PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 17 · 2024 · Pages 89–96

Note on admissible meshes on ball and simplex via Dubiner metric

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Dedicated to Len Bos, on the occasion of his 70th birthday

Abstract

In this paper, we present optimal admissible meshes for ball and simplex, based on Chebyshev nodes. We compare them with other recently studied point sets and give numerical evaluations, using the covering radius related to the Dubiner distance as the main metric of performance.

Keywords: admissible mesh, Chebyshev nodes, covering radius, Dubiner distance, Waldron points.

1 Introduction

In recent studies focusing on accurate discretization of compact sets $K \subset \mathbb{R}^d$ the concept of admissible meshes often plays a pivotal role. We briefly recall, following [4], that for a compact set $K \subset \mathbb{R}^d$, (polynomial) *admissible mesh* is a sequence of finite sets $X_n \subset K$ such that

$$\|p\|_K \le c\|p\|_{X_n}, \quad p \in \mathbb{P}_n^d \tag{1}$$

with constant $c > 0$ independent of p, where $\mathbb{P}_n^d$ denotes the space of polynomials in d variables, and at most n-th degree. We also require, for X_n to be $\mathbb{P}_n^d$-*determining* (meaning, any polynomial from $\mathbb{P}_n^d$ that vanishes on X_n must vanish on the entire set K) and for $\mathrm{card}(X_n)$ to grow at most polynomially with n, that is $\mathrm{card}(X_n) = O(n^\alpha)$ for $\alpha \ge d$. The polynomial admissible mesh is called *optimal* when $\alpha = d$.

We define the *Dubiner distance* on a compact set $K \subset \mathbb{R}^d$ as

$$d_D^K(x,y) := \sup\left\{\frac{1}{\deg p}|\arccos p(y) - \arccos p(x)| : \deg p \ge 1, \|p\|_K \le 1\right\}, \quad x,y \in K.$$

The Dubiner distance, was originally introduced in [6]. It plays a significant role in multivariate polynomial interpolation, and found numerous applications in the construction of norming sets and admissible meshes e.g. [1], [9], [5]. Among its many interesting properties, it is worth noting that, the Dubiner distance is invariant under invertible affine transformations. Indeed, for an invertible linear transformation $T : K \to T(K)$, we have

$$d_D^K(x,y) = d_D^{T(K)}(T(x), T(y)).$$

In the paper [8], Piazzon and Vianello showed a connection between Dubiner distance and admissible meshes. They demonstrated that for compact sets $X \subset K \subset \mathbb{R}^d$, if the *covering radius* $\rho_K(X)$ with respect to the Dubiner distance does not exceed θ/n for some $\theta \in (0, \pi/2), n \ge 1$, i.e.

$$\rho_K(X) := \sup_{x\in K}\inf_{a\in X} d_D^K(x,a) \le \frac{\theta}{n} \tag{2}$$

then the following inequality holds

$$\|p\|_K \le \frac{1}{\cos\theta}\|p\|_X, \quad p \in \mathbb{P}_n^d.$$

In this paper, our goal is to develop designs for polynomial admissible meshes over simplices and balls with potentially low cardinality, satisfying (1) with constant

$$c_m := 1/\cos(\pi/(2m)). \tag{3}$$

According to (2), it is enough to show that for admissible mesh X_{mn}, we have $\rho_K(X_{mn}) \le \pi/(2mn)$ and required constant in inequality (1) will then immediately follow. In Sections 2 and 3, we will outline constructions based on this principle, for meshes over simplices and balls, respectively. Additionally, we will provide a numerical comparison with other, recently studied admissible meshes. However, now we will state a lemma, that will become relevant in later parts of this paper.

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Lemma 1.1. *For any integer $d \geq 1$, and any $x \in [0,2]$ the following inequality holds*

$$\cos^d\left(\frac{x}{\sqrt{d}}\right) \geq \cos x.$$

Proof. The result is imminent for $d = 1$. Consider $d \geq 2$. By making use of Taylor series of $\cos(x)$, for all $x \in \mathbb{R}$, we have $1 - x^2/2 \leq \cos(x) \leq 1 - x^2/2 + x^4/24$. Consequently,

$$\cos^d\left(\frac{x}{\sqrt{d}}\right) \geq \left(1 - \frac{x^2}{2d}\right)^d = \sum_{k=0}^{d}(-1)^k\binom{d}{k}\left(\frac{x^2}{2d}\right)^k.$$

Consider the partial sum $\sum_{k=4}^{d}(-1)^k\binom{d}{k}\left(\frac{x^2}{2d}\right)^k$ starting from 4 and the following two cases.
1° If $d = 2l + 1$ for some $l \in \mathbb{N}$ then for $x \in [0,2]$

$$\sum_{k=4}^{d}(-1)^k\binom{d}{k}\left(\frac{x^2}{2d}\right)^k = \sum_{j=2}^{l}\left[\binom{d}{2j}\left(\frac{x^2}{2d}\right)^{2j} - \binom{d}{2j+1}\left(\frac{x^2}{2d}\right)^{2j+1}\right] = \sum_{j=2}^{l} d\binom{d-1}{2j}\left(\frac{x^2}{2d}\right)^{2j}\left[\frac{1}{d-2j} - \frac{x^2}{2d}\frac{1}{2j+1}\right]$$
$$\geq \sum_{j=2}^{l} d\binom{d-1}{2j}\left(\frac{x^2}{2d}\right)^{2j}\left[\frac{1}{d-2j} - \frac{2}{d}\frac{1}{2j+1}\right] = \sum_{j=2}^{l} d\binom{d-1}{2j}\left(\frac{x^2}{2d}\right)^{2j}\underbrace{\left[\frac{d(2j-1)+4j}{d(d-2j)(2j+1)}\right]}_{\geq 0} \geq 0.$$

2° In the case of $d = 2l$ for some $l \in \mathbb{N}$, we have

$$\sum_{k=4}^{d}(-1)^k\binom{d}{k}\left(\frac{x^2}{2d}\right)^k = \sum_{j=2}^{l}\left[\binom{d}{2j}\left(\frac{x^2}{2d}\right)^{2j} - \binom{d}{2j+1}\left(\frac{x^2}{2d}\right)^{2j+1}\right] + \left(\frac{x^2}{2d}\right)^d \geq 0.$$

the last inequality being a consequence of 1°. Since for $x \in [0,2]$ the partial sum starting from 4 is positive, we can write

$$\cos^d\left(\frac{x}{\sqrt{d}}\right) \geq \sum_{k=0}^{d}(-1)^k\binom{d}{k}\left(\frac{x^2}{2d}\right)^k \geq \sum_{k=0}^{3}(-1)^k\binom{d}{k}\left(\frac{x^2}{2d}\right)^k = 1 - \frac{x^2}{2} + \frac{d(d-1)}{2}\frac{x^4}{4d^2} - \frac{d(d-1)(d-2)}{6}\frac{x^6}{8d^3}$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{24}\left(\frac{3d(d-1)}{d^2} - \frac{(d-1)(d-2)}{d^2}\frac{x^2}{2}\right) \geq 1 - \frac{x^2}{2} + \frac{x^4}{24}\left(\frac{3d(d-1)}{d^2} - \frac{2(d-1)(d-2)}{d^2}\right)$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{24}\frac{d^2+3d-4}{d^2} \geq 1 - \frac{x^2}{2} + \frac{x^4}{24} \geq \cos x,$$

which completes the proof. □

2 Admissible meshes on a simplex

Let $q > 0$. By $\mathscr{C}_q$ we will denote $\lceil q \rceil$ *Chebyshev points* in $[-1,1]$, that is zeros of the $\lceil q \rceil$-th Chebyshev polynomial of the first kind, i.e.

$$\mathscr{C}_q := \left\{\cos\frac{(2j-1)\pi}{2\lceil q \rceil}, \quad 1 \leq j \leq \lceil q \rceil\right\}$$

where $\lceil \cdot \rceil$ is the usual ceiling function. It's worth noting that, since the Dubiner distance is invariant under invertible affine transformations, any results acquired for a given simplex can immediately be extended to any other simplex. We will use *Duffy transformation* between the cube $[-1,1]^d$ and the standard unit simplex

$$E = E_d := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1, \ldots, x_d \geq 0,\ x_1 + \ldots + x_d \leq 1\},$$

which can be defined as follows

$$\mathscr{D} : [-1,1]^d \ni t = (t_1, \ldots, t_d) \longmapsto x = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

where

$$x_1 = \frac{1+t_1}{2}, \quad x_i = \frac{1+t_i}{2}\prod_{j=1}^{i-1}\frac{1-t_j}{2}, \quad i = 2, \ldots, d.$$

Lemma 2.1. *The Duffy transformation defined above is a surjective function between $[-1,1]^d$ and the simplex E.*

Proof. Fix $[-1,1]^d \ni t = (\cos 2\phi_1, \ldots, \cos 2\phi_d)$ for some $\phi_1, \ldots, \phi_d \in [0, \pi/2]$. We have

$$x_1 = \frac{1+\cos 2\phi_1}{2} = \cos^2\phi_1 \geq 0, \quad x_i = \frac{1+\cos 2\phi_i}{2}\prod_{j=1}^{i-1}\frac{1-\cos 2\phi_j}{2} = \cos^2\phi_i\prod_{j=1}^{i-1}\sin^2\phi_j \geq 0, \quad i = 2, \ldots, d. \tag{4}$$

To prove that $\Sigma_{i=1}^d x_i \le 1$, for any $k \in \{2,\dots,d\}$ consider

$$\cos^2\phi_1 + \sum_{i=2}^k \cos^2\phi_i \prod_{j=1}^{i-1}\sin^2\phi_j + \prod_{j=1}^k \sin^2\phi_j = \cos^2\phi_1 + \sin^2\phi_1\left(\cos^2\phi_2 + \sum_{i=3}^k \cos^2\phi_i \prod_{j=2}^{i-1}\sin^2\phi_j + \prod_{j=2}^k \sin^2\phi_j\right)$$
$$= \cos^2\phi_1 + \sin^2\phi_1\left(\cos^2\phi_2 + \sin^2\phi_2\left(\cos^2\phi_3 + \sum_{i=4}^k \cos^2\phi_i \prod_{j=3}^{i-1}\sin^2\phi_j + \prod_{j=3}^k \sin^2\phi_j\right)\right) = \dots$$
$$= \cos^2\phi_1 + \sin^2\phi_1(\cos^2\phi_2 + \sin^2\phi_2(\cos^2\phi_3 + \ldots + \sin^2\phi_{k-2}(\cos^2\phi_{k-1} + \sin^2\phi_{k-1}(\cos^2\phi_k + \sin^2\phi_k)))) = 1.$$

As a result, we get $1 - \Sigma_{i=1}^k x_i = \prod_{j=1}^k \sin^2\phi_j$ for any $k \in \{2,\dots,d\}$. In particular,

$$1 - \Sigma_{i=1}^d x_i = \prod_{j=1}^d \sin^2\phi_j \ge 0. \tag{5}$$

Now, it is enough to show that any vector $(x_1,\dots,x_d) \in E$ is of the form (4). Indeed, $x_1 \in [0,1]$ implies that $x_1 = \cos^2\phi_1$ for some $\phi_1 \in [0,\pi/2]$. Since $0 \le x_k \le 1 - \sum_{j=1}^{k-1} x_j$ and, by the above, $1 - \sum_{j=1}^{k-1} x_j = \prod_{j=1}^{k-1}\sin^2\phi_j$, there exists $\phi_k \in [0,\pi/2]$ such that $x_k = \cos^2\phi_k \prod_{j=1}^{k-1}\sin^2\phi_j$. □

In the following part of the paper we will denote $\mathscr{C}^d_{\sqrt{d}mn} = (\mathscr{C}_{\sqrt{d}mn})^d \subset [-1,1]^d$ as the Cartesian product of $\mathscr{C}_{\sqrt{d}mn} \subset [-1,1]$. Additionally, from now on, to simplify the writing, we will assume that any empty product, in particular, one of the form $\prod_{i=j}^k$ for $k < j$ has the value 1.

Theorem 2.2. *Let $X_{\sqrt{d}mn} = \mathscr{D}(\mathscr{C}^d_{\sqrt{d}mn})$. Then, for the simplex $E = E_d \subset \mathbb{R}^d$, every integer $n \ge 1$ and real $m > 1$ the following inequality holds*

$$\rho_E(X_{\sqrt{d}mn}) \le \frac{\pi}{2mn}.$$

Consequently,

$$\|p\|_E \le c_m \|p\|_{X_{\sqrt{d}mn}}$$

for any polynomial $p \in \mathbb{P}^d_n$ where c_m is given in (3).

Proof. The points from the set $\mathscr{C}^d_{\sqrt{d}mn}$ can be expressed as

$$\{(\cos 2\theta_{i_1},\dots,\cos 2\theta_{i_d})\}_{(i_1,\dots,i_d)}, \quad 1 \le i_1,\dots,i_d \le \lceil \sqrt{d}mn \rceil,$$

where $\theta_{i_k} := \frac{(2i_k-1)\pi}{4\lceil\sqrt{d}mn\rceil} \in [0,\pi/2]$ for $k \in \{1,\dots,d\}$. Those points can then be transformed using the Duffy transformation to acquire the corresponding mesh for the simplex E,

$$\mathscr{D}(\mathscr{C}^d_{\sqrt{d}mn}) = \left\{(a_{i_1},\dots,a_{i_d}) : a_{i_k} = \cos^2\theta_{i_k}\prod_{j=1}^{k-1}\sin^2\theta_{i_j}, k = 1,\dots,d\right\}_{(i_1,\dots,i_d)} \subset E.$$

As in the proof of Lemma 2.1, any point $x = (x_1,\dots,x_d) \in E$, can be written in the form (4).

To estimate the covering radius, we will apply the known bound of Dubiner distance on a simplex, see [2], i.e.

$$d_D^E(a,b) \le 2\arccos(\tilde{a}\cdot\tilde{b}), \quad a,b \in E,$$

where $\cdot$ is the standard dot product, and

$$\tilde{a} := \left(\sqrt{a_1},\dots,\sqrt{a_d},\sqrt{1-\Sigma_{i=1}^d a_i}\right), \quad \tilde{b} := \left(\sqrt{b_1},\dots,\sqrt{b_d},\sqrt{1-\Sigma_{i=1}^d b_i}\right)$$

gives a mapping from the simplex E to the positive orthant of the unit sphere in $\mathbb{R}^{d+1}$. From (5), for $a, x \in E$ with parameters θ_i, ϕ_i, we have

$$d_D^E(x,a) \le 2\arccos(\widetilde{x}\cdot\widetilde{a}) = 2\arccos\left(\sum_{i=1}^d \sqrt{x_i\,a_i} + \sqrt{\left(1-\sum_{i=1}^d x_i\right)\left(1-\sum_{i=1}^d a_i\right)}\right)$$
$$= 2\arccos\left(\sum_{i=1}^d \cos\phi_i\cos\theta_i \prod_{j=1}^{i-1}\sin\phi_j\sin\theta_j + \prod_{j=1}^d \sin\phi_j\sin\theta_j\right).$$

Let's consider the sum

$$S := \sum_{i=1}^{d}\left(\cos\phi_i\cos\theta_i\prod_{j=1}^{i-1}\sin\phi_j\sin\theta_j\right)+\prod_{j=1}^{d}\sin\phi_j\sin\theta_j$$
$$= \cos\phi_1\cos\theta_1+\sin\phi_1\sin\theta_1\left[\sum_{i=2}^{d}\left(\cos\phi_i\cos\theta_i\prod_{j=2}^{i-1}\sin\phi_j\sin\theta_j\right)+\prod_{j=2}^{d}\sin\phi_j\sin\theta_j\right]=\ \dots$$
$$= \cos\phi_1\cos\theta_1+\sin\phi_1\sin\theta_1(\cos\phi_2\cos\theta_2+\sin\phi_2\sin\theta_2(\cos\phi_3\cos\theta_3+\dots$$
$$\dots+\sin\phi_{d-2}\sin\theta_{d-2}(\cos\phi_{d-1}\cos\theta_{d-1}+\sin\phi_{d-1}\sin\theta_{d-1}(\cos\phi_d\cos\theta_d+\sin\phi_d\sin\theta_d))))$$

Since $\cos\phi_{d-1}\cos\theta_{d-1}+\sin\phi_{d-1}\sin\theta_{d-1}(\cos\phi_d\cos\theta_d+\sin\phi_d\sin\theta_d)) = \cos\phi_{d-1}\cos\theta_{d-1}+\sin\phi_{d-1}\sin\theta_{d-1}\cos|\phi_d-\theta_d| \geq \cos|\phi_{d-1}-\theta_{d-1}|\cos|\phi_d-\theta_d|$, we get

$$S \geq \cos\phi_1\cos\theta_1+\sin\phi_1\sin\theta_1(\cos\phi_2\cos\theta_2+\sin\phi_2\sin\theta_2(\cos\phi_3\cos\theta_3+\dots+\sin\phi_{d-2}\sin\theta_{d-2}\cos|\phi_{d-1}-\theta_{d-1}|\cos|\phi_d-\theta_d|))$$
$$\geq \dots \geq \cos\phi_1\cos\theta_1+\sin\phi_1\sin\theta_1\prod_{j=2}^{d}\cos|\phi_j-\theta_j| \geq \prod_{j=1}^{d}\cos|\phi_j-\theta_j|,$$

and so we have

$$d_D^E(x,a)\leq 2\arccos\prod_{j=1}^{d}\cos|\phi_j-\theta_j|.$$

Now, observe that points $a=(a_{i_1},\dots,a_{i_d})\in X_{\sqrt{d}mn}=\mathscr{D}(\mathscr{C}^d_{\sqrt{d}mn})$ with $a_{i_k}=\cos^2\theta_{i_k}\prod_{j=1}^{k-1}\sin^2\theta_{i_j}$ are equidistributed with respect to $\theta_{i_1},\dots,\theta_{i_d}\in[0,\pi/2]$ with spacing of $\pi/(2\lceil\sqrt{d}mn\rceil)$ in between them. Therefore, for any $x=x(\phi_1,\dots,\phi_d)\in E$, $\phi_1,\dots,\phi_d\in[0,\pi/2]$, there exist $a_i\in X_{\sqrt{d}mn}$ with $i=(i_1,\dots,i_d)\in\{1,\dots,\lceil\sqrt{d}mn\rceil\}^d$, such that $|\phi_k-\theta_{i_k}|\leq\pi/(4\lceil\sqrt{d}mn\rceil)$ for all $k=1,\dots,d$. This leads to the following estimate of the covering radius

$$\rho_E(X_{\sqrt{d}mn})=\sup_{x\in E}\inf_{a\in X_{\sqrt{d}mn}}d_D^E(x,a)$$
$$\leq \sup_{\substack{\phi_k\in[0,\pi/2]\\ k=1,\dots,d}}\inf_{(i_1,\dots,i_d)} 2\arccos\prod_{k=1}^{d}\cos|\phi_k-\theta_{i_k}|$$
$$=2\arccos\prod_{k=1}^{d}\sup_{\phi_k\in[0,\pi/2]}\inf_{i_k}\cos|\phi_k-\theta_{i_k}|$$
$$\leq 2\arccos\prod_{k=1}^{d}\cos\left(\frac{\pi}{4\sqrt{d}mn}\right)=2\arccos\left(\cos^d\left(\frac{\pi}{4\sqrt{d}mn}\right)\right)$$

And now, by applying Lemma 1.1, we can finally write

$$2\arccos\left(\cos^d\left(\frac{\pi}{4\sqrt{d}mn}\right)\right)\leq 2\frac{\pi}{4mn}=\frac{\pi}{2mn}.$$

□

Remark 1. It is worth noting that for the plane, this mesh can be easily improved. Consider the subset of points from mesh $X_{\sqrt{2}mn}$ for which $\theta_{i_1}\in[\pi/4,\pi/2]$ and $\theta_{i_2}\in[0,\pi/4]$, which correspond to upper left quarter of the set $\mathscr{C}_{\sqrt{2}mn}\times\mathscr{C}_{\sqrt{2}mn}$. In other words, $\{(a',b')\in\mathscr{C}_{\sqrt{2}mn}\times\mathscr{C}_{\sqrt{2}mn}: a'\leq 0, b'\geq 0\}$. In this subset $\theta_{i_1},\theta_{i_2}$ are still equally distributed and the reasoning used in the proof of Theorem 2.2 holds for points $(x',y')\in\widetilde{Q}_1$ where

$$\widetilde{Q}_1:=\mathscr{D}(\{(x',y')\in[-1,1]^2: x'\leq 0, y'\geq 0\})=\{(x,y)\in E: x\leq 1/2, y\geq(1-x)/2\}.$$

Now, let's consider an affine transformation $\mathscr{T}$

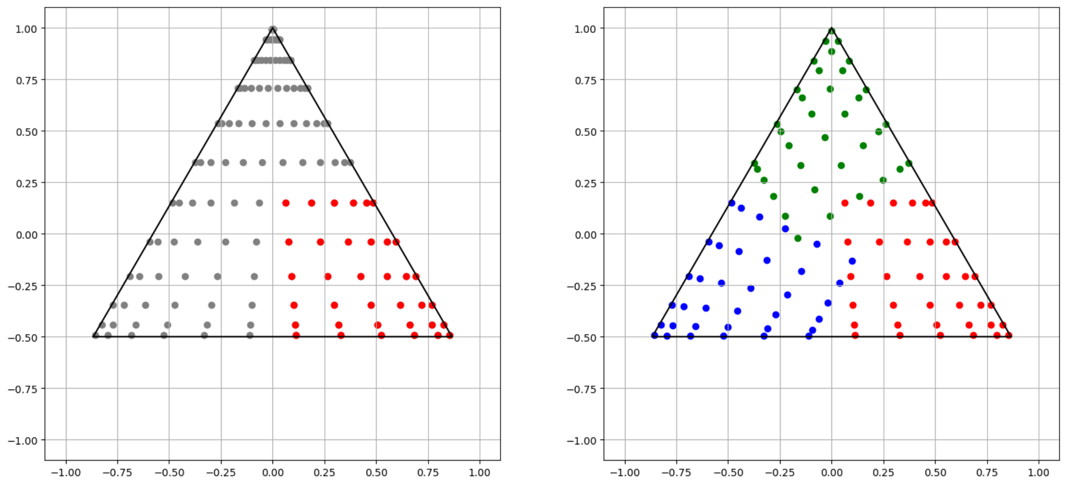
$$\mathscr{T}: E\ni(x,y)\longmapsto\left(\frac{\sqrt{3}}{2}(x+2y-1),\ \frac{3}{2}x-\frac{1}{2}\right)\in T$$

that maps the unit simplex E into an equilateral triangle T centered around the origin with vertices $V_1=(\sqrt{3}/2,-1/2)$, $V_2=(0,1)$, $V_3=(-\sqrt{3}/2,-1/2)$.

Since the Dubiner distance is invariant under invertible linear transformations, the reasoning above holds for points $(x,y)\in Q_1$ where

$$Q_1:=\mathscr{T}(\widetilde{Q}_1)=\{(x,y)\in T: x\geq 0, y\leq 1/4\}.$$

By rotating Q_1 by $2\pi/3$ and $4\pi/3$ around the origin, we can create sets Q_2, and Q_3 respectively. To show that $T=Q_1\cup Q_2\cup Q_3$, it is enough to see that the bottom right kite of equilateral triangle $K_1:=\{(x,y)\in T: x\geq 0, y\leq\sqrt{3}x/3\}$ is contained within Q_1.



(a) Admissible mesh $X_{\sqrt{2}mn}$ mapped onto the equilateral triangle with the transformation $\mathcal{T}$ for $mn = 8$. Red points indicate elements of the set Q_1.

(b) Improved grid with lower cardinality. Points marked in red, green and blue correspond to points from sets Q_1, Q_2 and Q_3, respectively.

Figure 1: Admissible meshes for a simplex.

Again rotating this kite by $2\pi/3$ and $4\pi/3$ around the origin, creates kites K_2 and K_3 contained within Q_2 and Q_3 respectively. Thus $T = K_1 \cup K_2 \cup K_3 \subset Q_1 \cup Q_2 \cup Q_3$. It follows that $Q_1 \cup Q_2 \cup Q_3$ is a covering of T and $\rho_T(\widetilde{X}_{\sqrt{2}mn}) \leq \frac{\pi}{2mn}$. This mesh contains only $3(\lceil \sqrt{2}mn/2 \rceil)^2$ points, compared to $(\lceil \sqrt{2}mn \rceil)^2$ in the original mesh, see Fig. 1b.

Corollary 2.3. *For any integer $n > 0$ and real number $m > 1$ the set $\widetilde{X}_{\sqrt{2}mn} := \widetilde{X}_1 \cup \widetilde{X}_2 \cup \widetilde{X}_3$ forms an admissible mesh on T with constant c_m given in (3), i.e.*

$$\|p\|_T \leq c_m \|p\|_{\widetilde{X}_{\sqrt{2}mn}}$$

for any polynomial $p \in \mathbb{P}_n^2$ where

$$\widetilde{X}_1 := \left\{ \left(\frac{\sqrt{3}}{4} y(1-x), \frac{3}{4}x + \frac{1}{4} \right) : (x,y) \in \mathcal{C}_{\sqrt{2}mn} \times \mathcal{C}_{\sqrt{2}mn}, x \leq 0, y \geq 0 \right\},$$

$$\widetilde{X}_2 := r_{\frac{2\pi}{3}}(\widetilde{X}_1) = \left\{ \left(-\frac{\sqrt{3}}{8}(y(1-x) + 3x + 1), \frac{1}{8}(3y(1-x) - 3x - 1) \right) : (x,y) \in \mathcal{C}_{\sqrt{2}mn} \times \mathcal{C}_{\sqrt{2}mn}, x \leq 0, y \geq 0 \right\},$$

$$\widetilde{X}_3 := r_{\frac{4\pi}{3}}(\widetilde{X}_1) = \left\{ \left(-\frac{\sqrt{3}}{8}(y(1-x) - 3x - 1), -\frac{1}{8}(3y(1-x) + 3x + 1) \right) : (x,y) \in \mathcal{C}_{\sqrt{2}mn} \times \mathcal{C}_{\sqrt{2}mn}, x \leq 0, y \geq 0 \right\}$$

with $r_\theta(x,y) := (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$ being the usual rotation around the origin.

2.1 Comparison of admissible meshes for a simplex

In the following sections, we refer to admissible mesh based on Duffy-like transformation, constructed in Theorem 2.2, and denoted by $X_{\sqrt{d}mn}$, as *Duffy points*. The points presented in Remark 1, denoted by $\widetilde{X}_{\sqrt{2}mn}$, are referred to as *Improved Duffy points*. In Figure 2, we compare these points with equidistant *Simplex points*, see [3], the recently studied *Waldron points* on the simplex, denoted as W_N, and the *Spherical Waldron points* on the sphere, which we project onto the simplex and refer to as *projected spherical Waldron points* or pSW_N. Both Waldron points and Spherical Waldron points were introduced in paper [3] via barycentric coordinates. Let $\alpha = (\alpha_1, \dots, \alpha_{d+1}) \in \mathbb{N}_0^{d+1}$ and $|\alpha| := \Sigma_{i=1}^{d+1} \alpha_i$. Here, we define:

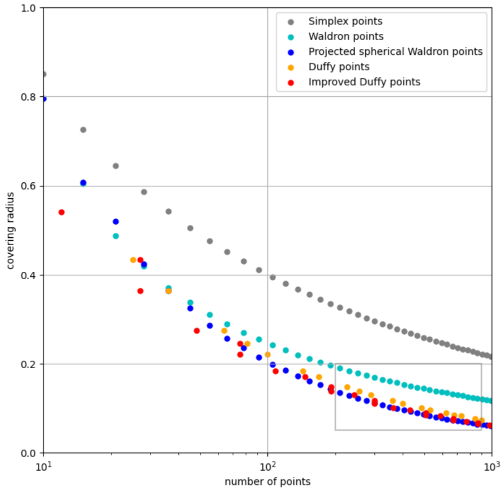
$$W_N := \left\{ x_\alpha = \sum_{i=1}^{3} w(\alpha_i/N) V_i : |\alpha| = N \right\}, \qquad pSW_N := \left\{ x_\alpha = \sum_{i=1}^{3} \frac{w(\alpha_i/N)}{\Sigma_{j=1}^{3} w(\alpha_j/N)} V_i : |\alpha| = N \right\}, \tag{6}$$

where $V_1 = (0,1)$, $V_2 = (-\sqrt{3}/2, -1/2)$, $V_3 = (\sqrt{3}/2, -1/2)$ are the vertices of the equilateral triangle, and $w(x) := (1 - \cos(\pi x))/2$ is the relevant *weight function*, which corresponds to Chebyshev-like points over the simplex.

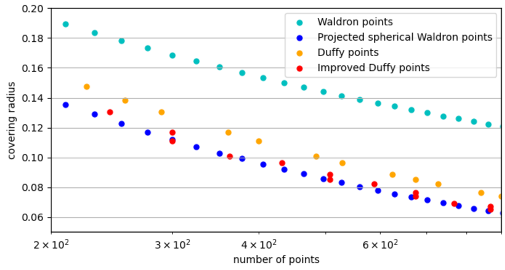
3 Admissible mesh on a ball

When working over a ball or a sphere, it is convenient to make use of generalized spherical coordinates. For the unit closed ball $B^d := \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$, $d \geq 2$, they correspond to the surjective transformation

$$\mathcal{G} : [0,1] \times [0,\pi]^{d-2} \times [0,2\pi] \ni (r, \theta_1, \dots, \theta_{d-1}) \mapsto (x_1, \dots, x_d) \in B^d$$



(a) Comparison of discussed admissible meshes over a simplex.



(b) Close up on the area outlined by the gray rectangle.

Figure 2: Numerical estimate of the covering radius based on the Baran distance.

such that

$$x_j = r\cos\theta_j \prod_{k=1}^{j-1}\sin\theta_k, \quad x_d = r\sin\theta_{d-1}\prod_{k=1}^{d-2}\sin\theta_k, \quad 1\le j\le d-1.$$

Additionally, let

$$\mathscr{U}(u_1,u_2,\dots,u_{d-1},u_d) := (|u_1|, \arccos(u_2),\dots,\arccos(u_{d-1}), 2\arccos(u_d))$$

for $u_1,\dots,u_d\in[-1,1]$. Then the composition $\mathscr{J} = \mathscr{G}\circ\mathscr{U}$ creates a mapping $\mathscr{J} : [-1,1]^d \to B^d$. It is well known that the Dubiner distance $d_D^{B^d}(a,b)$ coincides with the geodesic distance for points from the d-dimensional ball lifted to the $(d+1)$-dimensional hemisphere, see [2]. More precisely, for $a,b\in B^d$ we have

$$d_D^{B^d}(a,b) = \arccos(\tilde a\cdot\tilde b)$$

with

$$\tilde a := \left(a_1,\dots,a_d,\sqrt{1-\|a\|_2^2}\right), \quad \tilde b := \left(b_1,\dots,b_d,\sqrt{1-\|b\|_2^2}\right).$$

Theorem 3.1. *Let $Y_{\sqrt{2}mn} := \mathscr{J}((\mathscr{C}_{\sqrt{2}mn})^{d-1}\times\mathscr{C}_{2\sqrt{2}mn})$. Then for every integer $n>0$ and real $m>1$, the following inequality holds*

$$\rho_{B^d}(Y_{\sqrt{2}mn}) \le \frac{\pi}{2mn}.$$

Consequently,

$$\|p\|_{B^d} \le c_m\|p\|_{Y_{\sqrt{2}mn}}$$

for any polynomial $p\in\mathbb{P}_n^d$ where c_m is given in (3).

Proof. The points from the set $(\mathscr{C}_{\sqrt{2}mn})^{d-1}\times\mathscr{C}_{2\sqrt{2}mn}$ can be expressed as $(\cos\theta_1,\dots,\cos\theta_{d-1},\cos(\theta_d/2))$ where $(\theta_1,\dots,\theta_{d-1})\in[0,\pi]^{d-1}$ and $\theta_d\in[0,2\pi]$. By using the transformation $\mathscr{U}$, we can acquire new set of points:

$$a' = (|\cos\theta_1|,\theta_2,\dots,\theta_d)\in\mathscr{U}((\mathscr{C}_{\sqrt{2}mn})^{d-1}\times\mathscr{C}_{2\sqrt{2}mn}).$$

It is worth noting that this new set is spanning over d-ball B^d in spherical coordinates. Since $\mathscr{G}$ is surjective, for any point $x\in B^d$ there exist $x'\in[0,1]\times[0,\pi]^{d-2}\times[0,2\pi]$ such that $\mathscr{G}(x') = x$ where we can express x' as

$$x' = (r_{x'},\phi_2,\dots,\phi_d) \quad \text{where} \quad r_{x'}\in[0,1],\ (\phi_2,\dots,\phi_{d-1})\in[0,\pi]^{d-2},\ \phi_d\in[0,2\pi].$$

By choosing $\phi_1,\theta_1\in[0,\pi/2]$ such that $r_{x'} = \cos\phi_1$ and $|\cos\theta_1| = \cos\theta_1$, we can write

$$x' = (\cos\phi_1,\phi_2,\dots,\phi_d),$$
$$a' = (\cos\theta_1,\theta_2,\dots,\theta_d).$$

Let us take the rotation such that $A_{x'}(a) = A_{x'}(\mathcal{G}(a')) := \mathcal{G}(\cos\theta_1, \theta_2 - \phi_2, \dots, \theta_d - \phi_d)$, i.e. $A_{x'}$ corresponds to the rotation by $-\phi_i, i = 2, \dots, d$ in spherical coordinates. Now, we can estimate the Dubiner distance

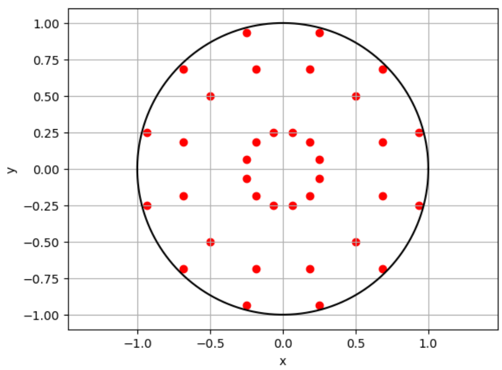
$$\begin{aligned}
d_D^{B^d}(x,a) &= d_D^{B^d}\left(\mathcal{G}(x'), \mathcal{G}(a')\right) \\
&= d_D^{B^d}\left(A_{x'}(\mathcal{G}(x')), A_{x'}(\mathcal{G}(a'))\right) \\
&= d_D^{B^d}\left(\mathcal{G}((\cos\phi_1, 0, \dots, 0)), \mathcal{G}((\cos\theta_1, \theta_2 - \phi_2, \dots, \theta_d - \phi_d))\right) \\
&= d_D^{B^d}\left((\cos\phi_1, 0, \dots, 0), (\cos\theta_1 \cos(\theta_2 - \phi_2), a_2, \dots, a_d)\right) \\
&= \arccos((\cos\phi_1, 0, \dots, 0, \sqrt{1-\cos^2\phi_1}) \cdot (\cos\theta_1 \cos(\theta_2 - \phi_2), a_2, \dots, a_d, \sqrt{1-\cos^2\theta_1})) \\
&= \arccos((\cos\phi_1, 0, \dots, 0, \sin\phi_1) \cdot (\cos\theta_1 \cos(\theta_2 - \phi_2), a_2, \dots, a_d, \sin\theta_1)) \\
&= \arccos(\cos\phi_1 \cos\theta_1 \cos(\theta_2 - \phi_2) + \sin\phi_1 \sin\theta_1) \\
&\le \arccos(\cos(\theta_2 - \phi_2)(\cos\phi_1 \cos\theta_1 + \sin\phi_1 \sin\theta_1)) = \arccos(\cos(\theta_1 - \phi_1)\cos(\theta_2 - \phi_2))
\end{aligned}$$

The mesh $(\mathcal{C}_{\sqrt{2}mn})^{d-1} \times \mathcal{C}_{2\sqrt{2}mn}$ is equidistant with respect to $\theta_1, \theta_2, \dots, \theta_d$ with a spacing of $\pi/(\lceil\sqrt{2}mn\rceil)$ between them, and so for every x', there exists a', such that $|\theta_i - \phi_i| \le \pi/(2\lceil\sqrt{2}mn\rceil)$ for $i = 1, \dots, d$. As in the case of the simplex

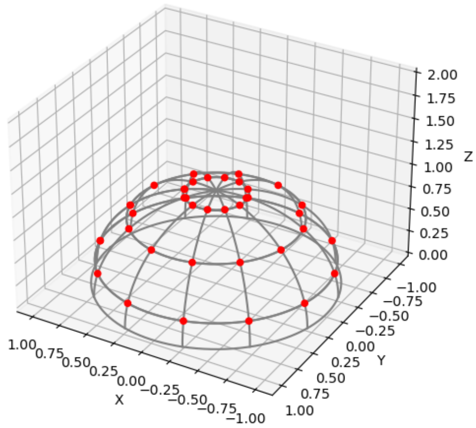
$$\begin{aligned}
\rho_{B^d}(Y_{\sqrt{2}mn}) &= \sup_{x \in B^d} \inf_{a \in Y_{\sqrt{2}mn}} d_D^{B^d}(x,a) \\
&\le \sup_{\phi_1,\phi_2} \inf_{\theta_1,\theta_2} \arccos(\cos(\theta_1 - \phi_1)\cos(\theta_2 - \phi_2)) \\
&= \arccos\left(\cos\left(\sup_{\phi_1}\inf_{\theta_1}|\theta_1 - \phi_1|\right)\cos\left(\sup_{\phi_2}\inf_{\theta_2}|\theta_2 - \phi_2|\right)\right) \\
&= \arccos\left(\cos\left(\frac{\pi}{2\lceil\sqrt{2}mn\rceil}\right)\cos\left(\frac{\pi}{2\lceil\sqrt{2}mn\rceil}\right)\right) \le \arccos\left(\cos^2\left(\frac{\pi}{2\sqrt{2}mn}\right)\right) \le \frac{\pi}{2mn}
\end{aligned}$$

by Lemma 1.1. □

Remark 2. Due to the symmetry of the Chebyshev nodes, the set $\{|u| : u \in \mathcal{C}_{\sqrt{2}mn}\}$ contains $\lceil\sqrt{2}mn/2\rceil$ points, and $Y_{\sqrt{2}mn} = \mathcal{J}((\mathcal{C}_{\sqrt{2}mn})^{d-1} \times \mathcal{C}_{2\sqrt{2}mn})$ creates an optimal admissible mesh, with at most $\lceil\sqrt{2}mn\rceil^d$ points, see Fig. 3.



(a) Admissible mesh $Y_{\sqrt{2}mn}$ with $mn = 4$.



(b) Elevation of $Y_{4\sqrt{2}}$ to the hemisphere.

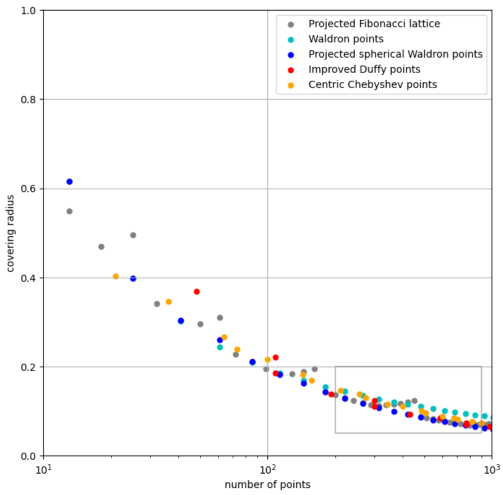
Figure 3: Admissible mesh for the unit disc.

3.1 Comparison of admissible meshes for a ball

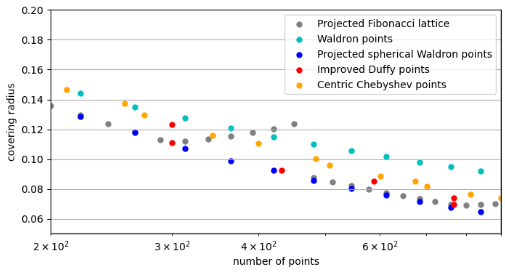
Remark 3. It is worth mentioning that any admissible mesh over a simplex, can be adapted to fit onto a ball. By elevating the simplex to positive orthant (as discussed in 2.2), by rotations around the elevation axis, it can be transformed onto an upper $(d+1)$-dimensional hemisphere and then projected back onto B^d

$$\left\{x \in \mathbb{R}^d : x_1, \dots, x_d \ge 0, \Sigma_{i=1}^d x_i \le 1\right\} \ni x \longmapsto x' = \left(\pm\sqrt{x_1}, \dots, \pm\sqrt{x_d}\right) \in B^d.$$

In this section, see Figure 4, we consider $d = 2$, the real unit disc. We compare the admissible mesh constructed in Theorem 3.1, denoted as $Y_{\sqrt{2}mn}$, which we refer to as *centric Chebyshev points*, together with points based on the *Fibonacci lattice*, which is known for producing nearly equidistant points on the sphere, see [7]. To adapt the Fibonacci lattice to our needs, we projected the upper hemisphere onto a disc, resulting in what we call the *projected Fibonacci lattice*. Additionally, we consider the improved Duffy points, constructed in Theorem 2.2, along with Waldron points and projected spherical Waldron points, defined in (6), and adapted to the unit disc as described in the above remark.



(a) Comparison of discussed admissible meshes over the unit disc.



(b) Close up on the area outlined by the gray rectangle.

Figure 4: Numerical comparison of admissible meshes for the unit disc.

Acknowledgement.
The work of the first author was partially supported by the National Science Centre, Poland, grant No. 2017/25/B/ST1/00906.

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