx)^{-2}\,dt = \frac{1}{w-1} \leq \frac{2}{w},$$

we finally deduce that:

$$I_2 \leq 8C_1\,\|f\|_\infty\,\frac{1}{w} =: K_2\,\frac{1}{w}.$$

Thus, the thesis follows by taking $C=\max\{K_1,K_2\}$. □

Remark 1. From Theorem 3.1, assuming f in suitable Lipschitz classes we can immediately deduce the corresponding qualitative order of approximation.

4 Quantitative estimates for the GS operators

Recalling the definition of the generalized sampling (GS) operators:

$$(G_w^\chi f)(x) := \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi(wx-k), \quad x\in\mathbb{R}, \quad w>0$$

for any bounded $f:\mathbb{R}\to\mathbb{R}$ and $\chi:\mathbb{R}\to\mathbb{R}$ a given kernel we can immediately state what follows, proceeding with the same strategy of Theorem 3.1.

Theorem 4.1. *Let χ be a given kernel satisfying (2). Then, for every $f\in C(\mathbb{R})$, we have:*

$$\left\|G_w^\chi f - f\right\|_\infty \leq C\left[\omega\left(f, \frac{\log(w)}{w}\right) + \frac{\|f\|_\infty}{w}\right],$$

for every sufficiently large $w>0$, where $C>0$ is a suitable absolute constant depending only on χ.

Proof. Noting that, from condition (χ2) and the property of the modulus of continuity, one has the following inequality:

$$\left|(G_w^\chi f)(x)-f(x)\right| \leq \sum_{k\in\mathbb{Z}}|f(k/w)-f(x)|\,|\chi(wx-k)|$$

$$\leq \sum_{k\in\mathbb{Z}}|f(k/w)-f(x)|\,|\chi(wx-k)| \leq \sum_{k\in\mathbb{Z}}\omega\left(f, \left|\frac{k}{w}-x\right|\right)|\chi(wx-k)|$$

$$\leq \omega\left(f, \frac{\log(w)}{w}\right)\sum_{k\in\mathbb{Z}}\left(1+\frac{w}{\log(w)}\left|\frac{k}{w}-x\right|\right)|\chi(wx-k)|,$$

hence the proof immediately follows arguing as in the proof of Theorem 3.1 and using the Euler-MacLaurin summation formula. □

Finally, we can also stress that also for the GS operators can be easily established a Jackson-type estimate and an inequality of the form (1) using the same strategy adopted respectively in [6] and [17] for the KS operators.

Finally, we can also observe that actually, in the proof of Theorem 4.1 the assumption that $\chi\in L^1(\mathbb{R})$ could be avoided, since it is used only when we deal with L^p-approximation results. This is also true when we consider the KS operators.

5 The critical case of the Fejér kernel and other examples

We recall that the Fejér kernel is defined as follows:

$$F(x) := \frac{1}{2}\,\mathrm{sinc}^2(x/2), \quad x \in \mathbb{R},$$

where:

$$\mathrm{sinc}(x) := \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & otherwise. \end{cases}$$

It is well-know that F is an example of band-limited (hence highly regular) kernel satisfying $(\chi 1)$ and $(\chi 2)$, see, e.g., [33, 7, 11, 1]. Moreover, it is also clear that:

$$F(x) = \mathcal{O}(|x|^{-2}), \quad as \quad |x| \to +\infty.$$

Now, we can also observe that, if we choose for instance $u = 1$:

$$M_1(F) = \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} F(u-k)\,|u-k| \geq \frac{2}{\pi^2} \sum_{n=0}^{+\infty} \sin^2\left(\frac{\pi}{2}(1+n)\right)(1+n)^{-1}$$
$$= \frac{2}{\pi^2} \sum_{n=0}^{+\infty} \frac{1}{2n+1} = +\infty.$$

This shows that the Fejér kernel is a typical example of kernel satisfying condition (2). Hence we can state the following corollary.

Corollary 5.1. *For every $f \in C(\mathbb{R})$ we have:*

$$\left\|G_w^F f - f\right\|_\infty \leq C_1^F \left[\omega\left(f, \frac{\log(w)}{w}\right) + \frac{\|f\|_\infty}{w}\right],$$

and

$$\left\|K_w^F f - f\right\|_\infty \leq C_2^F \left[\omega\left(f, \frac{\log(w)}{w}\right) + \frac{\|f\|_\infty}{w}\right],$$

for every sufficiently large $w > 0$ and for suitable absolute constants $C_i^F > 0$, $i = 1, 2$.

Other examples of kernels with the same behaviour can be generated if one can find the following asymptotic behaviour of χ:

$$M_1\,|u|^{-2} \leq |\chi(u)| \leq M_2\,|u|^{-2}, \quad as \quad |u| \to +\infty, \tag{3}$$

for suitable positive constants $M_1 \leq M_2$. Indeed, condition (3) immediately implies assumption (2). An example of kernel function χ satisfying (3) can be found in [14], p.7 or in [15], p. 4591.

6 Conclusions and future developments

In this paper we estimated the rate of approximation for the KS and GS operators, under the general assumption (2) on χ. These results allow to "fully complete" the topic of studying the rate of convergence of the KS and GS operators with respect to the uniform norm, and employing the classical modulus of continuity of f.

As future work, one could try to estimate the error of approximation by means of the second order modulus of smoothness (without requiring, as usual, suitable Strang-Fix type assumptions on the discrete algebraic moment of the kernel), or to extend the present results (for the KS operators) for functions belonging to $L^p(\mathbb{R})$. Also extensions for the cases of the Durrmeyer and Steklov sampling operators could be considered.

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Conflicts of interests/Competing interests

The authors declare that they have not conflict of interest and/or competing interest.

Availability of data and material and Code availability

Not applicable.

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