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Density of measures related to orthogonal polynomials

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To our good friend Len Bos

Abstract

Let $K \subset \mathbb{C}^d,\ d \geq 1$ be compact and $\mathcal{M}(K)$ be the convex set of Borel probability measures supported in K. We show that various natural subclasses of $\mathcal{M}(K)$ associated with regular asymptotic n—th root behavior of corresponding orthonormal polynomials are dense in the standard weak topology under mild assumptions on K.

1 Introduction

In the theory of orthogonal polynomials on the complex plane, one starts with a (positive) Borel measure μ of finite total mass and compact support K in the complex plane $\mathbb{C}$. Without loss of generality, one may assume μ is a probability measure, $\mu(K) = 1$; we write $\mathcal{M}(K)$ for the set of all Borel probability measures with support in K. When K is not polar, a key issue, as carefully described in the book [8], is to determine when the sequence of orthonormal polynomials $\{p_{n,\mu}\}$ in $L^2(\mu)$, where deg $p_n = n$, exhibit regular asymptotic (n—th root) behavior. This means that the sequence of subharmonic functions $\{\frac{1}{n}\log|p_{n,\mu}(z)|\}$ approximate in a precise way the Green function $V_K^*(z) := \limsup_{\zeta\to z} V_K(\zeta)$, where

$$V_K(z) = \sup\Big(\frac{1}{\deg(p)}\log|p(z)| : p \in \bigcup_n \mathcal{P}_n,\ ||p||_K := \sup_K |p| \leq 1\Big) \tag{1}$$

and $\mathcal{P}_n$ denotes the (holomorphic) polynomials of degree at most n. One writes that such a μ belongs to the class **Reg**; see Definition 2.6. Several sufficient conditions are known which imply that $\mu \in$ **Reg**. The definition (1) makes sense for compact sets $K \subset \mathbb{C}^d,\ d > 1$. For $\mu \in \mathcal{M}(K)$, a theory of orthogonal polynomials associated to compact sets in $\mathbb{C}^d$ for $d > 1$ was developed in [3], including an appropriate notion of **Reg** class. In this higher dimensional setting, much less is known regarding sufficient conditions for μ to belong to **Reg**.

In the next section, we state some known sufficient conditions for $\mu \in$ **Reg** in both the one and several variable settings. Our goal in this note is to show that under some natural hypotheses on K, these conditions, and hence μ belonging to **Reg**, is a generic condition within $\mathcal{M}(K)$.

2 Sufficient conditions for $\mu \in$ Reg

For the moment, we work in $\mathbb{C}^d$ where $d \geq 1$. Recall that a real-valued function u on a domain $D \subset \mathbb{C}^d$ is *plurisubharmonic* (psh) in D if it is uppersemicontinuous (usc) and $u|_{D\cap L}$ is subharmonic on components of $D \cap L$ for each complex line $L \subset \mathbb{C}^d$. A set $E \subset \mathbb{C}^d$ is *pluripolar* (polar if $d = 1$) if there exists $u \not\equiv -\infty$ psh (subharmonic if $d = 1$) on a neighborhood of E with $E \subset \{u = -\infty\}$. We write $L(\mathbb{C}^d)$ for the set of u psh in $\mathbb{C}^d$ of at most logarithmic growth; i.e.,

$$L(\mathbb{C}^d) = \{u \text{ psh in } \mathbb{C}^d : u(z) \leq \log|z| + \mathcal{O}(1),\ |z| \to \infty\}.$$

In particular, given a polynomial $p \in \bigcup_n \mathcal{P}_n$, the function $u(z) = (\deg(p))^{-1}\log|p(z)| \in L(\mathbb{C}^d)$. For a bounded subset E of $\mathbb{C}^d$, we define

$$V_E(z) := \sup\{u(z) :\ u \in L(\mathbb{C}^n),\ u \leq 0 \text{ on } E\}. \tag{2}$$

Note that if $E \subset F$ then $V_F \leq V_E$. If K is compact, (2) agrees with the function defined by (1) (cf., [5], Theorem 5.1.7). In this setting, the usc regularization $V_K^*(z) := \limsup_{\zeta\to z} V_K(\zeta)$ is either identically $+\infty$ or else $V_K^* \in L(\mathbb{C}^d)$. We assume throughout

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this note that $V_K^* \not\equiv +\infty$; this occurs precisely when K is not (pluri)polar (cf., [5], Corollary 5.2.2). Moreover, if $V_K^* \not\equiv +\infty$ it is known that $V_K^* \in L^+(\mathbb{C}^d)$ where

$$L^+(\mathbb{C}^d) = \{u \in L(\mathbb{C}^d) : u(z) - \log|z| = \mathcal{O}(1),\ |z| \to \infty\},$$

and that $V_K^* = 0$ on $K \setminus Z$ where Z is a pluripolar set.

Next, we say K is a regular compact set if V_K^* is continuous. There are several known sufficient conditions for regularity of a compact set $K \subset \mathbb{C}^d$; if $d = 1$, this is equivalent to the unbounded component Ω of the complement of K being a regular domain for the Dirichlet problem. We denote by $\mu_K \in \mathcal{M}(K)$ the *equilibrium measure* of K; if $d = 1$, this is the unique measure in $\mathcal{M}(K)$ of minimal logarithmic energy; if $d \geq 1$ it is the Monge-Ampère measure $(dd^c V_K^*)^d$ associated to V_K^*. For $d > 1$ and u a real-valued function of class C^2 on a domain in $\mathbb{C}^d$, $(dd^c u)^d = c_d \det H(u) dV$ where $c_d > 0$ is a dimensional constant,

$$H(u) := [\frac{\partial^2 u}{\partial z_j \bar{\partial} z_k}]_{j,k=1,\ldots,d}$$

is the complex Hessian of u, and dV is Lebesgue measure on $\mathbb{C}^d = \mathbb{R}^{2d}$. In particular, if u is psh, $(dd^c u)^d$ is a positive, absolutely continuous measure. For a locally bounded psh function u, such as V_K^*, one can still define $(dd^c u)^d$ as a positive measure which puts no mass on pluripolar sets, cf., [5]. The support S_K of μ_K is contained in $\partial\Omega$ if $d = 1$ and, in general, is contained in the Shilov boundary of K with respect to the uniform algebra $P(K)$ generated by the polynomials restricted to K, i.e., the smallest closed subset E of K such that for all $p \in P(K)$, $||p||_K = ||p||_E$. In particular, S_K has empty interior. Finally, since $V_K^* \in L^+(\mathbb{C}^d)$ if K is not pluripolar, for such sets, in particular, for regular compact sets, the measure μ_K puts no mass on pluripolar sets.

Given $\mu \in \mathcal{M}(K)$ where $V_K^* \in L(\mathbb{C}^d)$, we consider the following subclasses of $\mathcal{M}(K)$ to which μ may or may not belong:

Definition 2.1. (**Erdös-Turan**) We say $\mu \in \mathcal{M}(K)$ satisfies the Erdös-Turan condition if

$$\frac{d\mu}{d\mu_K} > 0 \text{ almost everywhere with respect to } \mu_K;$$

i.e., writing $d\mu = f\, d\mu_K + d\mu_s$ where $f \in L^1(\mu_K)$ we have $f > 0$ μ_K-a.e. We write $\mu \in E(K)$.

Definition 2.2. (**Szegö**) We say $\mu \in \mathcal{M}(K)$ satisfies the Szegö property if $\mu \in E(K)$ and $\int_K \log f\, d\mu_K > -\infty$. We write $\mu \in S(K)$.

Definition 2.3. (**Determining** [9, 7]) We say $\mu \in \mathcal{M}(K)$ is determining for K if for each Borel subset E of K with $\mu(E) = \mu(K)$ (E is called a *carrier* of μ), we have $V_E^* = V_K^*$. We write $\mu \in D(K)$.

Definition 2.4. (**Bernstein-Markov**) We say that (K, μ) satisfies a Bernstein-Markov property if for all $p_k \in \mathcal{P}_k$,

$$||p_k||_K := \sup_{z \in K} |p_k(z)| \leq M_k ||p_k||_{L^2(\mu)} \text{ with } \limsup_{k \to \infty} M_k^{1/k} = 1.$$

Equivalently, for any sequence $\{p_j\}$ of nonzero polynomials with $\deg(p_j) \to +\infty$,

$$\limsup_{j \to \infty} \left(\frac{||p_j||_K}{||p_j||_{L^2(\mu)}} \right)^{1/\deg(p_j)} \leq 1.$$

We write $\mu \in B(K)$.

Definition 2.5. (**Psh Bernstein-Markov** [2]) We say that (K, μ) satisfies a plurisubharmonic Bernstein-Markov property if for all $\epsilon > 0$, there exists $C = C(\epsilon, K)$ such that for all $p \geq 1$,

$$\sup_K e^{pu} = (\sup_K e^u)^p \leq C(1+\epsilon)^p ||e^u||^p_{L^p(\mu)}$$

for all $u \in L(\mathbb{C}^n)$. We write $\mu \in PB(K)$.

Definition 2.6. (**Reg**) We say $\mu \in$ **Reg** if for any sequence $\{p_j\}$ of nonzero polynomials with $\deg(p_j) \to +\infty$,

$$\limsup_{j \to \infty} \left(\frac{|p_j(z)|}{||p_j||_{L^2(\mu)}} \right)^{1/\deg(p_j)} \leq \exp(V_K^*(z))$$

1. locally uniformly for $z \in \mathbb{C}$; i.e., if $d = 1$;
2. pointwise for $z \in \mathbb{C}^d$ if $d > 1$.

Remark 1. If $d = 1$, $\mu \in$ **Reg** if and only if

$$\limsup_{n \to \infty} \frac{1}{n} \log|p_{n,\mu}(z)| \leq V_K^*(z) \text{ locally uniformly for } z \in \mathbb{C}$$

where $\{p_{n,\mu}\}$ with $\deg(p_{n,\mu}) = n$ are the orthonormal polynomials in $L^2(\mu)$ (Theorem 3.2.1 [8]); if $d > 1$, an analogous pointwise inequality characterization exists (Theorem 3.3 [3]).

Note these definitions make sense in $\mathbb{C}^d$ for any $d \geq 1$ although we have not seen $E(K)$ nor $S(K)$ used unless $d = 1$. Following [6, p.134], we will say that a measure $\mu \in \mathcal{M}(K)$ is C-absolutely continuous if it does not charge pluripolar sets, that is $\mu(Z) = 0$ if Z is pluripolar. We shall denote by $\mathcal{M}'(K)$ the subset of $\mathcal{M}(K)$ of C-absolutely continuous measures, and by $\mathcal{M}_a(K)$ the subset of $\mathcal{M}(K)$ of absolutely continuous measures with respect to μ_K.

We summarize what is known relating the above criteria; cf., [8], Chapter 4, [2], [3], and [4] for the proofs.

Proposition 2.1. *Let $K \subset \mathbb{C}^d$ with $V_K^* \in L(\mathbb{C}^d)$ where $d \geq 1$.*
1) If K is regular, then

$$D(K) \subset B(K) = \mathbf{Reg},$$

and the inclusion is proper. Moreover,

$$D(K) \cap \mathcal{M}'(K) = PB(K) \cap \mathcal{M}'(K) \subset B(K)$$

and the last inclusion is proper.
2) If $d = 1$, for a general compact set K,

$$S(K) \subset E(K) \subset D(K) \subset \mathbf{Reg},$$

and each inclusion is proper.

In Proposition 3.5 below, we prove that for a compact set $K \subset \mathbb{C}^d$, the inclusion $E(K) \subset D(K)$ also holds true.

3 Results

Throughout this section we let $K \subset \mathbb{C}^d$ compact with $V_K^* \in L(\mathbb{C}^d)$. Here, we say a sequence of measures converges weakly (or in the weak-* topology) to a measure μ, and we write $\mu_n \to \mu$, if

$$\lim_{n\to\infty} \int_K f\, d\mu_n = \int_K f\, d\mu, \quad \text{for all } f \in C(K).$$

Lemma 3.1. *For $K \subset \mathbb{C}^d$ as above, $E(K)$ is dense in $\mathcal{M}(K)$; i.e., given $\mu \in \mathcal{M}(K)$, there exists $\{\mu_n\} \subset E(K)$ with $\mu_n \to \mu$ weakly.*

Proof. Let $\mu \in \mathcal{M}(K) \setminus E(K)$. Since measures from $E(K)$ can have arbitrary singular parts, we may assume, without loss of generality, that μ is absolutely continuous with respect to μ_K. Write $d\mu = f\, d\mu_K$ where $f \in L^1(\mu_K)$. Let A be a Borel subset of K with $f > 0$ a.e. with respect to μ_K on $K \setminus A$, $f = 0$ a.e. with respect to μ_K on A, and $\mu_K(A) > 0$. Thus

$$1 = \mu(K) = \int_K f\, d\mu = \int_{K\setminus A} f\, d\mu.$$

Define

$$f_n(z) = 1/n,\ z \in A; \qquad f_n(z) = \big(1 - \mu_K(A)/n\big) f(z),\ z \in K \setminus A,$$

and set $d\mu_n := f_n d\mu_K$. Clearly $f_n \in L^1(\mu_K)$ and $\mu_n \in E(K)$ for each n. If $h \in C(K)$, we have

$$\begin{aligned}\int_K h d\mu_n &= \frac{1}{n}\int_A h d\mu_K + \big(1-\mu_K(A)/n\big)\int_{K\setminus A} hf\, d\mu_K \\ &= \frac{1}{n}\int_A h d\mu_K + \big(1-\mu_K(A)/n\big)\int_K h d\mu \\ &= \frac{1}{n}[\int_A h d\mu_K - \mu_K(A)\int_K h d\mu] + \int_K h d\mu \to \int_K h d\mu, \quad \text{as } n \to \infty.\end{aligned}$$

□

Remark 2. The proof shows in particular that $E(K) \cap \mathcal{M}_a(K)$ is dense in $\mathcal{M}_a(K)$.

In fact, the smaller Szegö class is sufficient to approximate $\mathcal{M}(K)$.

Proposition 3.2. *For $K \subset \mathbb{C}^d$ as above, $S(K)$ is dense in $\mathcal{M}(K)$; i.e., given $\mu \in \mathcal{M}(K)$, there exists $\{\mu_n\} \subset S(K)$ with $\mu_n \to \mu$ weakly.*

Proof. Let $\mu \in \mathcal{M}(K) \setminus S(K)$. Since measures from $S(K)$ can have arbitrary singular parts, we may again assume, without loss of generality, that μ is absolutely continuous with respect to μ_K. Write $d\mu = f\, d\mu_K$ where $f \in L^1(\mu_K)$ with $f \geq 0$ and $\int_K f\, d\mu_K = 1$. Then $\mu \in S(K)$ if and only if $f > 0$ a.e. μ_K and

$$\int_K \log f\, d\mu_K > -\infty.$$

To prove our result, from Lemma 3.1 it suffices to show for $\mu \in E(K) \setminus S(K)$ that there exists $\{\mu_n\} \subset S(K)$ with $\mu_n \to \mu$ weakly. Thus we suppose $d\mu = f\, d\mu_K$ satisfies $f > 0$ a.e. with respect to μ_K but $\int_K \log f\, d\mu_K = -\infty$. To construct μ_n, we let $A_n \subset K$

be a Borel set such that $f > 1/n$ a.e. μ_K on A_n and $f \le 1/n$ a.e μ_K on $K \setminus A_n$. Then $\mu \in E(K)$ implies that $\mu_K(A_n) \nearrow 1$ and $\cup_n A_n =: K \setminus P$ where $\mu_K(P) = 0$. We define

$$f_n(z) = 1/n \text{ for } z \in K \setminus (A_n \cup P) \text{ and}$$

$$f_n(z) = \alpha_n f(z) \text{ for } z \in A_n \cup P \text{ where } \alpha_n := \frac{1-(1/n)\mu_K(K \setminus (A_n \cup P))}{\mu(A_n \cup P)}.$$

It is straightforward that $0 < \alpha_n < 1$ and $\lim_{n\to\infty} \alpha_n = 1$. We now define

$$d\mu_n := f_n d\mu_K.$$

Since $\alpha_n > 0$ and $f > 0$ a.e. with respect to μ_K, we have $f_n > 0$ a.e. with respect to μ_K. To see that $\mu_n \in S(K)$, note by construction $\mu_n \in \mathcal{M}(K)$. Moreover, we have

$$\int_K \log f_n d\mu_K = \int_{K\setminus(A_n\cup P)} \log f_n d\mu_K + \int_{A_n\cup P} \log f_n d\mu_K$$
$$\ge (\log \frac{1}{n}) \cdot \mu_K(K \setminus (A_n \cup P)) + (\log \frac{\alpha_n}{n}) \cdot \mu_K(A_n \cup P) > -\infty.$$

Finally, to show $\mu_n \to \mu$ weakly, let $h \in C(K)$. Then

$$\int_K h d\mu_n = \frac{1}{n} \int_{K\setminus(A_n\cup P)} h d\mu_K + \alpha_n \int_{A_n\cup P} hf\, d\mu_K \to \int_K hf\, d\mu_K = \int_K h d\mu, \quad \text{as } n \to \infty$$

since $\lim_{n\to\infty} \alpha_n = 1$ and $\mu_K(A_n) \nearrow 1$. □

From Proposition 2.1, since $E(K) \subset D(K) \subset B(K)$ for $K \subset \mathbb{C}$ regular, we have the following.

Corollary 3.3. *For $K \subset \mathbb{C}$ regular, $B(K)$ is dense in $\mathcal{M}(K)$.*

Remark 3. The class $B(K)$ is a much larger class than $E(K)$ as, e.g., there exist discrete measures $\mu \in B(K)$, i.e., consisting entirely of atoms. Indeed, any compact set K admits a Bernstein-Markov measure μ with a countable (and hence pluripolar) carrier (so $\mu \in B(K) \setminus D(K)$ [4].

For our next result, we will need a version of the domination principle; cf., [1].

Proposition 3.4. *Let $u \in L(\mathbb{C}^d)$ and $v \in L^+(\mathbb{C}^d)$. Then*

$$u \le v,\ (dd^c v)^d\text{-a.e.} \quad \Longrightarrow \quad u \le v \ \text{ in } \mathbb{C}^d.$$

Proposition 3.4 is not necessarily true if only $v \in L(\mathbb{C}^d)$; indeed, in this case, if $d > 1$, the Monge-Ampère measure $(dd^c v)^d$ is not necessarily well-defined.

We have the following.

Proposition 3.5. *For $K \subset \mathbb{C}^d$ as above, $E(K) \subset D(K)$. Thus, from Lemma 3.1, $D(K)$ is dense in $\mathcal{M}(K)$.*

Proof. Let $\mu \in E(K)$; i.e., $d\mu = f d\mu_K + d\mu_s$ where $f \in L^1(\mu_K)$ with $f \ge 0$ and $f > 0$ μ_K-a.e. Let $E \subset K$ be a Borel set with $\mu(E) = \mu(K) = 1$. Clearly $V_K^* \le V_E^*$. To prove the reverse inequality, let $u \in L(\mathbb{C}^d)$ with $u \le 0$ on E; we want to show that $u \le V_K^*$ in $\mathbb{C}^d$. Suppose first that $\mu_s = 0$. Since $\mu(E) = \mu(K)$, i.e.,

$$\int_E f\, d\mu_K = \int_K f\, d\mu_K,$$

we have, on one hand, $f = 0$ μ_K-a.e. on $K \setminus E$. On the other hand, $f > 0$ μ_K-a.e. Thus,

$$\mu_K(K \setminus E) = \mu_K((K \setminus E) \cap \{f = 0\}) \le \mu_K(\{f = 0\}) = 0$$

which implies that $u \le 0$ μ_K-a.e. Since $V_K^* = 0$ μ_K-a.e. and $V_K^* \in L^+(\mathbb{C}^d)$, by Proposition 3.4 we get $u \le V_K^*$ in $\mathbb{C}^d$.

In the general case, we note that the absolutely continuous part of μ with respect to μ_K, i.e., $f\, d\mu_K$, is determining for K since $f > 0$ μ_K-a.e. Precisely, one should renormalize $f\, d\mu_K$ to get a probability measure; i.e., using $cf\, d\mu_K$ for $c > 1$ so that $\int_K cf\, d\mu_K = \int_K d\mu_K$ so that $cf\, d\mu_K$ is determining for K (the carriers for $f\, d\mu_K$ and $cf\, d\mu_K$ are the same). But if $\mu_s \ne 0$, a subset E of K is a carrier of $d\mu = f\, d\mu_K + d\mu_s$ if and only if it is a carrier of $f\, d\mu_K$ and a carrier of $d\mu_s$. Hence, for E a carrier of $d\mu$, we have $V_E^* \le V_K^*$ by the first part of the proof. □

Remark 4. If $d = 1$, Proposition 3.5 can be proved using logarithmic energy; see pp. 102-103 of [8].

Corollary 3.6. *For $K \subset \mathbb{C}^d$ regular, $PB(K) \cap \mathcal{M}_a(K)$ is dense in $\mathcal{M}_a(K)$.*

Proof. Since $\mu_K \in \mathcal{M}'(K)$, we have $\mathcal{M}_a(K) \subset \mathcal{M}'(K)$. Also, by item 1) of Proposition 2.1, $D(K) \cap \mathcal{M}'(K) = PB(K) \cap \mathcal{M}'(K)$, hence $D(K) \cap \mathcal{M}_a(K) = PB(K) \cap \mathcal{M}_a(K)$. Thus,

$$E(K) \cap \mathcal{M}_a(K) \subset D(K) \cap \mathcal{M}_a(K) = PB(K) \cap \mathcal{M}_a(K),$$

which, together with Remark 2, proves the result. □

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