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Sharp L_p Bernstein type inequalities for certain cuspidal domains

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Dedicated to Professor Len Bos on the occasion of his 70th birthday

Abstract

The aim of this work is to prove two kinds of Bernstein type inequalities on certain cuspidal sets. The first one is related to the plurisubharmonic extremal function. The second type concerns star-shaped, centrally symmetric domains and uses the Euclidean distance (in directions parallel to the system axis) from the boundary.

1 Introduction

Let $\mathcal{P}_n^d$ be the class of all algebraic polynomials in d variables with real coefficients of degree at most n. Further, let $C(\Omega)$ be the real space of all real valued continuous functions f defined on a compact set $\Omega \subset \mathbb{R}^d$ with the norm $\|f\|_\Omega := \sup_{x\in\Omega} |f(x)|$, and let $L_s(\Omega)$, $1 \le s < \infty$, be the space of all Lebesgue-measurable functions f on $\Omega \subset \mathbb{R}^d$ such that

$$\|f\|_{L_s(\Omega)} := \left(\int_\Omega |f(x)|^s\, dx\right)^{1/s} < \infty \quad \text{if} \quad 1 \le s < \infty.$$

We denote by D_j the partial derivative with respect to the variable x_j, whereas the rth-order partial derivative is denoted by $D_j^{(r)}$. Moreover, $\mathbb{N} = \{1, 2, 3, \ldots\}$. Throughout this paper, we adopt the convention that the letter c denotes a constant that depends on fixed parameters, such as d, s but independent of the degree n of polynomials.

The classical Bernstein inequality, for univariate algebraic polynomials of degree n, gives the following sharp upper bound for their derivatives:

$$\|\sqrt{1-x^2}\,p_n'(x)\|_{[-1,1]} \le n\|p_n\|_{[-1,1]}. \tag{1}$$

It is well known (see [19]) that Bernstein's inequality holds for higher derivatives i.e. there exists a constant $c(r)$ depending only on r such that

$$\|(\sqrt{1-x^2})^r p_n^{(r)}(x)\|_{[-1,1]} \le c(r)n^r\|p_n\|_{[-1,1]}. \tag{2}$$

It is also well known (see for instance [8]) that inequalities (1) and (2) extend to the L_s norm on $[-1,1]$ with some constants depending on s

$$\|\sqrt{1-x^2}\,p_n'(x)\|_{L_s([-1,1])} \le c(s)n\|p_n\|_{L_s([-1,1])}, \tag{3}$$

$$\|(\sqrt{1-x^2})^r p_n^{(r)}(x)\|_{L_s([-1,1])} \le \tilde{c}(s)\tilde{c}(r)n^r\|p_n\|_{L^s([-1,1])}. \tag{4}$$

The inequalities mentioned above, along with their many extensions (see, for example, [3, 5, 6, 7, 8, 9, 10, 11, 12, 13], and the survey [18]), hold a fundamental position in various realms of analysis and approximation theory.

In order to understand the principal claims, we discuss known results that are relevant and related to our work. Let E be a compact set in $\mathbb{C}^m$. Define

$$V_E(z) := \sup\{u(z) : u \in \mathcal{L},\, u \le 0 \text{ on } E\}, \quad z \in \mathbb{C}^m,$$

where $\mathcal{L}$ is the Lelong class of plurisubharmonic (briefly, psh) functions in $\mathbb{C}^m$ with logarithmic growth: $u(z) < const + \log(1+|z|)$, see, for example [15], page 184. Then the upper semieontinuous regularization $V_E^*(z) := \limsup_{\xi\to z} V_E(\xi)$ is called the (Siciak)

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extremal function of E. Let K be a compact set in $\mathbb{R}^d$ with nonempty interior. Then, by Theorem 1.4 from [1] (see also [2]), the following inequalities hold for every $x \in \text{int}(K)$ and $p \in \mathcal{P}_n^d$:

$$|D_j p(x)| \leq n D_j^+ V_K(x)(\|p\|_K^2 - p^2(x))^{1/2}, \tag{5}$$

where $D_j^+ V_E(x) := \liminf_{\epsilon \to 0^+} \frac{V_E(x+i\epsilon e_j)}{\epsilon}$ for $j = 1, \dots, d$, and $\{e_1, \dots, e_d\}$ is the standard orthogonal basis in $\mathbb{R}^d$. If K is the closure of an open bounded subset of $\mathbb{R}^d$, then the above inequality implies that

$$\left\| \frac{D_j p}{D_j^{+,*} V_K} \right\|_K \leq n \|p\|_K \quad \text{for every} \quad p \in \mathcal{P}_n^d, \tag{6}$$

where $D_j^{+,*} V_K(y) := \limsup_{x \to y} D_j^+ V_K(x)$. Moreover, it was shown therein (see Theorem 3.7 in [1]) that if K is a compact, convex subset of $\mathbb{R}^d$ such that $0 \in \text{int}(K)$, then there is a constant $c(K) > 0$ such that, for every polynomial $p \in \mathcal{P}_n^d$, we have the Bernstein inequality

$$|D_j p(x)| \leq c(K) n (\text{dist}(x, \partial K))^{-1/2} (\|p\|_K^2 - p^2(x))^{1/2}, \quad \text{for } x \in \text{int}(K),\ j = 1, \dots, d, \tag{7}$$

where ∂K is the boundary of K. The inequality (7) yields

$$\left\| (\text{dist}(x, \partial K))^{\frac{1}{2}} D_j p \right\|_K \leq c(K) n \|p\|_K. \tag{8}$$

In a recent paper, Kroó [17] (see also [16]) extended the inequality (8) to the case of the L_s space, and considering the model case of a convex polytope, verified that the Euclidean distance to the boundary cannot be replaced by an essentially larger function. In another recent paper [14] (see also [21]), the authors considered the simplex defined by

$$\triangle^d := \{x \in \mathbb{R}^d : x_1 \geq 0, \dots, x_d \geq 0, |x| \leq 1\}, \quad |x| := x_1 + \dots + x_d. \tag{9}$$

and proved that, for

$$\phi_j(x) := \frac{\sqrt{x_j}\sqrt{1-|x|}}{\sqrt{x_j + 1 - |x|}}, \tag{10}$$

a doubling weight w on $\triangle^d$, $n, r \in \mathbb{N}$ and $1 \leq s \leq \infty$, there is a constant $c > 0$, depending on d, s, r, w, such that

$$\|\phi_j^r D_j^{(r)} p\|_{L_s(\triangle^d, w)} \leq c(d, r, s, w) n^r \|p\|_{L_s(\triangle^d, w)}, \quad 1 \leq j \leq d, \tag{11}$$

for every polynomial $p \in \mathcal{P}_n^d$. Here, as usual, $\|\cdot\|_{L_s(\triangle^d, w)}$ stands for the weighted L_s norm. Since

$$D_j^+ V_{\triangle^d}(x) = \frac{\sqrt{x_j + 1 - |x|}}{\sqrt{x_j}\sqrt{1-|x|}} = \frac{1}{\phi_j(x)},$$

the inequality (11) is a generalization of (5) in the case when $E = \triangle^d$.

Let $K = \overline{\text{int}(K)}$ be a star-shaped, centrally symmetric (with respect to the origin) domain in $\mathbb{R}^d$ and let $m \geq 1$. Define for $v \in \mathbb{S}^{d-1}$ and $x \in K$

$$\rho_v(K, x) := \sup\{t \geq 0 : [x - tv, x + tv] \subset K\}.$$

Assume that there exist constants M_j, m_j, $j = 1, \dots, d$, such that

$$\rho_{e_j}(K, tx) \geq M_j (1 - |t|)^{m_j}$$

for $t \in [-1, 1]$, $x \in \partial K$, $j = 1, \dots, d$. If we put $\rho_*(K, x) := \min_{1 \leq j \leq d} \rho_{e_j}(K, x)$, $m := \max_{1 \leq j \leq d} m_j$, $M := \min_{1 \leq j \leq d} M_j$. Then the following analogue of the classical Bernstein inequality was given in [4]

$$\left\| \rho_*(K, x)^{1 - \frac{1}{2m}} D_j p \right\|_K \leq \sqrt{2} M^{-\frac{1}{2m}} n \|p\|_K, \quad 1 \leq j \leq d, \tag{12}$$

for every polynomial $p \in \mathcal{P}_n^d$.

The aim of this work is to prove new Bernstein type inequalities on certain cuspidal sets. More precisely, for $k \in \mathbb{N}$, we consider the following domains

$$\Theta_k = \{x \in \mathbb{R}^d : x_1 \geq 0, \dots, x_d \geq 0, |x|_k \leq 1\}, \quad |x|_k := x_1^{1/k} + \dots + x_d^{1/k},$$
$$\Omega_k = \{x \in \mathbb{R}^d : |x_1|^{1/k} + \dots + |x_d|^{1/k} \leq 1\}.$$

In the present paper we establish an L_s analogue of (5) for Θ_k and an L_s analogue of (12) for Ω_k. Moreover, we show that our estimates are asymptotically sharp.

2 Main results

This section addresses main theorems.

Theorem 2.1. *Let k be a natural number, and let $\Theta_k = \{x \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0,\ x_1^{1/k} + \ldots + x_d^{1/k} \leq 1\}$. Then, for every $1 \leq s < \infty$ and each $r \in \mathbb{N}$, there exists a positive constant c such that for every polynomial $p \in \mathcal{P}_n^d$,*

$$\left\| \left(x_j^{1-1/k} \sqrt{1-|x|_k} \right)^r D_j^{(r)} p \right\|_{L_s(\Theta_k)} \leq c(d,k,r,s) n^r \|p\|_{L_s(\Theta_k)}, \quad j = 1, \ldots, d. \tag{13}$$

Proof. Taking into account the fact that the following methods do not actually depend on the number of variables, we will perform the proof in the case of $d = 2$. The proof for each j is the same, hence we only consider the case $j = 2$. First, we define

$$E_1 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ (1 - x_1^{1/k})^k 2^{-k} \leq x_2 \leq (1 - x_1^{1/k})^k\},$$
$$E_2 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq (1 - x_1^{1/k})^k 2^{-k}\}.$$

Then, using the change of variables $x_1 = t^k$, $x_2 = z^k$, we have

$$\left\| \left(x_2^{1-1/k} \sqrt{1 - x_1^{1/k} - x_2^{1/k}} \right)^r D_2^{(r)} p \right\|_{L_s(E_1)}^s = \int_0^1 \int_{\frac{1-t}{2}}^{1-t} |z^{r(k-1)} (1-t-z)^{\frac{r}{2}} D_2^{(r)} p(t^k, z^k)|^s w(t,z)\, dz dt,$$

where $w(t,z) = k^2 t^{k-1} z^{k-1}$. Is is clear that, for $t \in [0,1)$ and $z \leq 1-t \leq 2z$,

$$\sqrt{1-t-z} \leq \sqrt{2} \frac{\sqrt{z}\sqrt{1-t-z}}{\sqrt{1-t}} = \sqrt{2}\phi_2(t,z),$$

see (10).

Therefore,

$$\int_0^1 \int_{\frac{1-t}{2}}^{1-t} |z^{r(k-1)} (1-t-z)^{\frac{r}{2}} D_2^{(r)} p(t^k, z^k)|^s w(t,z)\, dz dt \leq \int_0^1 \int_0^{1-t} |z^{r(k-1)} (\sqrt{2}\phi_2(t,z))^r D_2^{(r)} p(t^k, z^k)|^s w(t,z)\, dz dt.$$

Applying inequality (11) to $D_2^{(r-1)} p(t^k, z^k)$, with the weight

$$w_1(t,z) = z^{(r-1)(k-1)s} (\phi_2(t,z))^{s(r-1)} 2^{\frac{rs}{2}} w(t,z),$$

there exists a positive constant $c_1 > 0$ such that

$$\int_0^1 \int_0^{1-t} |k z^{k-1} \phi_2(t,z) D_2^{(r)} p(t^k, z^k)|^s w_1(t,z)\, dz dt \leq c_1 n^s \int_0^1 \int_0^{1-t} |D_2^{(r-1)} p(t^k, z^k)|^s w_1(t,z)\, dz dt.$$

Applying inequality (11) to $D_2^{(r-l)} p(t^k, z^k)$, with the weight

$$w_{l+1}(t,z) = \frac{w_l(t,z)}{z^{(k-1)s} \phi_2^s(t,z)} = z^{(r-l-1)(k-1)s} (\phi_2(t,z))^{s(r-l-1)} 2^{\frac{rs}{2}} w(t,z),$$

for $l = 1, 2, \ldots, r-2$, there are positive constants $c_{l+1} > 0$ such that

$$\int_0^1 \int_0^{1-t} |k z^{k-1} \phi_2(t,z) D_2^{(r-l)} p(t^k, z^k)|^s w_{l+1}(t,z)\, dz dt \leq c_{l+1} n^s \int_0^1 \int_0^{1-t} |D_2^{(r-l-1)} p(t^k, z^k)|^s w_{l+1}(t,z)\, dz dt.$$

Once again by (11), applied to $p(t^k, z^k)$, with the weight $w_r(t,z) = \frac{w_{r-1}(t,z)}{z^{(k-1)s} \phi_2^s(t,z)} = 2^{\frac{rs}{2}} w(t,z)$, there exists a positive constant $c_r > 0$ so that

$$\int_0^1 \int_0^{1-t} |k z^{k-1} \phi_2(t,z) D_2 p(t^k, z^k)|^s w_r(t,z)\, dz dt \leq c_r n^s \int_0^1 \int_0^{1-t} |p(t^k, z^k)|^s w_r(t,z)\, dz dt.$$

Therefore,

$$\int_0^1 \int_0^{1-t} |z^{r(k-1)} (\sqrt{2}\phi_2(t,z))^r D_2^{(r)} p(t^k, z^k)|^s w(t,z)\, dz dt \leq \Lambda 2^{\frac{rs}{2}} n^{sr} \int_0^1 \int_0^{1-t} |p(t^k, z^k)|^s w(t,z)\, dz dt = \Lambda 2^{\frac{rs}{2}} n^{sr} \|p\|_{E_1 \cup E_2}^s.$$

Here $\Lambda := \prod_{l=1}^r c_l$.

The above reasoning then shows that

$$\left\| \left(x_2^{1-1/k} \sqrt{1 - x_1^{1/k} - x_2^{1/k}} \right)^r D_2^{(r)} p \right\|_{L_s(E_1)}^s \leq \Lambda 2^{\frac{rs}{2}} n^{sr} \|p\|_{E_1 \cup E_2}^s. \tag{14}$$

Now we wish to prove a similar result for the set E_2. Using the change of variables $x_1 = t^k$, $x_2 = z(1-t)^{k-1}$, we have

$$\left\|\left(x_2^{1-1/k}\sqrt{1-x_1^{1/k}-x_2^{1/k}}\right)^r D_2^{(r)}p\right\|_{L_s(E_2)}^s = \int_0^1\int_0^{\frac{1-t}{2^k}} |(u(t,z))^r D_2^{(r)}p(t^k,z(1-t)^{k-1})|^s v(t)\,dz\,dt,$$

where $u(t,z) = z^{1-\frac{1}{k}}(1-t)^{k-2+\frac{1}{k}}\sqrt{1-t-(z(1-t)^{k-1})^{\frac{1}{k}}}$, $v(t) = kt^{k-1}(1-t)^{k-1}$. Now we observe that for any $0 \le 2^k z \le 1-t$ and $0 \le t \le 1$,

$$u(t,z) \le \sqrt{z}(1-t)^{k-1} \le \frac{(1-t)^{k-1}}{\sqrt{1-2^{-k}}}\frac{\sqrt{z}\sqrt{1-t-z}}{\sqrt{1-t}} = \frac{(1-t)^{k-1}}{\sqrt{1-2^{-k}}}\phi_2(t,z).$$

Hence

$$\int_0^1\int_0^{\frac{1-t}{2^k}} |(u(t,z))^r D_2^{(r)}p(t^k,z(1-t)^{k-1})|^s v(t)\,dz\,dt$$
$$\le (1-2^{-k})^{-\frac{rs}{2}}\int_0^1\int_0^{1-t} |((1-t)^{k-1}\phi_2(t,z))^r D_2^{(r)}p(t^k,z(1-t)^{k-1})|^s v(t)\,dz\,dt.$$

By applying inequality (11) to $q(t,z) = p(t^k,z(1-t)^{k-1})$, with the weight $v(t)$, there exists a positive constant κ so that

$$\int_0^1\int_0^{1-t} |((1-t)^{k-1}\phi_2(t,z))^r D_2^{(r)}p(t^k,z(1-t)^{k-1})|^s v(t)\,dz\,dt \le \kappa n^{rs}\int_0^1\int_0^{1-t} |p(t^k,z(1-t)^{k-1})|^s v(t)\,dz\,dt = \kappa n^{rs}\|p\|_{E_1\cup E_2}^s.$$

Thus,

$$\left\|\left(x_2^{1-1/k}\sqrt{1-x_1^{1/k}-x_2^{1/k}}\right)^r D_2^{(r)}p\right\|_{L_s(E_2)}^s \le \frac{\kappa n^{rs}}{(1-2^{-k})^{\frac{rs}{2}}}\|p\|_{E_1\cup E_2}^s. \tag{15}$$

Since $\Theta_k = E_1 \cup E_2$, (14) and (15) implies (13). □

Remark 1. Let $E = \{(x,y) \in \mathbb{R}^2 : x,y \ge 0,\ \sqrt{x}+\sqrt{y} \le 1\}$. By applying a result of Klimek (see, e.g. [15], Theorem 5.3.1) to the set $[-1,1]\times[-1,1]$ and the mapping $f(z_1,z_2) = \frac{1}{4}((z_1-z_2)^2,(z_1+z_2)^2)$, with appropriate choice of the branch of the square roots, we find that

$$V_E(z_1,z_2) = 2\max\{\log|h(\sqrt{z_1}+\sqrt{z_2})|, \log|h(\sqrt{z_1}-\sqrt{z_2})|\},$$

for $(z_1,z_2) \in \mathbb{C}^2$ such that $|\sqrt{z_1}+\sqrt{z_2}| < 1$, $|\sqrt{z_1}-\sqrt{z_2}| < 1$. Here the function h is the inverse function to the Joukowski function, $g(z) = \frac{1}{2}(z+\frac{1}{z})$ for $z \in \mathbb{C}\setminus\{0\}$. Then, if $(x_1,x_2) \in \mathrm{int}(E)$, one can calculate that

$$D_j^+ V_E((x_1,x_2)) = \frac{1}{\sqrt{x_j}\sqrt{1-(\sqrt{x_1}+\sqrt{x_2})^2}}, \quad j = 1,2.$$

Hence, if $(x_1,x_2) \in \mathrm{int}(E)$, we have

$$\sqrt{x_j}\sqrt{1-\sqrt{x_1}-\sqrt{x_2}} \le \frac{1}{D_j^+ V_E((x_1,x_2))} \le 2\sqrt{x_j}\sqrt{1-\sqrt{x_1}-\sqrt{x_2}}.$$

Therefore (13) can be considered as an extension of (5).

The second main theorem is as follows.

Theorem 2.2. *Let k be a natural number, and let $\Omega_k = \{x \in \mathbb{R}^d : |x_1|^{1/k}+\ldots+|x_d|^{1/k} \le 1\}$. Then, for every $1 \le s < \infty$ and each $r \in \mathbb{N}$, there exists a positive constant c such that for every polynomial $p \in \mathcal{P}_n^d$,*

$$\left\|\left(\rho_*(\Omega_k,x)^{1-\frac{1}{2k}}\right)^r D_j^{(r)}p\right\|_{L_s(\Omega_k)} \le c(d,k,r,s)n^r\|p\|_{L_s(\Omega_k)}, \quad j = 1,\ldots,d. \tag{16}$$

Proof. For similar reasons as before, we will perform the proof in the case of two variables. First, let us examine the case when $k = 1$. Given any $x \in \Omega_1$ we denote by $\tau_{\Omega_1}(x) := \inf_{y\in\partial\Omega_1}\|x-y\|_2$ the Euclidean distance from x to the boundary of Ω_1. Since

$$\tau_{\Omega_1}(x) \le \rho_*(\Omega_1,x) \le \sqrt{d}\,\tau_{\Omega_1}(x),$$

the result follows from Corollary 2 in [17].

Let $k \ge 2$ and, for $a \in [0,1)$, define

$$\Omega_k^+(a) := \{(x_1,x_2) \in \mathbb{R}^2 : a \le x_1 \le 1,\ -(1-\sqrt[k]{x_1})^k \le x_2 \le (1-\sqrt[k]{x_1})^k\},$$
$$\Omega_k^-(a) := \{(x_1,x_2) \in \mathbb{R}^2 : -1 \le x_1 \le -a,\ -(1-\sqrt[k]{-x_1})^k \le x_2 \le (1-\sqrt[k]{-x_1})^k\}.$$

Then, after the change of variables $x_1 = t^k$, $x_2 = s(1-t)^{k-1}$, we can write

$$\left\|\left((\rho_*(\Omega_k,x))^{1-\frac{1}{2k}}\right)^r D_2^{(r)}p\right\|_{L_s(\Omega_k^+(a))}^s = \int_{\sqrt[k]{a}}^1 \int_{t-1}^{1-t} \left|\left((\rho_*(\Omega_k,(t^k,s(1-t)^{k-1})))^{1-\frac{1}{2k}}\right)^r D_2^{(r)}p(t^k,s(1-t)^{k-1})\right|^s v(t)dsdt,$$

where $v(t) = k(t(1-t))^{k-1}$. From the definition of $\rho_*(\Omega_k,x)$, for $(x_1,x_2)\in\Omega_k^+(a)$, we have

$$\rho_*(\Omega_k,x) \le (1-\sqrt[k]{x_1})^k - |x_2|.$$

Hence, if $0\le t\le 1$, $t-1\le s\le 1-t$, we obtain

$$(\rho_*(\Omega_k,(t^k,s(1-t)^{k-1})))^{1-\frac{1}{2k}} \le (1-t)^{k-\frac{3}{2}+\frac{1}{2k}}(1-t-|s|)^{1-\frac{1}{2k}} \le (1-t)^{k-1}\sqrt{1-t-|s|}.$$

Thus, for $\frac{1}{2}\le t\le 1$, $t-1\le s\le 1-t$,

$$(\rho_*(\Omega_k,(t^k,s(1-t)^{k-1})))^{1-\frac{1}{2k}} \le (1-t)^{k-1}\sqrt{1-t-|s|} = (1-t)^{k-1}\sqrt{\sqrt{2}\tau_{\Omega_1^+(0)}(t,s)}.$$

Then, if $q(t,s) = p(t^k,s(1-t)^{k-1})$, by Corollary 2 in [17] (applied to $\Omega_1^+(0)$ with the weight v), there exists a positive constant κ such that

$$\left\|\left((\rho_*(\Omega_k,(x_1,x_2)))^{1-\frac{1}{2k}}\right)^r D_2^{(r)}p\right\|_{L_s(\Omega_k^+(\frac{1}{2^k}))} \le \left\|\left(\sqrt{\sqrt{2}\tau_{\Omega_1^+(0)}(t,s)}\right)^r D_2^{(r)}q\right\|_{L_s(\Omega_1^+(0),v)} \le \kappa n^r\|q\|_{L_s(\Omega_1^+(0),v)} = \kappa n^r\|p\|_{L_s(\Omega_k^+(0))}. \quad (17)$$

Now let $\Upsilon = \{(x_1,x_2)\in\mathbb{R}^2 : 0\le x_1\le \frac{1}{2^k}, -(1-\sqrt[k]{x_1})^k \le x_2 \le (1-\sqrt[k]{x_1})^k\}$. Then

$$(\rho_*(\Omega_k,(x_1,x_2)))^{1-\frac{1}{2k}} \le \sqrt{(1-\sqrt[k]{x_1})^k-|x_2|} \le 2^{\frac{k}{2}}\sqrt{\tilde{d}((x_1,x_2),(0,1))}, \quad (x_1,x_2)\in\Upsilon, \quad (18)$$

where

$$\tilde{d}((x_1,x_2),(0,1)) = \sup_{0<\lambda:(x_1,\lambda x_2)\in\Upsilon}\|(x_1,x_2)-(x_1,\lambda x_2)\|_2 \cdot \sup_{\lambda<0:(x_1,\lambda x_2)\in\Upsilon}\|(x_1,x_2)-(x_1,\lambda x_2)\|_2.$$

By Theorem 2.2 in [12], there exists a positive constant $c(r,s)>0$ such that

$$\|(\tilde{d}((x_1,x_2),(0,1)))^{r/2}D_2^{(r)}p\|_{L_s(\Upsilon)} \le c(r,s)n^r\|p\|_{L_s(\Upsilon)}.$$

Thus, by (18), and by the fact that $\Upsilon\subset\Omega_k^+(0)$, we have

$$\left\|\left((\rho_*(\Omega_k,(x_1,x_2)))^{1-\frac{1}{2k}}\right)^r D_2^{(r)}p\right\|_{L_s(\Upsilon)} \le \sigma^r c(r,s)n^r\|p\|_{L_s(\Upsilon)} \le \sigma^r c(r,s)n^r\|p\|_{L_s(\Omega_k^+(0))}. \quad (19)$$

By (17) and (19), there exists a positive constant $c>0$ so that

$$\left\|\left((\rho_*(\Omega_k,(x_1,x_2)))^{1-\frac{1}{2k}}\right)^r D_2^{(r)}p\right\|_{L_s(\Omega_k^+(0))} \le cn^r\|p\|_{L_s(\Omega_k^+(0))}.$$

Similarly,

$$\left\|\left((\rho_*(\Omega_k,(x_1,x_2)))^{1-\frac{1}{2k}}\right)^r D_2^{(r)}p\right\|_{L_s(\Omega_k^-(0))} \le cn^r\|p\|_{L_s(\Omega_k^-(0))}.$$

The proof is completed by the fact that $\Omega_k^+(0)\cup\Omega_k^-(0) = \Omega_k$. □

3 Sharpness of the Bernstein type inequalities

In this section we shall prove sharpness of Theorems 2.1 and 2.2.
To prove the statements we consider the following sequence of polynomials

$$U_n(x_1,\dots,x_d) = x_d^r P_n^{(\alpha,\alpha)}(x_1).$$

Here $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree n associated with parameters α,β. Using the identity $\int_0^1(1-t^b)^a\,dt = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}$, we obtain

$$\|U_n\|_{L_s(\Theta_k)}^s = \frac{1}{rs+1}\prod_{i=2}^{d-1}\frac{\Gamma(k(rs+i)+1)\Gamma(k+1)}{\Gamma(k(rs+i)+k+1)}\int_0^1(1-x_1^{1/k})^{k(rs+d-1)}|P_n^{(\alpha,\alpha)}(x_1)|^s\,dx_1. \quad (20)$$

We now apply Bernoulli's inequality to deduce that

$$\left(\frac{1-z}{l}\right)^l \le \left(1-z^{1/l}\right)^l \le (1-z)^l$$

for each positive integer l and $z \in [0,1]$. Therefore,

$$\|U_n\|^s_{L_s(\Theta_k)} \sim \int_0^1 (1-x_1)^{k(rs+d-1)} |P_n^{(\alpha,\alpha)}(x_1)|^s \, dx_1 \tag{21}$$

Now a result proved by Szegö (see [20, Chap. VII]) comes into play. With $\mu_{\alpha,s} = \alpha s - 2 + s/2$, we have

$$\int_0^1 \left|P_n^{(\alpha,\alpha)}(x)\right|^s (1-x)^l \, dx \sim n^{\alpha s - 2l - 2} \quad \text{whenever} \quad 2l < \mu_{\alpha,s}. \tag{22}$$

If $2k(s+1)(rs+d) < \mu_{\alpha,s}$, then we can combine (21) and (22) to see that

$$\|U_n\|^s_{L_s(\Theta_k)} \sim n^{\alpha s - 2k(rs+d-1)-2}. \tag{23}$$

By the symmetry relation $P_n^{(\alpha,\beta)}(-z) = (-1)^n P_n^{(\alpha,\beta)}(z)$, we have

$$\|U_n\|^s_{L_s(\Omega_k)} = 4\|U_n\|^s_{L_s(\Theta_k)} \sim n^{\alpha s - 2k(rs+d-1)-2}. \tag{24}$$

Now let $y_d = (1 - x_1^{1/k} - \ldots - x_{d-1}^{1/k})^k$. Then

$$\int_0^{y_d} \left(x_d^{1-1/k} \sqrt{1 - x_1^{1/k} - \ldots - x_{d-1}^{1/k} - x_d^{1/k}} \right)^{rs} dx_d = y_d^{rs(1-\frac{1}{2k})+1} \int_0^1 t^{rs(1-\frac{1}{k})} (1-t^{\frac{1}{k}})^{\frac{rs}{2}} \, dt.$$

Moreover, if $y_{d-1} = (1 - x_1^{1/k} - \ldots - x_{d-2}^{1/k})^k$,

$$\int_0^{y_{d-1}} y_d^{rs(1-\frac{1}{2k})+1} dx_{d-1} = y_{d-1}^{rs(1-\frac{1}{2k})+2} \int_0^1 (1-t^{\frac{1}{k}})^{krs(1-\frac{1}{2k})+k} \, dt = y_{d-1}^{rs(1-\frac{1}{2k})+2} \frac{\Gamma(krs(1-\frac{1}{2k})+k+1)\Gamma(k+1)}{\Gamma(krs(1-\frac{1}{2k})+2k+1)}.$$

Repeating this argument, we obtain that there exists a positive constant c, independent of n, such that

$$\int_0^{(1-x_1^{1/k})^k} \cdots \int_0^{y_d} \left(x_d^{1-1/k} \sqrt{1 - x_1^{1/k} - \ldots - x_{d-1}^{1/k} - x_d^{1/k}} \right)^{rs} dx_d \ldots dx_2 = c(1-x_1^{1/k})^{krs(1-\frac{1}{2k})+k(d-1)}.$$

Hence, in the same manner as before,

$$\left\| \left(x_d^{1-1/k} \sqrt{1-|x|_k} \right)^r D_d^{(r)} U_n \right\|^s_{L_s(\Theta_k)} \sim n^{\alpha s - 2k(rs+d-1)-2+rs}. \tag{25}$$

Thus, by (23) and (25), for fixed k, r, s, we have

$$\frac{\left\| \left(x_d^{1-1/k} \sqrt{1-|x|_k} \right)^r D_d^{(r)} U_n \right\|_{L_s(\Theta_k)}}{\|U_n\|_{L_s(\Theta_k)}} \sim n^r$$

which proves that the estimate (13) is asymptotically sharp. To prove the analogous property for the set Ω_k, we need to consider the function $\rho_*(\Omega_k, x) = \min_{1 \le j \le d} \rho_{e_j}(\Omega_k, x)$. If $x \in \Omega_k$ and each $x_i \ge 0$ then, by definition,

$$\rho_{e_j}(\Omega_k, x) = (1 - |x|_k + x_j^{1/k})^k - x_j.$$

Hence, for any such x and each j,

$$\rho_{e_j}(\Omega_k, x) \ge (1-|x|_k)^k.$$

We therefore conclude that

$$\rho_*(\Omega_k, x) \ge (1-|x|_k)^k \quad \text{for} \quad x \in \Omega_k^+ = \{x \in \Omega_k : x_i \ge 0, \, i = 1, \ldots, d\}. \tag{26}$$

Proceeding as before,

$$\left\| (1-|x|_k)^{r(k-\frac{1}{2})} D_d^{(r)} U_n \right\|^s_{L_s(\Omega_k^+)} = c_1 \int_0^1 (r!)^s (1-x_1)^{krs(1-\frac{1}{2k})+k(d-1)} |P_n^{(\alpha,\alpha)}(x_1)|^s \, dx_1,$$

where $c_1 = \prod_{l=1}^{d-1} \frac{\Gamma(rs(k-\frac{1}{2})+(l-1)k+1)\Gamma(k+1)}{\Gamma(rs(k-\frac{1}{2})+lk+1)}$. Using (22) again yields

$$\left\| (1-|x|_k)^{r(k-\frac{1}{2})} D_d^{(r)} U_n \right\|^s_{L_s(\Omega_k^+)} \sim n^{\alpha s - 2k(rs+d-1)-2+rs}. \tag{27}$$

Thus, by (24), (26), (27), and the symmetry of Ω_k and $\rho_*(\Omega_k, x)$, for fixed k, r, s, there exists a positive constant c_2 so that

$$\frac{\left\|\left((\rho_*(\Omega_k, x))^{1-\frac{1}{2k}}\right)^r D_d^{(r)} U_n\right\|_{L_s(\Omega_k)}}{\|U_n\|_{L_s(\Omega_k)}} \geq c_2 n^r$$

which gives the asymptotic optimality of statement (16).

At the end of this work, we formulate a hypothesis that naturally arises in the context of the obtained results.

Conjecture Let $\alpha \geq 1$, and let $\Omega_\alpha = \{x \in \mathbb{R}^d : |x_1|^{1/\alpha} + \ldots + |x_d|^{1/\alpha} \leq 1\}$. Then, for every $1 \leq s < \infty$ and each $r \in \mathbb{N}$, there exists a positive constant c such that for every polynomial $p \in \mathcal{P}_n^d$,

$$\left\|\left(\rho_*(\Omega_\alpha, x)^{1-\frac{1}{2\alpha}}\right)^r D_j^{(r)} p\right\|_{L_s(\Omega_\alpha)} \leq c(d, k, r, s) n^r \|p\|_{L_s(\Omega_\alpha)}, \quad j = 1, \ldots, d.$$

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References

[1] M. Baran. Bernstein type theorems for compact sets in R^n, *J. Approx. Theory*, 69:156–166, 1992.

[2] M. Baran. Bernstein type theorems for compact sets in R^n revisited, *J. Approx. Theory*, 79:190–198, 1994.

[3] M. Baran, W. Pleśniak. Bernstein and van der Corput-Schaake type inequalities on semialgebraic curves, *Studia Math.*, 125:83–96, 1997.

[4] M. Baran, G. Sroka. On Extremal Functions and V. Markov Type Polynomial Inequality for Certain Subsets of $\mathbb{R}^N$, *Constr. Approx.*, 60:135–163, 2024.

[5] L. Białas-Cież. Equivalence of Markov's property and Hölder continuity of the Green function for Cantor-type sets, *East J. Approx.*, 1:249–253, 1995.

[6] L. Białas-Cież, A. Kowalska. Bernstein-Chebyshev inequality and Baran's radial extremal function on algebraic sets. *Dolomites Research Notes on Approximation*, 14.3:16–26, 2021.

[7] L. Białas-Cież, J.-P. Calvi, A. Kowalska. Markov and Division Inequalities on Algebraic Sets, *Results Math.*, 79:135, 2024.

[8] P. Borwein and T. Erdélyi, Polynomials and Polynomial Inequalities, *Springer*, 1995.

[9] L.P. Bos, N. Levenberg, P. Milman, B. A. Taylor. Tangential Markov inequalities characterize algebraic submanifolds of $\mathbb{R}^N$, *Indiana Univ. Math. J.*, 44:115–138, 1995.

[10] D. Burns, N. Levenberg, S. Ma'u, Sz. Révész. Monge-Ampère measures for convex bodies and Bernstein-Markov type inequalities, *Trans. Amer. Math. Soc.*, 362:6325–6340, 2010.

[11] F. Dai, A. Prymak. L^p-Bernstein inequalities on C^2-domains and applications to discretization, *Trans. Amer. Math. Soc.*, 375:1933–1976, 2022.

[12] Z. Ditzian. Multivariate Bernstein and Markov inequalities, *J. Approx. Theory*, 70:273–283, 1992.

[13] M. Ganzburg. A note on sharp multivariate Bernstein- and Markov-type inequalities, *J. Approx. Theory*, 286:105847, 2023.

[14] Y. Ge, Y. Xu. Sharp Bernstein inequalities on simplex, *Constr. Approx.*, 2024. https://doi.org/10.1007/s00365-024-09680-6

[15] M. Klimek. Pluripotential Theory, *Clarendon Press*, 1991.

[16] A. Kroó. Sharp L_p Bernstein type inequality for cuspidal domains in $\mathbb{R}^d$, *J. Approx. Theory*, 267:105593, 2021.

[17] A. Kroó. L_p Bernstein type inequalities for star like Lip α domains, *J. Math. Anal. Appl.*, 532:127986, 2024.

[18] A. Kroó. Multivariate Bernstein-Markov type inequalities in L^p norm, *Constructive Theory Of Functions*, Lozenets 2023, (B. Draganov, K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 101-111 Professor Marin Drinov Publishing House of BAS, Sofia, 2024.

[19] G. Mastroianni, V. Totik. Weighted polynomial inequalities with doubling and A_∞ weights, *Constr. Approx.*, 16:37–71, 2000.

[20] G. Szegö. Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ. 23, *Amer. Math. Soc.*, Providence, 1975.

[21] Y. Xu. Bernstein inequality on conic domains and triangle, *J. Approx. Theory*, 290:105889, 2023.