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Around some extremal problems for multivariate polynomials

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Dedicated to Len Bos, on the occasion of his 70th birthday

Abstract

Let E be a compact subset of $\mathbb{C}^N$ and V_E be the pluricomplex Green's function of E. The Hölder continuity property, HCP for short, is one of the most interesting features of V_E. By means of a radial modification of V_E, we give some equivalent conditions to HCP connected with the Pleśniak property and the Markov inequality for polynomials. Moreover, we consider a capacity, a Chebyshev constant and a transfinite diameter with respect to a fixed norm on the space of polynomials of N variables. We prove that this capacity is not greater than a corresponding Chebyshev constant. One section is devoted to economisation procedure of approximation by telescoping series.

1 Introduction.

The *pluricomplex Green's function* of a compact set $E \subset \mathbb{C}^N$ can be defined by

$$V_E(z) := \sup\{u(z) : u \in \mathcal{L}_N \text{ and } u \le 0 \text{ on } E\} \quad \text{for } z \in \mathbb{C}^N, \tag{1}$$

where $\mathcal{L}_N$ is the Lelong class of all plurisubharmonic functions in $\mathbb{C}^N$ of logarithmic growth at the infinity, i.e.

$$\mathcal{L}_N := \{u \in PSH(\mathbb{C}^N) : u(z) - \log\|z\|_2 \le \mathcal{O}(1) \text{ as } \|z\|_2 \to \infty\} \tag{2}$$

(for background information, see [23]). Here $\|z\|_2$ stands for the Euclidean norm on $\mathbb{C}^N$. Some examples of V_E can be found in [23].

Let V_E^* be the upper semi continuous regularisation of V_E. By Siciak's theorem (see [32]), either $V_E^* \in \mathcal{L}_N$ or $V_E^* \equiv +\infty$. This is equivalent to the fact that E is either a *nonpluripolar* or *pluripolar* set (either *nonpolar* or *polar* for $N = 1$). For a nonpolar set $E \subset \mathbb{C}$, the upper semi continuous regularisation V_E^* coincides with the Green's function g_E of the unbounded component of $\hat{\mathbb{C}} \setminus E$ with logarithmic pole at infinity (as usual $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$). A Hölder continuity of V_E is called the *HCP property* of E, see (10) for a precise definition.

The L-*capacity* of $E \subset \mathbb{C}^N$ is defined by

$$C(E) := \liminf_{\|z\|_2 \longrightarrow \infty} \frac{\|z\|_2}{\exp V_E(z)}.$$

The set E is pluripolar if and only if $C(E) = 0$. In the one-dimensional case, $C(E)$ equals the logarithmic capacity of E (see [32]) and lim inf can be replaced by the limit.

A deep connection of the above defined quantities with polynomial approximation is given by the well known *Zakharyuta-Siciak theorem* [32] (cf. [23]):

$$\exp V_E(z) = \sup_{n\ge1}\left(\sup\{|P(z)| : P \in \mathcal{P}_n(\mathbb{C}^N),\ \|P\|_E \le 1\}\right)^{1/n} = \sup_{n\ge1} \Phi_n(E,z)^{1/n} =: \Phi(E,z), \ \ z \in \mathbb{C}^N, \tag{3}$$

where

$$\Phi_n(E,z) := \sup\{|P(z)| : P \in \mathcal{P}_n(\mathbb{C}^N),\ \|P\|_E \le 1\}.$$

Here $\mathcal{P}_n(\mathbb{C}^N)$ is the vector space of polynomials of N variables with complex coefficients of degree at most $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\mathcal{P}(\mathbb{C}^N) := \bigcup_{n\in\mathbb{N}_0} \mathcal{P}_n(\mathbb{C}^N)$ and $\|\cdot\|_E$ is the maximum norm on E. The symbol $\mathbb{N}$ will denote $\mathbb{N}_0 \setminus \{0\}$. The above defined $\Phi(E,\cdot)$ is the famous *Siciak extremal function*. Obviously, for all $z \in \mathbb{C}^N$ the estimate

$$\Phi_n(E,z) \le \exp(nV_E(z))$$

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holds and is known as the *Bernstein-Walsh-Siciak inequality*.

For an arbitrary fixed norm $\mathcal{N}$ on $\mathbb{C}^N$ and $E = \{z \in \mathbb{C}^N : \mathcal{N}(z) \le r\}$ we have the following formula (cf. [32])

$$\Phi(E,z) = \max\{1, \mathcal{N}(z)/r\}, \quad z \in \mathbb{C}^N. \tag{4}$$

We can also consider a modification of the Siciak extremal function with respect to $\mathcal{N}$

$$\varphi_{\mathcal{N}}(E,r) := \sup\{\Phi(E,z+w) : z \in E, \ \mathcal{N}(w) \le r\}, \quad r \ge 0.$$

In the case $\mathcal{N}(z) = ||z||_2$ we write simply $\varphi(E,r)$ and call it the *radial modification of Siciak extremal function* or, shortly, the *radial extremal function*. The function $\varphi_{\mathcal{N}}(E,r)$ was first studied in [4] where the following fact was proved

$$\frac{r}{\varphi(E,r)} \nearrow C(E) \quad \text{as } r \to \infty.$$

Various generalisations of Siciak extremal function have been studied over the past decades. The most interesting of them are related to certain norms considered on $\mathcal{P}(\mathbb{C}^N)$ instead of the supremum one in the definition of Φ_E. As an example, the Alexander capacity of $\mathbb{R}\mathbb{P}^n$ was computed in [14]. Moreover, since the Siciak extremal function is closely connected with approximation and estimates, it is worth noting that polynomial inequalities have long been investigated in various norms, e.g. [22], [25], [6], [7].

The idea of a radial modification of $\Phi(E,\cdot)$ permits us to consider some extremal functions and capacities connected with certain norms on $\mathbb{C}^N$ and on $\mathcal{P}(\mathbb{C}^N)$, and to find some relations between them. This leads us to new estimates of capacities that can become a useful tool for computing a new approximation of transfinite diameters or Chebyshev constants of some concrete compact sets (the problem of giving the exact value of capacities for concrete sets is usually very complicated). By means of specific norms, new bounds for classical one (Euclidean or sup norm) can be obtained, see e.g. [8, Cor. 2.10]. Moreover, approximation results concerning unusual norms may be useful in solving problems related to interpolation by complex Calderon method. As a step towards the eventual equivalence of HCP and the Markov inequality for polynomials (a longstanding open problem), we give some equivalent conditions to Hölder continuity property of V_E, see Thm. 4.3, Cor. 4.4).

In the second section we present basic properties of extremal functions and capacities related to given norms. We prove that the capacity $C(\mathcal{R})$ is not greater than the Chebyshev constant $t(\mathcal{R})$ for any norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$. Moreover, we also consider a transfinite diameter $\tau(\mathcal{R})$ and show that $C(\mathcal{R}) = t(\mathcal{R}) = \tau(\mathcal{R})$ for the maximum norm $\mathcal{R}(P) = ||P||_E$ on a set $E \subset \mathbb{C}^N$ that is the Cartesian product of N compact sets. This part of the paper is partially motivated by [19] where the authors studied the transfinite diameter, the Chebyshev constant, the Wiener energy and the links between them.

The third section is devoted to another extremal problem of multivariate approximation. Namely, we consider an extremal family of polynomials that can be used in an economisation procedure and telescoping approximation series. This will allow us to choose some elements from a sequence of approximation polynomials in order to get a faster approximation of functions.

The forth section deals with the relationship between Markov-type inequalities, Pleśniak property (see [27, Thm. 3.3.ii]), and the growth of the Green's function near a set E. This problem has been considered in a large number of papers published over the last few years by Andrievskii, Carleson, Ransford, Totik and others (see e.g. [1], [2], [3], [9], [17], [18], [21], [31], [34], [33]). The Markov-type inequalities come from the classical Andrei Markov inequality on $[-1,1]$ and are widely investigated owing to its connections with polynomial approximation, constructive theory of functions and some applications in numerical analysis (e.g. [27], [28], [29], [15], [30], [16]). On the other hand, a Vladimir Markov estimate on $[-1,1]$ has been generalised in another way than the A. Markov inequality, see [5]. The equivalence of HCP and a V. Markov-type inequality proved in [5], permits us to show some estimates of Chebyshev constants of sets with the Hölder continuity property of their pluricomplex Green's function, see Proposition 4.1. Moreover, we prove that HCP and the Pleśniak property are equivalent, see Corollary 4.4.

For the reader's convenience, we list below the notation that we will use later on. Let $\mathcal{N}$, $\mathcal{R}$ be fixed norms on $\mathbb{C}^N$ and $\mathcal{P}(\mathbb{C}^N)$, respectively. We are interested in the following terms:

- radial extremal functions
  $\varphi_{\mathcal{N}}(\mathcal{R},r)$, see Def. 2.1,
  $\varphi_{\mathcal{N}}(E,r) := \varphi_{\mathcal{N}}(\mathcal{R},r)$ for $\mathcal{R}(P) = ||P||_E$,
  $\varphi(E,r) := \varphi_{\mathcal{N}}(\mathcal{R},r)$ for $\mathcal{R}(P) = ||P||_E$, $\mathcal{N}(z) = ||z||_2$,
- capacities
  $C_{\mathcal{N}}(\mathcal{R})$, see Def. 2.1,
  $C(E) := C_{\mathcal{N}}(\mathcal{R})$ for $\mathcal{R}(P) = ||P||_E$, $\mathcal{N}(z) = ||z||_2$,
  $C_\infty(\mathcal{R}) := C_{\mathcal{N}}(\mathcal{R})$ for $\mathcal{N}(z) = ||z||_\infty := \max\{|z_j| : 1 \le j \le N\}$,
  $C_\infty(E) := C_{\mathcal{N}}(\mathcal{R})$ for $\mathcal{R}(P) = ||P||_E$, $\mathcal{N}(z) = ||z||_\infty$,
- Chebyshev constants
  $t(\mathcal{R})$, see (6),

$t(E) := t(\mathcal{R})$ for $\mathcal{R}(P) = \|P\|_E$,
$t^*(\mathcal{R})$, see (9),
$t^*(E) := t^*(\mathcal{R})$ for $\mathcal{R}(P) = \|P\|_E$,

- transfinite diameters
  $\tau(\mathcal{R})$, see (8),
  $\tau(E) := \tau(\mathcal{R})$ for $\mathcal{R}(P) = \|P\|_E$.

Throughout the paper, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} z_1 \ldots \partial^{\alpha_N} z_N}$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \ldots \alpha_N!$, $\beta^\alpha = \beta_1^{\alpha_1} \ldots \beta_N^{\alpha_N}$ for $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\beta = (\beta_1, \ldots, \beta_N)$. The notation $\alpha \leq \beta$ means that $\alpha_1 \leq \beta_1, \ldots, \alpha_N \leq \beta_N$. Let us note that $D^{\alpha(j)} z^{\alpha(j)} = \alpha(j)!$ and $D^{\alpha(j)} z^{\alpha(l)} = 0$ for $l < j$.

2 Capacities and Chebyshev constants.

In this section we consider the *radial extremal functions* and *capacities* defined as follows.

Definition 2.1. For a norm $\mathcal{N}$ on $\mathbb{C}^N$ and a norm (or a seminorm) $\mathcal{R}$ on the space $\mathcal{P}(\mathbb{C}^N)$ set

$$\varphi_{\mathcal{N}}(\mathcal{R}, r) := \sup\left\{\mathcal{R}(P(x+z))^{1/\deg P} : \deg P \geq 1,\ \mathcal{R}(P) \leq 1,\ \mathcal{N}(z) \leq r\right\},\ r > 0,$$

$$C_{\mathcal{N}}(\mathcal{R}) := \liminf_{r\to\infty} \frac{r}{\varphi_{\mathcal{N}}(\mathcal{R}, r)},$$

where $P(x+z)$ denotes the polynomial $x \mapsto P(x+z)$ with a fixed z.

Proposition 2.1. *If $\mathcal{N}$ is a norm on $\mathbb{C}^N$ and $\mathcal{R}$ is a norm on $\mathcal{P}(\mathbb{C}^N)$ then*

$$(0, +\infty) \ni r \longmapsto \frac{\varphi_{\mathcal{N}}(\mathcal{R}, r)}{r} \tag{5}$$

is a nonincreasing function.

Proof. Let $\mathcal{R}_n^*$ be the norm on the dual space $\left(\mathcal{P}_n(\mathbb{C}^N), \mathcal{R}\right)^*$. It is well known that

$$\mathcal{R}(P(x+\zeta)) = \sup\{|\Lambda(P(x+\zeta))| : \Lambda \in (\mathcal{P}_n(\mathbb{C}^N))^*,\ \mathcal{R}_n^*(\Lambda) = 1\},$$

where $n = \deg P \geq 1$. Now fix n and a polynomial P such that $\deg P = n$ and $\mathcal{R}(P) = 1$. For a (complex) linear functional Λ, $\mathcal{R}_n^*(\Lambda) = 1$, set $Q(\zeta) = \Lambda(P(x+\zeta)) \in \mathcal{P}_n(\mathbb{C}^N)$. Let $\zeta \in \mathbb{C}^N$, $\mathcal{N}(\zeta) = r_1 \geq r$. From (3) and (4) we have

$$|Q(\zeta)| \leq \left(\frac{r_1}{r}\right)^n \sup\{|Q(\eta)| : \mathcal{N}(\eta) \leq r\},$$

whereby

$$\frac{\sup\{|\Lambda(P(x+\zeta))|^{1/n} : \mathcal{N}(z) \leq r_1\}}{r_1} \leq \frac{\sup\{|\Lambda(P(x+\zeta))|^{1/n} : \mathcal{N}(z) \leq r\}}{r}.$$

Taking the supremum, first with respect to Λ, next over P, and finally over n, we get the expected result. □

Corollary 2.2. *The function $\varphi_{\mathcal{N}}(\mathcal{R}, r)$ has the following properties*

(a) $\frac{r}{\varphi_{\mathcal{N}}(\mathcal{R}, r)} \nearrow C_{\mathcal{N}}(\mathcal{R})$ *as* $r \longrightarrow \infty$ *and thus* $\log \varphi_{\mathcal{N}}(\mathcal{R}, r) - \log r \searrow -\log C_{\mathcal{N}}(\mathcal{R})$.

(b) The function $\mathbb{R} \ni t \longmapsto \log \varphi_{\mathcal{N}}(\mathcal{R}, e^t)$ *is convex nondecreasing, while the function* $\mathbb{R} \ni t \longmapsto \log \varphi_{\mathcal{N}}(\mathcal{R}, e^t) - t$ *is convex nonincreasing.*

Now, following [10], we introduce the notion of *Chebyshev constant $t(\mathcal{R})$ associated to a given norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$.*

Let $z^{\alpha(1)}, z^{\alpha(2)}, \ldots$ be all monomials in $\mathcal{P}(\mathbb{C}^N)$ ordered so that $|\alpha(j)| \leq |\alpha(k)|$ if $j \leq k$, and the monomials of a fixed degree are ordered lexicographically (where $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$). Set

$$M_j = M_j(\mathcal{R}) := \inf\left\{\mathcal{R}(P) : P(z) = z^{\alpha(j)} + \sum_{l=1}^{j-1} c_l z^{\alpha(l)},\ c_l \in \mathbb{C}\right\},$$

$$\tau_j = \tau_j(\mathcal{R}) := M_j^{1/|\alpha(j)|}, \quad t(\mathcal{R}) := \liminf_{j\to\infty} \tau_j(\mathcal{R}). \tag{6}$$

In the case of $\mathcal{R}(P) = \|P\|_E$ for a compact set $E \subset \mathbb{C}^N$, we simply write $M_j(E)$, $\tau_j(E)$ and $t(E)$. It is worth noticing that for every $\mathcal{R}$ and j there exists a polynomial $\mathcal{T}_j(z) = z^{\alpha(j)} + \sum_{l=1}^{j-1} c_l z^{\alpha(l)}$ such that $M_j(\mathcal{R}) = \mathcal{R}(\mathcal{T}_j)$. Usually, a polynomial $\mathcal{T}_j$ is not unique, i.e. the set

$$\Delta_j(\mathcal{R}) := \left\{\mathcal{T}_j(z) = z^{\alpha(j)} + \sum_{l=1}^{j-1} c_l z^{\alpha(l)} : c_l \in \mathbb{C},\ M_j(\mathcal{R}) = \mathcal{R}(\mathcal{T}_j)\right\}$$

has more than one element. As an example, consider $S = \{(x,y) \in \mathbb{R}^2 : \; x,y \geq 0, x+y \leq 1\}$ and $\mathcal{R}(P) = ||P||_S$. Then $\mathcal{T}_1(x,y) = x - \frac{1}{2}$, while $\Delta_2(S) = \{x + ay - \frac{1}{2},\; a \in [0,1]\}$. In the general case, the set $\Delta_j(\mathcal{R})$ is a convex, compact subset of $\mathcal{P}_{|\alpha_j|}(\mathbb{C}^N)$.

Proposition 2.3. *For any compact set $E \subset \mathbb{C}^N$*

$$t(E) = \inf\{\tau_j(E) : \; j \in \mathbb{N}\}.$$

Proof. Obviously $\inf\{\tau_j(E) : \; j \in \mathbb{N}\} \leq t(E)$. The opposite inequality is a consequence of the following observation: for each $j \in \mathbb{N}$ there exists $\omega(j) \in \mathbb{N}$ such that $M_j(E)^2 \geq M_{\omega(j)}(E)$ and $\deg \mathcal{T}_{\omega(j)} = 2 \deg \mathcal{T}_j$. □

Remark 1. If $N = 1$ then $\inf\{\tau_j(E) : \; j \in \mathbb{N}\} = \lim_{j\to\infty} \tau_j(E)$. This is not true for $N > 1$ and, in the general case, we cannot replace the lim inf by the limit. However, the following useful fact holds: *if $\mathcal{R}_1, \mathcal{R}_2$ are two norms on $\mathcal{P}(\mathbb{C}^N)$ such that*

$$C_1(\deg P)^\alpha \mathcal{R}_1(P) \leq \mathcal{R}_2(P) \leq C_2(\deg P)^\beta \mathcal{R}_1(P) \tag{7}$$

with C_1, C_2, α, β independent of $P \in \mathcal{P}(\mathbb{C}^N)$ then $t(\mathcal{R}_1) = t(\mathcal{R}_2)$.

We will compare the Chebyshev constant $t(E)$ defined above for $\mathcal{R}(P) = ||P||_E$ with the capacity

$$C_\infty(E) := \liminf_{z\to\infty} \frac{||z||_\infty}{\exp V_E(z)},$$

where $||z||_\infty = \max_{1\leq j\leq N} |z_j|$.

Proposition 2.4. *For any compact set $E \subset \mathbb{C}^N$*

$$C_\infty(E) \; \leq \; t(E).$$

To prove this inequality, we need the following fact that is an easy consequence of Taylor's theorem and Cauchy's integral formula for the polydisc $\mathbb{T}^N := \{||z||_\infty \leq 1\}$.

Lemma 2.5. *If $P(z) = \sum_{|\alpha|\leq n} a_\alpha z^\alpha \in \mathcal{P}(\mathbb{C}^N)$ and $||P||_{\mathbb{T}^N} < 1$, then $|a_\alpha| < 1$ for all α.*

Proof of Proposition 2.4. Fix an arbitrary $j \in \mathbb{N}$. We will use a Chebyshev polynomial $\mathcal{T}_j$ from $\Delta_j(E)$. By the Zaharjuta-Siciak theorem, we have

$$C_\infty(E) = \liminf_{z\to\infty} \frac{||z||_\infty}{\Phi_E(z)} \leq \liminf_{z\to\infty} ||z||_\infty \left(\frac{||\mathcal{T}_j||_E}{|\mathcal{T}_j(z)|}\right)^{1/|\alpha(j)|}$$
$$= \tau_j \liminf_{z\to\infty} \frac{||z||_\infty}{|\mathcal{T}_j(z)|^{1/|\alpha(j)|}}.$$

Consider the *homogenous part* of $\mathcal{T}_j$ denoted by $\widehat{\mathcal{T}_j}$. Since the coefficient of $z^{\alpha(j)}$ is equal to 1 in the polynomial $\widehat{\mathcal{T}_j}$, by means of Lemma 2.5, we can choose $w \in \mathbb{T}^N$ such that $|\widehat{\mathcal{T}_j}(w)| \geq 1$. Consequently,

$$\liminf_{z\to\infty} \frac{||z||_\infty}{|\mathcal{T}_j(z)|^{1/|\alpha(j)|}} \leq \lim_{n\to\infty} \frac{||nw||_\infty}{|\mathcal{T}_j(nw)|^{1/|\alpha(j)|}}$$
$$= \lim_{n\to\infty} \frac{n}{\left|\sum_{|\alpha(\ell)|=|\alpha(j)|} c_\ell n^{|\alpha(j)|} w^{\alpha(\ell)} + \sum_{|\alpha(\ell)|<|\alpha(j)|} c_\ell n^{|\alpha(\ell)|} w^{\alpha(\ell)}\right|^{1/|\alpha(j)|}}$$
$$= \frac{1}{|\sum_{|\alpha(\ell)|=|\alpha(j)|} c_\ell w^{\alpha(\ell)}|^{1/|\alpha(j)|}} = \frac{1}{|\widehat{\mathcal{T}_j}(w)|^{1/|\alpha(j)|}} \leq 1$$

and the proof is complete. □

Corollary 2.6. *Let $E = E_1 \times ... \times E_N \subset \mathbb{C}^N$ and $E_1, ..., E_N$ be compact sets in $\mathbb{C}$. Then*

$$C_\infty(E) = t(E) = \min\{c(E_1), ..., c(E_N)\},$$

where $c(F)$ is the logarithmic capacity of a compact set $F \subset \mathbb{C}$.

Proof. We use the formula proved in [4, Cor.2.8] concerning product property for the capacity $C_\infty(E)$, i.e.

$$C_\infty(E_1 \times ... \times E_N) = \min\{c(E_1), ..., c(E_N)\}.$$

By the definition of the Chebyshev constant $t(E)$, we have

$$t(E_1 \times ... \times E_N) \leq \min\{t(E_1), ..., t(E_N)\} = \min\{c(E_1), ..., c(E_N)\}$$

the last equality being a consequence of the fact that in the one-dimensional case the logarithmic capacity and the Chebyshev constant are equal. We apply now Proposition 2.4 to get the desired statement. □

Proposition 2.4 can also be formulated in a more general situation. Consider a norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$ and $\mathcal{N}(z) = ||z||_\infty$. In such a case, we write

$$C_\infty(\mathcal{R}) = C_{\mathcal{N}}(\mathcal{R}).$$

Theorem 2.7. *For any norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$*

$$C_\infty(\mathcal{R}) \le t(\mathcal{R}).$$

Proof. Fix $j \in \mathbb{N}$ and take $\mathcal{T}_j \in \Delta_j(\mathcal{R})$. We have

$$C_\infty(\mathcal{R}) = \lim_{z\to\infty} \frac{||z||_\infty}{\varphi_{\mathcal{N}}(\mathcal{R}, ||z||_\infty)} \le \liminf_{z\to\infty} ||z||_\infty \left(\frac{\mathcal{R}(\mathcal{T}_j)}{\mathcal{R}(\mathcal{T}_j(w+z))} \right)^{1/|\alpha(j)|}$$

$$= \tau_j \liminf_{z\to\infty} \frac{||z||_\infty}{\mathcal{R}(\mathcal{T}_j(w+z))^{1/|\alpha(j)|}} \le \tau_j \liminf_{z\to\infty} \frac{||z||_\infty}{|\Lambda(\mathcal{T}_j(w+z))|^{1/|\alpha(j)|}},$$

for any functional $\Lambda \in (\mathcal{P}_{|\alpha_j|}(\mathbb{C}^N))^*$ with $||\Lambda|| = 1$. Take Λ such that $|\Lambda(1)| = \mathcal{R}(1)$. Let z_0 be a vector from $\mathbb{T}^N$ such that $|\widehat{\mathcal{T}_j}(z_0)| \ge 1$. Applying an argument similar to the proof of Proposition 2.4, and picking $z = nz_0$, we get

$$\liminf_{z\to\infty} \frac{||z||_\infty}{|\Lambda(\mathcal{T}_j(w+z))|^{1/|\alpha(j)|}} \le \frac{1}{\left(|\Lambda(1)||\widehat{\mathcal{T}_j}(z_0)|\right)^{1/|\alpha(j)|}} \le \frac{1}{\mathcal{R}(1)^{1/|\alpha_j|}}$$

and letting j tend to infinity, finishes the proof. □

Conjecture 1. For any norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$

$$C_\infty(\mathcal{R}) = t(\mathcal{R}).$$

In particular, $C_\infty(E) = t(E)$ for any compact set $E \subset \mathbb{C}^N$.

By means of the sequence of constants $\tau_j(\mathcal{R})$ we can define $\tau(\mathcal{R})$ the *transfinite diameter of* $\mathcal{R}$, following Zaharjuta, see [35]. Let

$$\Sigma_N := \left\{ \theta \in \mathbb{R}^N : \sum_{k=1}^N \theta_k = 1,\ \theta_k \ge 0 \right\}, \qquad \Sigma_N^0 := \{\theta \in \Sigma_N : \theta_k > 0 \text{ for all } k\},$$

$$\tau(\mathcal{R}, \theta) := \limsup_{j\to\infty,\ \frac{\alpha(j)}{|\alpha(j)|}\to\theta} \tau_j(\mathcal{R}),$$

$$\tau(\mathcal{R}) := \exp\left[\frac{1}{\sigma(\Sigma_N)} \int_{\Sigma_N} \log \tau(\mathcal{R}, \theta) d\sigma(\theta) \right], \tag{8}$$

where the limit $\tau(\mathcal{R}, \theta)$ exists for $\theta \in \Sigma_N^0$ and σ is the Lebesgue surface measure on the hyperplane $\{\theta \in \mathbb{R}^N : \sum_{k=1}^N \theta_k = 1\}$. One can easily check that $\sigma(\Sigma_N) = \sqrt{N}/(N-1)!$.

By the above definition and Theorem 2.7, we get inequalities

$$C_\infty(\mathcal{R}) \le t(\mathcal{R}) \le \tau(\mathcal{R}).$$

We close this section with the definition of other constants related to the supremum norm on E or to another norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$. Let

$$t^*(E) := \liminf_{j\to\infty} \tau_{\nu(j)}(E) = \inf\{\tau_{\nu(j)}(E) : j \in \mathbb{N}\},$$

where $\nu : \mathbb{N} \to \mathbb{N}_0$ is such a sequence that $\alpha(\nu(j)) \in \bigcup_{k=1}^N \mathbb{N}e_k$. More generally,

$$t^*(\mathcal{R}) := \liminf_{j\to\infty} \tau_{\nu(j)}(\mathcal{R}). \tag{9}$$

Obviously, $t^*(\mathcal{R})$ is easier to compute than $t(\mathcal{R})$ and therefore, we are interested in a relationships between them. We have the following

Proposition 2.8. *For an arbitrary norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$ the constant $t^*(\mathcal{R})$ has the following properties*

(a) $C_\infty(\mathcal{R}) \le t^*(\mathcal{R})$,

(b) $t(\mathcal{R}) \le t^*(\mathcal{R})$ *and, in general, we do not have the equality,*

(c) if $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are comparable in the sense of (7) then* $t^*(\mathcal{R}_1) = t^*(\mathcal{R}_2)$,

(d) if $E_1, ..., E_N$ *are compact subsets of* $\mathbb{C}$ *and* $E = E_1 \times \cdots \times E_N$ *then*

$$C(E) = C_\infty(E) = t(E) = t^*(E) = \min\{c(E_1), \ldots, c(E_N)\}.$$

Remark 2. Observe that for a spectral norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$ we get $\mathcal{R}(P^k) = \mathcal{R}(P)^k$ for all polynomials P, $k \in \mathbb{N}$, see [7], and hence, $t(\mathcal{R}) = t^*(\mathcal{R})$. If $\mathcal{R}$ is not a spectral norm, computing the exact value of $t(\mathcal{R})$ is a difficult problem. In some cases, condition (c) in Proposition 2.8 can be useful. If we take $\mathcal{R}_1$ equal to the sup norm on E and an $L^q(\mu)$ norm with $q \geq 1$ and a measure μ satisfying the Bernstein-Markow property (see e.g. [13]), as $\mathcal{R}_2$, then $t^*(\mathcal{R}_1) = t^*(\mathcal{R}_2)$. For example, $t^*(\mathcal{R}) = \frac{1}{2}$ for the $L^q([-1,1])$ norm $\mathcal{R}$ with $q \geq 1$ and the Lebesgue measure on $[-1,1]$. Consequently, $t(\mathcal{R}) = \frac{1}{2}$ but Chebyshev polynomials in this case seems to be not known, apart from the case of $q = 2$ and $q = \infty$.

Example 2.1. For the set $E = \mathbb{B}_2 = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 1\}$ and the norms

$$\mathcal{R}_1(P) = ||P||_E, \qquad \mathcal{R}_2(P) = \left(\frac{1}{\pi}\int_E |P(x,y)|^2 dxdy\right)^{1/2},$$

we have $t^*(\mathcal{R}_1) = t^*(\mathcal{R}_2)$.

To generalise the above example and to formulate a conjecture, recall that a norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$ has the *generalised Nikolski property* if for an arbitrary polynomial $P \in \mathcal{P}(\mathbb{C})^N$ there exists the limit

$$\lim_{j\to\infty} \mathcal{R}(P^j)^{1/j} =: \mathcal{R}_\infty(P),$$

and $\mathcal{R}_\infty$ and $\mathcal{R}$ are comparable in the sense of (7).

Observe that for

$$E = \mathbb{B}_N = \{x \in \mathbb{R}^N : \ x_1^2 + \cdots + x_N^2 \leq 1\}, \qquad \mathcal{R}(P) = \left(\frac{1}{\mathrm{vol}(E)}\int_E |P(x)|^2 dx\right)^{1/2},$$

and $\mathcal{R}_\infty(P) = ||P||_E$ we have $t^*(\mathcal{R}) = t^*(\mathcal{R}_\infty) = \frac{1}{2} = C_\infty(\mathcal{R}_\infty)$ (the details will be explained in a next paper).

Conjecture 2. For any norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$ with generalised Nikolski property the following equality holds

$$t^*(\mathcal{R}) = C_\infty(\mathcal{R}_\infty).$$

In particular, $t^*(E) = C_\infty(E)$ for any compact set $E \subset \mathbb{C}^N$.

Finally, we recall a notion of minimal polynomials $\mathbb{P}(\alpha)$ and define a kind of Chebyshev constant related to them. Following Bloom and Calvi [12], for a multindex α of length d, let

$$\mathbb{P}(\alpha) := x^\alpha + \mathcal{P}_{d-1}(\mathbb{C}^N),$$

$$\mathbb{T}(\alpha,\mathcal{R}) := \inf\{\mathcal{R}(P) : \ P \in \mathbb{P}(\alpha)\}, \ \mathbb{T}(\mathcal{R}) := \inf\left\{\mathbb{T}(\alpha,\mathcal{R})^{1/|\alpha|} : \ \alpha > 0\right\}$$

(we write $\mathbb{T}(\alpha, E)$ and $\mathbb{T}(E)$ in the case $\mathcal{R}(P) = ||P||_E$). By means of known results (cf. [12] and [11]), we can calculate

$E \subset \mathbb{R}^2$	$\mathbb{T}(E)$	$C_\infty(E)$	$\tau(E)$
$K_2 = [-1,1] \times [-1,1]$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$B_2 = \{x : \ x_1^2 + x_2^2 \leq 1\}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{\sqrt{2e}}$
$S_2 = \{x : \ x_1, x_2 \geq 0, \ x_1 + x_2 \leq 1\}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2e}$

We can easily get the following property.

Proposition 2.9. *If*

$$\mathcal{D}(\mathcal{R}; n, \alpha) := \sup\left\{(\mathcal{R}(D^\alpha P)/\alpha!)^{1/|\alpha|} : \ \deg P = n, \ \mathcal{R}(P) = 1\right\}, \ 0 < |\alpha| \leq n.$$

then

$$\mathcal{D}(\mathcal{R}; n, \alpha) \geq (1/\mathbb{T}(\mathcal{R}, \alpha))^{1/|\alpha|}, \ |\alpha| = n,$$

$$\sup\{\mathcal{D}(\mathcal{R}; n, \alpha) : \ |\alpha| = n, n \geq 1\} \geq 1/\mathbb{T}(\mathcal{R}).$$

3 Telescoping series and economisation procedure.

Consider an arbitrary norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$, and a Banach space $\mathcal{X}_{\mathcal{R}}$ generated by $\mathcal{R}$ as the completion of the space of polynomials. Usually, it is rather difficult to describe the whole $\mathcal{X}_{\mathcal{R}}$.

Let $f \in \mathcal{X}_{\mathcal{R}}$ be the limit of a Cauchy sequence with respect to $\mathcal{R}$ of polynomials p_n where $\deg p_n = n$. Since we have only one norm $\mathcal{R}$ in this section, we denote it by $\|\cdot\|$. We get a *telescoping series*

$$||f|| = \lim_{n\to\infty} ||p_n||, \quad ||f - p_n|| = \lim_{m\to\infty} ||p_m - p_n||,$$

$$f = p_0 + \sum_{n=0}^{\infty}(p_{n+1} - p_n).$$

Now, we describe an *economisation procedure* of the telescoping series. For a fixed n we want to find a polynomial $p_{n,m-n}$ of degree n, starting with a polynomial p_m, $m > n$, such that $||f - p_{n,m-n}|| < ||f - p_n||$ and $p_{n,m-n}$ is obtained in the following way.

Let $\mathcal{T} = (P_\alpha)_{\alpha\in\mathbb{N}_0^N}$ be a family of polynomials satisfying two conditions:

- $P_\alpha \in \mathbb{P}(\alpha)$,
- $\limsup_{|\alpha|\to\infty} ||P_\alpha||_{|\alpha|}^{1/|\alpha|} < 1$.

Due to the last condition, we get an appropriate estimate of an error when we replace x^α by a monomial of lower degree, see below. This estimate is important to control an error, e.g. of numerical approximation of function f.

We can write p_m as a sum $p_m = \sum_{j=0}^{m} \mathrm{Hom}_j(p_m)$ where $\mathrm{Hom}_j(p_m)$ is the homogeneous part of p_m of degree j or $\mathrm{Hom}_j(p_m)$ is equal 0. Then $p_{m,1}$ is a polynomial of degree $m-1$, a modification of p_m: we replace $\mathrm{Hom}_m(p_m) = \sum_{|\alpha|=m} c_\alpha x^\alpha$ by $\sum_{|\alpha|=m} c_\alpha(x^\alpha - P_\alpha)$ and let

$$p_{m,1} = \sum_{j=0}^{m-1} \mathrm{Hom}_j(p_m) + \sum_{|\alpha|=m} c_\alpha(x^\alpha - P_\alpha).$$

Next we repeat this procedure (if $\mathrm{Hom}_{m-1}(p_{m,1}) = 0$ we modify the first nonzero homogeneous part of $p_{m,1}$) and we obtain a polynomial $p_{m,m-n}$ of degree n. We expect that $||f - p_{m,m-n}|| < ||f - p_n||$ and $p_{m,m-n}$ is near a best approximant to f.

The economisation procedure related to $\mathcal{T} = (P_\alpha)_{\alpha\in\mathbb{N}_0^N}$ can be expressed by the family of projections $\mathcal{T}_n$ on $\mathcal{P}(\mathbb{C}^N)$ defined by

$$\mathcal{T}_n(P_\alpha) := \begin{cases} P_\alpha & \text{if } |\alpha| \le n \\ 0 & \text{if } |\alpha| > n \end{cases}$$

for $n \ge 1$. It is easy to see that $p_{m,m-n} = \mathcal{T}_n(p_m)$.

In practice, a first problem is to find a "good" family $(P_\alpha)_{\alpha\in\mathbb{N}_0^N}$. Especially interesting is a problem of finding a proper family of orthogonal polynomials. *Favards type theorems* seem to be a helpful tool. The next task is to give a description of the operator $L_n(f) = \lim_{m\to\infty} p_{m,m-n}$ and to estimate $||f - L_n(f)||$ to obtain "almost best approximant".

Example 3.1. Let $\mathcal{T} = (\widehat{T}_n)_{n\ge 0}$ be the family of monic Chebyshev polynomials of the first kind, i.e. $\widehat{T}_n := \frac{1}{2^{n-1}} T_n$, $n \ge 1$, $\widehat{T}_0 := T_0 = 1$.

Since $(\widehat{T}_n)_{n\ge 0}$ forms an orthogonal system on $[-1,1]$ with weight $\frac{1}{\pi\sqrt{1-x^2}}$, we define $\mathcal{F}_k$ by

$$\mathcal{F}_k(f)(x) = \sum_{j=0}^{k} \widehat{f}_j \widehat{T}_j(x),$$

where

$$\widehat{f}_j = \begin{cases} \frac{1}{\pi}\int_{-1}^{1} f(x)\frac{dx}{\sqrt{1-x^2}}, \ j = 0, \\ 2^{2j-1}\frac{1}{\pi}\int_{-1}^{1} f(x)\widehat{T}_j(x)\frac{dx}{\sqrt{1-x^2}}, \ 1 \le j \le k. \end{cases}$$

It follows, by the orthogonality of monic Chebyshev polynomials, that $\mathcal{F}_k(\widehat{T}_j(x)) = 0$ for $j > k$ and $\mathcal{F}_k(\widehat{T}_j(x)) = \widehat{T}_j(x)$ for $1 \le j \le k$. Thus $\mathcal{T}_k = \mathcal{F}_k$ on $\mathcal{P}(\mathbb{R})$, $k \in \mathbb{N}$.

Let f be a continuous function on $[-1,1]$ and $p_n \rightrightarrows f$ then

$$\lim_{n\to\infty} \mathcal{T}_k(p_n)(x) = \mathcal{F}_k(f).$$

A sense of the above equality is the following: we can approximate f by $\mathcal{F}_k(f)$ and if k is fixed then we can approximate $\mathcal{F}_k(f)$ by $\mathcal{T}_k(p_n)$ if we properly choose n. Very often n in not so big, especially if f is a real analytic function. Thus the economisation procedure is so good as good is the orthogonal projection related to the considered family of polynomials. The approximation by a telescoping series can be quite easily applied by programs to symbolic computations like Maxima (even by means of Wolfram Alpha we get a good result).

As a concrete example, consider $f(x)=\cos(x)=\lim_{n\to\infty}\sum_{j=0}^{n}(-1)^n\frac{x^{2n}}{(2n)!}$. Then

$$\mathcal{T}_2(p_3)=\mathcal{T}_2(1-x^2/2+x^4/24-x^6/720)=\frac{4585}{4608}-\frac{2118}{4608}x^2$$
$$\approx 0.9950086-0.4596354x^2,$$
$$\mathcal{F}_2(\cos)(x)=J_0(1)+2J_2(1)-4J_2(1)x^2\approx 0.995005-0.459614x^2.$$

Here J_0 and J_2 are Bessel's J functions. We also have

$$\max\{|\cos(x)-\mathcal{T}_2(p_3)(x)|:\ x\in[-1,1]\}\approx 0.0049913,$$
$$\max\{|\cos(x)-\mathcal{F}_2(\cos)(x)|:\ x\in[-1,1]\}\approx 0.0049949,$$
$$E_2(\cos)=\max\{|\cos(x)-P_2(\cos,[-1,1])(x)|:\ x\in[-1,1]\}\approx 0.0049536,$$

where $P_2(\cos,[-1,1])(x)$ is the polynomial of the best approximation.

The economisation procedure (also known as *telescoping*) related to the Chebyshev polynomials of the first kind was originally introduced by C. Lanchos (cf. [24], p. 457–463 and [20] p. 70-74) and is known to be a remarkable optimisation in some applications.

Consider now the family of monic Chebyshev polynomials of the second kind $\mathcal{T}=(\widehat{U}_n)_{n\geq 0}$ orthogonal in $[-1,1]$ with the weight $\sqrt{1-x^2}$ and $\mathcal{T}_2$, $\mathcal{F}_2$ (with respect to $\mathcal{T}$) for the same function as above, i.e. $\cos(x)$. We have

$$\mathcal{T}_2(p_3)(x)=\frac{11491}{11520}-\frac{601}{1280}x^2\approx 0.9974826-0.4695312x^2,$$
$$\mathcal{F}_2(\cos)(x)=2J_1(1)+6J_3(1)-24J_3(1)x^2\approx 0.997481-0.469520x^2,$$

where $J_1(1), J_3(1)$ are values of Bessel J function. The polynomial of degree 2 with the minimum property in least-first-power approximation is given by $P_2(\cos)(x)=\alpha+\beta x^2$ where

$$\alpha=\left(\frac{1}{2}-\frac{3}{10}\sqrt{5}\right)\cos\left(\frac{\sqrt{5}+1}{4}\right)+\left(\frac{1}{2}+\frac{3}{10}\sqrt{5}\right)\cos\left(\frac{\sqrt{5}-1}{4}\right)$$
$$\approx 0.99746017,$$
$$\beta=\frac{4}{\sqrt{5}}\left(\cos\left(\frac{\sqrt{5}+1}{4}\right)-\cos\left(\frac{\sqrt{5}-1}{4}\right)\right)\approx -0.4694364$$

We can calculate

$$\frac{1}{2}\int_{-1}^{1}|\cos(x)-\mathcal{T}_2(p_3)(x)|dx\approx 0.00249763,$$
$$\frac{1}{2}\int_{-1}^{1}|\cos(x)-\mathcal{P}_2(\cos)(x)|dx\approx 0.00249762,$$
$$\frac{1}{2}\int_{-1}^{1}|\cos(x)-P_2(\cos)(x)|dx\approx 0.00249758.$$

Remark 3. In the above example polynomials $P_n=\widehat{T}_n$ or $P_n=\widehat{U}_n$ can be obtained in the following manner. Define $P_0=1$, $P_1(x)=x-\beta_1$ with $||x-\beta_1||=\inf\{||x-\beta||:\ \beta\in\mathbb{R}\}$, and

$$P_{n+1}(x)=xP_n(x)-\beta_{n+1}P_{n-1}(x)$$

with

$$||xP_n(x)-\beta_{n+1}P_{n-1}(x)||=\inf\{||xP_n(x)-\beta P_{n-1}(x)||:\ \beta\in\mathbb{R}\}.$$

We are interested only in the case where $\limsup_{n\to\infty}||P_n||^{1/n}<1$. The last property is a kind of capacity condition.

4 Hölder continuity of Green's function and Pleśniak property

We start with definitions of main notions of this section: the Hölder continuity property of the pluricomplex Green function (see e.g. [27]) and two Markov inequalities. A Pleśniak property will be defined after a brief introduction, see Def. 4.4.

Definition 4.1. The set $E \subset \mathbb{C}^N$ admits the *Hölder continuity property* with exponent $\alpha \in (0,1]$ (we write $HCP(\alpha)$ for short) if

$$|V_E(w) - V_E(z)| \leq A\|w - z\|_2^{\alpha} \tag{10}$$

with a positive constant A independent of $w, z \in \mathbb{C}^N$.

Definition 4.2. The set $E \subset \mathbb{C}^N$ admits the *A. Markov inequality* with exponent $m \geq 1$ (we write $AMI(m)$ for short) if there exists a constant $M > 0$ such that for every polynomial $P \in \mathcal{P}(\mathbb{C}^N)$

$$\|D_j P\|_E \leq M(\deg P)^m \|P\|_E, \ j = 1, \ldots, N, \tag{11}$$

where $\deg P$ is the total degree of the polynomial P. If E admits inequality (11) with some constants m, M then it is said to be a *Markov set*. Considering monomials, can be easily showed that necessarily $m \geq 1$.

It may be interesting to the reader that, in the case $E \subset \mathbb{R}^N$ (this case is the most important from the point of view of applications), Markov property with exponent m is equivalent to the bound of the Laplace operator Δ

$$||\Delta P||_E \leq M_1 (\deg P)^{2m} ||P||_E,$$

see [8]. Moreover, in the general case, inequality (11) is equivalent to the existence of constants M', m' such that for all polynomials P

$$\left\| \frac{\partial^N P}{\partial z_1 \ldots \partial z_N} \right\|_E \leq M' (\deg P)^{m'} ||P||_E.$$

One can easily prove by means of Zakharyuta-Siciak theorem (3) and Cauchy's integral formula, that the A. Markov inequality is a consequence of the Hölder continuity property

$$HCP(\tfrac{1}{m}) \implies AMI(m).$$

Observe that the A. Markov inequality (11) is equivalent to the following property

$$\|D^{\alpha} P\|_E \leq M_0^{|\alpha|} (\deg P)^{m|\alpha|} \|P\|_E \tag{12}$$

with a constant $M_0 > 0$ independent of $P \in \mathcal{P}(\mathbb{C}^N)$, $\alpha \in \mathbb{N}_0^N$ and with the same exponent $m \geq 1$ as in (11).

Definition 4.3. (see [5]) A compact set $E \subset \mathbb{C}^N$ admits the *V. Markov inequality* with exponents $m, k \geq 1$ ($VMI(m,k)$ in short) if for every $\alpha \in \mathbb{N}_0^N$, $P \in \mathcal{P}(\mathbb{C}^N)$

$$\|D^{\alpha} P\|_E \leq B^{|\alpha|} \frac{(\deg P)^{m|\alpha|}}{|\alpha|!^{k-1}} \|P\|_E \tag{13}$$

with a constant $B \geq 1$ independent of α and P.

Since $\alpha! \leq |\alpha|! \leq N^{|\alpha|}\alpha!$, the above condition is equivalent to the existence of a constant A such that

$$\|D^{\alpha} P\|_E \leq A^{|\alpha|} \frac{(\deg P)^{m|\alpha|}}{\alpha!^{k-1}} \|P\|_E. \tag{14}$$

Although m and k seems to be independent in the above definition, the exponent k should be less than or equal to m by the following argument. Applying condition (14) to polynomials $\mathcal{T}_j$ such that $||\mathcal{T}_j||_E = M_j(E)$ we get the inequality

$$M_j(E) \geq \left(\frac{1}{A}\right)^{|\alpha(j)|} \frac{(\alpha(j)!)^k}{|\alpha(j)|^{|\alpha(j)|m}} \geq \left(\frac{1}{NA}\right)^{|\alpha(j)|} \frac{(|\alpha(j)|!)^k}{|\alpha(j)|^{|\alpha(j)|m}}, \tag{15}$$

and hence

$$\tau_j(E) \geq \frac{1}{NAe^k} |\alpha(j)|^{k-m} \tag{16}$$

since $n! > (n/e)^n$. If we suppose $k > m$ then we get $t(E) = \infty$, which is impossible because E is compact. Additionally, in the case $k = m$ we get the following.

Proposition 4.1. *If there exist positive constants A, m such that inequality (14) holds with $k = m$, then*

$$t(E) \geq 1/(Ne^m A) > 0, \tag{17}$$

and therefore, condition $t(E) > 0$ is necessary for E to have V. Markov's property. Similarly, if we assume that (14) holds for all $\alpha \in \bigcup_{l=1}^{N} \mathbb{N}e_l$ then

$$t^*(E) \geq 1/(e^m A) > 0. \tag{18}$$

Moreover, if $k = m = 1$ we get the following stronger inequalities (cf. [10])

$$t(E) \geq \frac{1}{NA}, \qquad t^*(E) \geq \frac{1}{A}.$$

We can easily see that $VMI(m,1) \iff AMI(m)$. The property $VMI(m,k)$ for $k=m$ has been investigated in [5] where the following equivalence has been shown.

Theorem 4.2. *([5, Thm.2.9]) For every compact set E in $\mathbb{C}^N$ and $m \geq 1$*

$$HCP(\tfrac{1}{m}) \iff VMI(m,m).$$

Remark 4. We can consider A. Markov's and V. Markov's inequalities with respect to a fixed norm $\mathcal{R}$ on $\mathcal{P}(\mathbb{C}^N)$ by replacing the supremum norm in Definitions 3.4 and 3.5 by the norm $\mathcal{R}$. In such a situation A. Markov's inequality need not imply $t(\mathcal{R}) > 0$ (cf. [8]).

Pleśniak has proved in [27] that $AMI(m)$ is equivalent to the existence of a positive constant D such that for every $n \geq 1$ and for all polynomial P of degree at most n we have

$$|P(z)| \;\leq\; D\,||P||_E \quad \text{whenever } \operatorname{dist}(z,E) \leq \frac{1}{n^m} \tag{19}$$

which is equivalent to

$$||P||_{E(1/n^m)} \leq D,\ ||P||_E = 1,$$

where

$$E(r) := \left\{z \in \mathbb{C}^N \,:\, \operatorname{dist}(z,E) \leq r\right\}.$$

Consequently, it is easily seen that Pleśniak's inequality is equivalent to the following condition

$$\sup_{n\geq 1} ||\Phi_n(E,\cdot)||_{E(1/n^m)} < \infty.$$

Now, for a nonpluripolar set E consider the stronger condition

$$\sup_{n\geq 1} ||\exp(nV_E)||_{E(1/n^m)} =: H < \infty$$

We have

$$||V_E||_{E(1/n^m)} \leq \frac{1}{n}\log H = \left(\frac{1}{n^m}\right)^{1/m}\log H, \quad n \geq 1.$$

If $r \in (0,1]$ then $1/(n+1)^m < r \leq 1/n^m$ with an $n \geq 1$ and we obtain

$$||V_E||_{E(r)} \leq \left(\frac{1}{n^m}\right)^{1/m}\log H \leq r^{1/m}\log H^2$$

that is equivalent to

$$||P||_{E(r)} \leq \exp\left(Dr^{1/m}n\right)||P||_E, \quad r \in (0,1], \quad D = \log H^2.$$

Remark 5.

(1) The last condition is also satisfied for $r > 1$ (with a constant D_1). To account for this, fix $z \in E$, $w \in \mathbb{C}^N$, $||w||_2 = 1$ and a polynomial P of degree $\leq n$. Next, consider a polynomial of one variable $Q(\zeta) = P(z+\zeta w)$. For $|\zeta| = 1$, we have the bound $|Q(\zeta)| \leq \exp(Dn)||P||_E$. Hence, by the Bernstein inequality, for $|\zeta| = r > 1$ we get

$$|Q(\zeta)| \leq |\zeta|^n \exp(nD)||P||_E \leq \exp\left(D_1 r^{1/m} n\right)||P||_E,\ D_1 = \max(D,m).$$

(2) In a similar way, we can check the following conclusion

If $||P||_{B(z,r)} \leq \exp\left(Dr^\alpha n^\beta\right)||P||_E$ *for* $r \in (0,1]$
then $||P||_{B(z,r)} \leq \exp\left(\max(D,1/\alpha)r^\alpha n^\beta\right)||P||_E$ *for all* $r > 0$,
where $\deg P \leq n$, $D,\alpha > 0$, $\beta \geq 1$ *and* $B(z,r)$ *is the Euclidean ball* $B(z,r) := \{w \in \mathbb{C}^N \,:\, ||w-z||_2 \leq r\}$.

We consider a slightly generalised inequality.

Definition 4.4. A compact set $E \subset \mathbb{C}^N$ has the *Pleśniak property* with exponents m, $k \geq 1$ ($P(m,k)$ in short) if for every $n \in \mathbb{N}$

$$||P||_{E(r)} \;\leq\; \exp\left(Dr^{1/k}n^{m/k}\right)||P||_E \tag{20}$$

with a constant $D > 0$ independent of $P \in \mathcal{P}_n(\mathbb{C}^N)$ and $r \in (0,1]$.

Remark 6.

(1) Inequality (20) implies $k \leq m$. Indeed, if we take $k > m$ then $m/k < 1$. Replacing P by P^l we get the inequality $||P||_{E(r)} \leq ||P||_E$ for all $r \in (0,1]$, which is impossible if $\deg P \geq 1$.

(2) By Remark 5, we can assume, that condition (20) is satisfied for all $r > 0$ with constant $\max(D,k)$ instead D.

(3) If $k = m$ then, by the observation before Remark 5, we get an equivalent condition to HCP.

Theorem 4.3. *Let E be a compact set in $\mathbb{C}^N$, $z \in E$ and $m \geq k \geq 1$, $D, B \geq 1$. Fix $n \in \mathbb{N}$ and $P \in \mathcal{P}_n(\mathbb{C}^N)$. If*

$$\forall\ \alpha \in \mathbb{N}_0^N \quad |D^\alpha P(z)| \ \leq\ B^{|\alpha|} \frac{n^{m|\alpha|}}{|\alpha|!^{k-1}} \|P\|_E$$

then

$$\forall\ r > 0\ \|P\|_{B(z,r)} \ \leq\ \exp\left(Dr^{1/k}\, n^{m/k}\right) \|P\|_E \quad with\ \ D = k(BN)^{1/k}.$$

Conversely, if

$$\forall\ r > 0 \quad \|P\|_{B(z,r)} \ \leq\ \exp\left(Dr^{1/k}\, n^{m/k}\right) \|P\|_E$$

then

$$\forall\ \alpha \in \mathbb{N}_0^N \quad |D^\alpha P(z)| \leq B^{|\alpha|} \frac{n^{m|\alpha|}}{|\alpha|!^{k-1}} \|P\|_E \quad with\ \ B = e\sqrt{N}D^k.$$

Proof. In order to show the first implication, fix $r > 0$ and $w \in B(z, r)$. By Taylor's theorem and the assumption,

$$|P(w)| \leq \sum_{|\alpha|\leq n} \frac{1}{\alpha!} |D^\alpha P(z)|\, \|w - z\|_2^{|\alpha|}$$

$$\leq \sum_{|\alpha|\leq n} \frac{1}{\alpha!} B^{|\alpha|} \frac{n^{m|\alpha|}}{|\alpha|!^{k-1}} \|P\|_E\, r^{|\alpha|} = \sum_{\nu=0}^{n} B^{\nu} \frac{n^{m\nu}}{\nu!^{k-1}} r^{\nu} \|P\|_E \sum_{|\alpha|=\nu} \frac{1}{\alpha!}\,.$$

Since $\sum_{|\alpha|=\nu} \frac{1}{\alpha!} = \frac{N^\nu}{\nu!}$, we get

$$\|P\|_{B(z,r)} \leq \sum_{\nu=0}^{n} \frac{B^\nu N^\nu n^{m\nu} r^\nu}{\nu!^k} \|P\|_E = \sum_{\nu=0}^{n} \left(\frac{(BNrn^m)^{\nu/k}}{\nu!}\right)^k \|P\|_E$$

$$\leq \left(\sum_{\nu=0}^{n} \frac{(BNrn^m)^{\nu/k}}{\nu!}\right)^k \|P\|_E \ \leq\ \exp\left[k(BNrn^m)^{1/k}\right] \|P\|_E.$$

To prove the second implication, fix $\alpha \in \mathbb{N}_0^N$ and $z \in E$. By Cauchy's formula for the polydisc $P(z, cr) = \{w = (w_1, \ldots, w_N) \in \mathbb{C}^N : |w_j - z_j| \leq cr \ \text{ for } j = 1, \ldots, N\}$ with $c = 1/\sqrt{N}$ and $r \in (0, 1]$, we have

$$|D^\alpha P(z)| \leq \frac{\alpha!}{(cr)^{|\alpha|}} \|P\|_{P(z,cr)} \leq \sqrt{N}^{|\alpha|} \frac{\alpha!}{r^{|\alpha|}} \|P\|_{B(z,r)}.$$

The assumption leads us to

$$|D^\alpha P(z)| \ \leq\ \sqrt{N}^{|\alpha|} \frac{\alpha!}{r^{|\alpha|}} \exp\left(Dr^{1/k}\, n^{m/k}\right) \|P\|_E. \tag{21}$$

But

$$\inf_{r>0} \frac{1}{r^{|\alpha|}} \exp\left(Dr^{1/k}\, n^{m/k}\right) = \left(eD^k\right)^{|\alpha|} \left(n^m/|\alpha|^k\right)^{|\alpha|}.$$

This finishes the proof because $\alpha! \leq |\alpha|!$. □

Corollary 4.4. *For every compact set E in $\mathbb{C}^N$ and $m \geq k \geq 1$*

$$P(m,k) \iff VMI(m,k)\,.$$

In particular,

$$P(m,m) \Leftrightarrow HCP(\tfrac{1}{m}) \quad \text{and} \quad P(m,1) \Leftrightarrow AMI(m).$$

Remark 7. As in the proof of Proposition 2.2, we can show that $VMI(m,k)$ is equivalent to the following discrete version of inequality (20)

$$\forall\ n \in \mathbb{N},\ \ell \in \{1, \ldots, n\} \quad \forall\ P \in \mathcal{P}_n(\mathbb{C}^N) \quad \|P\|_{E(k,\ell,m)} \ \leq\ D^\ell\, \|P\|_E$$

with a constant D depending only on E and k where

$$E(k,\ell,m) := \left\{z \in \mathbb{C}^N \ :\ \mathrm{dist}\,(z, E) \leq \tfrac{\ell^k}{n^m}\right\}.$$

This equivalence clearly shows a connection between the original Pleśniak inequality (19) and the Pleśniak property $P(m,k)$ defined by (20).

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