On minimizing cyclists' ascent times

Len Bos[a] · Michael A. Slawinski[b] · Raphaël A. Slawinski[c] · Theodore Stanoev[b]

Abstract

We prove that, given an average power, the ascent time is minimized if a cyclist maintains a constant ground speed regardless of the slope. Herein, minimizing the time is equivalent to maximizing — for a given uphill — the corresponding mean ascent velocity (VAM: *velocità ascensionale media*), which is a common training metric. We illustrate the proof with numerical examples, and show that, in general, maintaining a constant instantaneous power results in longer ascent times; both strategies result in the same time if the slope is constant. To remain within the athlete's capacity, we examine the effect of complementing the average-power constraint with a maximum-power constraint. Even with this additional constraint, the ascent time is the shortest with a modified constant-speed — not constant-power — strategy; as expected, both strategies result in the same time if the maximum and average powers are equal to one another. Given standard available information — including level of fitness, quantified by the power output, and ascent profile — our results allow to formulate reliable and convenient strategies of uphill timetrials.

1 Introduction

In this article, we formulate a strategy to minimize the time of climbing an uphill with a bicycle, which is the goal for timetrial competitions composed of an uphill; in Italy, they are referred to with the specific name of a *cronoscalata*. We consider that the ascent profile, including distance and steepness, is known. The weight of the cyclist, together with the bicycle, as well as fitness level, quantified by the power output, are also known. With these — and other information, such as the air, rolling and drivetrain resistances — we formulate the aforementioned strategy. The formulation is based on power considerations, which — for the last couple of decades — has been the key metric in training and race preparation. Notably, most modern racing bicycles have power meters.

We begin this article by invoking a phenomenological model to estimate the power required to maintain a given speed. This model is analogous to the one used and discussed by Bos et al. (2023, 2024). Subsequently, we seek the riding strategy that results in the least ascent time, which — under stated constraints and assumptions — we show to be constant speed. We proceed to exemplify this result for several distinct ascent profiles. We conclude by discussing the obtained result, comparing it to another plausible strategy: constant power, and commenting on possible future developments. In the appendices, we provide a method for imposing an optimization constraint, namely, the value of the available average power, justify an aspect of our model simplification, namely, neglecting the power used to increase kinetic energy, examine the effect of imposing the second constraint, namely, the value of the available maximum power, comment on numerical optimization, and conclude with an insight into empirical adequacy of the model.

2 Phenomenological model

We consider a phenomenological model for a bicycle-cyclist system of mass m moving with ground speed V.[1] The effort expended by the cyclist is quantified by power P, which is composed of power P_K, required to change the kinetic energy, $\frac{1}{2}mV^2$, and the power to overcome opposing forces, P_F, as well as to overcome the drivetrain resistance,

$$P = \frac{\overbrace{m\dfrac{dV}{dt}V}^{P_K} + \overbrace{FV}^{P_F}}{1-\lambda}, \tag{1}$$

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[a]Dipartimento di Informatica, Università di Verona, Italy
[b]Department of Earth Sciences, Memorial University of Newfoundland, Canada
[c]Department of Chemistry and Physics, Mount Royal University, Canada
[1]To be consistent with our previous study, (Bos et al., 2023, 2024), where we distinguish between the wheel speed, v, and the centre-of-mass speed, V, herein we use implicitly the latter. In this model, formally, $v \equiv V$; however, conceptually, we consider the motion of the centre of mass.

Figure 1: A cyclist ascending a segment with slope θ

where λ is the drivetrain-resistance coefficient. P_F is affected by change in elevation, as well as the resistance of rolling and of air. Hence, explicitly,

$$P = \cfrac{\overbrace{m\,\dfrac{\mathrm{d}V}{\mathrm{d}t}}^{\text{change in speed}} + \overbrace{m\,g\sin\theta}^{\text{change in elevation}} + \underbrace{\overbrace{\mathrm{C_{rr}}m\,g\cos\theta}^{\text{rolling resistance}} + \overbrace{\tfrac{1}{2}\,\mathrm{C_d A}\,\rho\,V^2}^{\text{air resistance}}}_{\text{normal force}}}{\underbrace{1-\lambda}_{\text{drivetrain efficiency}}}\,V\,, \qquad (2)$$

where g is the acceleration due to gravity, θ is the slope, as illustrated in Figure 1, and ρ is the air density. Thus, the three model parameters are the aforementioned λ, together with the rolling-resistance coefficient, $\mathrm{C_{rr}}$, and the air-resistance coefficient, $\mathrm{C_d A}$.

3 Shortest ascent time

3.1 Formulation

To determine the optimal strategy for minimizing the ascent time, we allow the instantaneous power to vary within the constraint of a given average power to be specified *a priori* based on the cyclist's capacities for the expected duration of the effort. This power can be obtained from the cyclist's power profile (e.g., Leo et al., 2022, Figure 2), as discussed in Appendix A. We assume the ascent to be monotonic, and model the hill by N straight-line segments. Furthermore, we assume that each segment is traversed at a constant speed and we allow speed to be discontinuous between segments.[2] In the limit — as $N \to \infty$ — any realistic hill as well as any speed profile can be approximated with arbitrary accuracy. Yet, our argument presented in this section is independent of the value of N.

We neglect P_K, the power associated with changes in kinetic energy, which is the first term in the numerator of expression (2). This is justified by the fact that, riding uphill, speeds are typically low, and changes in kinetic energy are much smaller than

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[2]The discontinuous speed results in a simplified model that is easier to understand, yet captures the essence of the physics of more sophisticated models. Also, a discretized model makes it easier to impose additional constraints, such as that of maximum power, discussed in Appendix C. We have analyzed models with continuous speed — for which there does not seem to be an analytical solution — and have confirmed by numerical experimentation that the conclusions remain substantially unchanged.

the work done against opposing forces, especially against gravity. An insight into this issue, albeit indirect and *a posteriori*, is presented in Appendix B. Given the low speeds, including changes in kinetic energy would be expected to modify the resulting values only slightly, without substantially altering the conclusions. Incorporating these changes is a subject of future work.

Given the model and assumptions described above, we wish to find—for each segment—the speed, with its corresponding power, that minimize the total ascent time, subject to the constraint of a given average power. In Section 3.2, below, we find the solution that optimizes the time. In Section 3.3, we show that the optimum solution is indeed a global minimum.

3.2 Critical point

Let the jth segment, with length L_j and slope angle θ_j, be traversed with speed V_j and corresponding power $P_j = P_j(\theta_j, V_j)$, where θ_j is a parameter and V_j is a variable. Hence, the ascent time, for the jth segment, is $t_j = L_j/V_j$, and the total time is

$$T = \sum_{j=1}^{N} t_j = \sum_{j=1}^{N} \frac{L_j}{V_j}. \tag{3}$$

We wish to minimize T, subject to the constraint of a given average power,

$$\overline{P} = \frac{W}{T}, \tag{4}$$

where the total work is

$$W = \sum_{j=1}^{N} P_j t_j = \sum_{j=1}^{N} P_j \frac{L_j}{V_j}. \tag{5}$$

Mathematically, the problem is to optimize $T\big(\{V_j\}\big)$ as a function of segment speeds, $\{V_j\}$, subject to the constraint

$$\overline{P}\big(\{V_j\}\big) - P_0 = 0, \tag{6}$$

where P_0 is the specified average power.

Such a constrained optimization problem can be solved using the method of Lagrange multipliers; herein, the Lagrangian function is

$$\mathcal{L}\big(\{V_j\}, \Lambda\big) = T + \Lambda\big(\overline{P} - P_0\big), \tag{7}$$

where Λ is a Lagrange multiplier introduced to enforce the constraint. A necessary condition for a constrained optimum is that the gradient of $\mathcal{L}$ taken with respect to vector $\boldsymbol{\beta} = [V_1, \ldots, V_N, \Lambda]^T$ be zero,

$$\nabla_{\boldsymbol{\beta}} \mathcal{L} = \mathbf{0}; \tag{8}$$

which we write explicitly as

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial V_j} = 0, & j = 1, \ldots, N \\[2mm] \dfrac{\partial \mathcal{L}}{\partial \Lambda} = 0 \end{cases}. \tag{9}$$

Equivalently, condition (8) may be expressed in terms of the gradient taken with respect to vector $\boldsymbol{V} = [V_1, \ldots, V_N]^T$,

$$\begin{cases} \nabla_{\boldsymbol{V}} \mathcal{L} = 0 \\[2mm] \overline{P} - P_0 = 0 \end{cases}. \tag{10}$$

Using definition (7), condition (10) may be written as

$$\begin{aligned} \nabla_{\boldsymbol{V}} T &= -\Lambda \nabla_{\boldsymbol{V}}\big(\overline{P} - P_0\big) \\ &= -\Lambda\left(\frac{1}{T}\nabla_{\boldsymbol{V}} W - \frac{W}{T^2}\nabla_{\boldsymbol{V}} T\right) \\ &= -\frac{\Lambda}{T}\big(\nabla_{\boldsymbol{V}} W - P_0 \nabla_{\boldsymbol{V}} T\big), \end{aligned} \tag{11}$$

where to obtain the second equality we use definition (4), and to obtain the third equality, we use expression (6). Using equations (3) and (5) for T and W, respectively, expressed in terms of the segment speeds, V_j, we obtain

$$\frac{\partial T}{\partial V_j} = -\frac{L_j}{V_j^2}, \qquad j = 1, \ldots, N \tag{12}$$

and

$$\frac{\partial W}{\partial V_j} = \frac{\partial P_j}{\partial V_j}\frac{L_j}{V_j} - P_j\frac{L_j}{V_j^2}, \qquad j = 1, \ldots, N. \tag{13}$$

Substituting into equation (11) and simplifying, we obtain

$$T = \Lambda \left(V_j \frac{\partial P_j}{\partial V_j} - P_j + P_0 \right), \qquad j = 1, \ldots, N. \tag{14}$$

Using model (2) — without the first term in the numerator — and the assumption that the speed for each segment is constant, the power for the jth segment may be written as

$$P_j = \frac{m\,g\,\sin\theta_j\,V_j + m\,g\,C_{rr}\cos\theta_j\,V_j + \frac{1}{2}C_d A \rho\,V_j^3}{1 - \lambda}, \qquad j = 1, \ldots, N. \tag{15}$$

Using expressions (15), we can write conditions (14) as

$$T = \Lambda \left(\frac{C_d A \rho}{1 - \lambda} V_j^3 + P_0 \right), \qquad j = 1, \ldots, N, \tag{16}$$

where the total ascent time, stated in expression (3) is a function of all the segment speeds, $T = T\left(\{V_j\}\right)$. System (16) may be solved numerically for the segment speeds that optimize the ascent time. However, an immediate consequence of conditions (16) is that

$$T = \Lambda \left(\frac{C_d A \rho}{1 - \lambda} V_i^3 + P_0 \right) = \Lambda \left(\frac{C_d A \rho}{1 - \lambda} V_j^3 + P_0 \right), \tag{17}$$

for any pair of segments i and j. It follows immediately that $V_i = V_j$, for any pair of segments. In other words, *for a given average power, the critical ascent time is achieved with a constant ground speed regardless of the slope angle*. This conclusion is a consequence of the following:

1. The power expended against gravity and rolling resistance, both of which depend on the slope angle of segment θ_j, is linear in the segment speed, V_j.

2. The coefficients of air resistance, together with the air density, $C_d A \rho$, as well as the drivetrain efficiency, $1 - \lambda$, are segment-independent.

If either of these conditions were not satisfied, the critical speed — while it could still be found by numerically solving system (14) — would not be constant.

3.3 Minimum

System (16) is a necessary condition for a point in the N-dimensional space of segment speeds $\{V_j\}$ to be a critical point of Lagrangian (7), in other words, to be an extremum of time (3), subject to constraint (6). To prove that the solution found in Section 3.2 is indeed a constrained minimum, we note that there is only one critical point, which is a consequence of the cubic relationship between power and velocity of model (2). Hence, the minimum we seek is either at this critical point or else is on the boundary of the region of feasible candidates.

The lower limit on a segment speed, V_j, is the requirement that the cyclist not go backwards, $V_j \geqslant 0$. The upper limit is that value of V_j for which the work done over the jth segment, $W_j = P_j t_j$, is maximized. Mathematically, there is no such maximum; yet, in practice, the capacity of a rider for work is finite. Let us call this maximum work W_{max} and, hence, the resulting subspace of the space of segment speeds is defined by the condition that

$$W\left(\{V_j\}\right) \leqslant W_{max}, \tag{18}$$

where the total work is given by expression (5). The work done over the kth segment is maximized if the speed on that segment, V_k, is a maximum and the speeds on the other segments, V_j, where $j \neq k$, are zero, subject to condition (18). Hence, on the boundary of the feasibility region, at least one of the speeds is zero. However, if at least one of the speeds is zero, the total ascent time is infinity. Since, by hypothesis, the total work on the boundary of the feasibility region is finite, $W = W_{max} < \infty$, the average power on the boundary is zero,

$$\overline{P} = \frac{W}{T} = 0. \tag{19}$$

Therefore the constraint of a given average power cannot be satisfied on the boundary of the feasibility region, and so the minimum we seek cannot lie on the boundary. It follows that the constant-speed solution is a global minimum of the constrained ascent time.
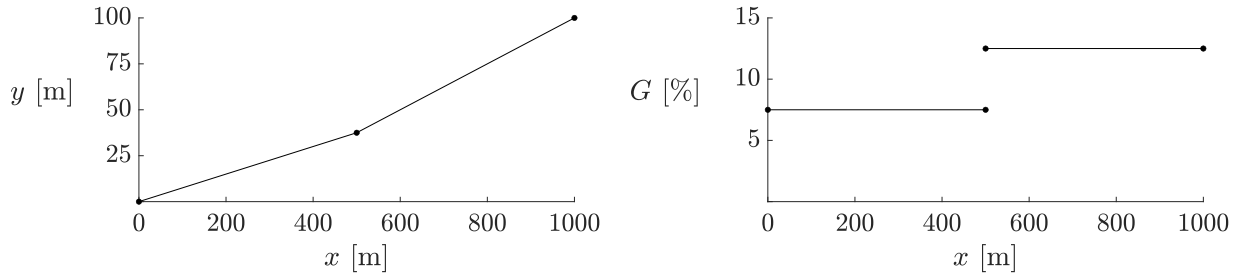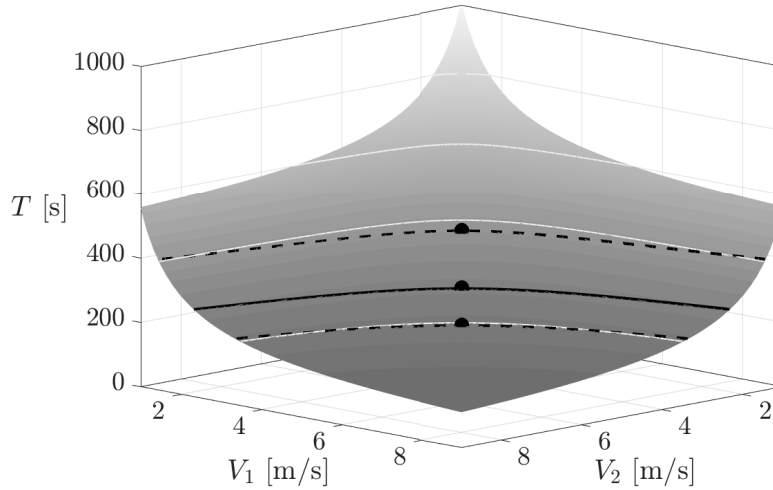
Figure 2: Two-segment ascent and grade profile

Figure 3: Ascent-time surface: black lines correspond to curves of a given time-averaged power; black dots therein correspond to the $V_1 = V_2$ speeds that are solutions of the ascent-time minimization with a constraint of a given power; white lines are level curves of ascent time.

3.4 Ascent-time surface

To illustrate the result that a constant speed results in the shortest traveltime, we consider a two-segment ascent, $N = 2$. We parameterize each segment by its grade, $G_j = \tan\theta_j \cdot 100\%$, where $j = 1, 2$, which is the tangent of the slope angle, θ_j. We set the initial coordinates at the origin, $x_0 = 0\,\mathrm{m}$ and $y_0 = 0\,\mathrm{m}$, and the final ones at $x_2 = 1000\,\mathrm{m}$ and $y_2 = 100\,\mathrm{m}$. The intermediate coordinates are $x_1 = 500\,\mathrm{m}$ and $y_1 = 37.5\,\mathrm{m}$, which is tantamount to grade percentages of $G_1 = 7.5\%$ and $G_2 = 12.5\%$ and an average of 10%. With these ascent parameters, the length of each segment is $L_1 = 501.4043\,\mathrm{m}$ and $L_2 = 503.8911\,\mathrm{m}$, which amounts to a total ascent distance of 1005.2954 m. We plot the ascent, and its associated percent grades, in Figure 2.

To calculate the ascent time, stated in expression (3), we let $m = 70\,\mathrm{kg}$, $g = 9.81\,\mathrm{m/s^2}$, $\rho = 1.2\,\mathrm{kg/m^3}$, $\mathrm{C_dA} = 0.3\,\mathrm{m^2}$, $\mathrm{C_{rr}} = 0.005$ and $\lambda = 0.02$. In Figure 3, we plot the resulting ascent-time surface as a function of segment speeds, V_1 and V_2. Therein, the black curves correspond to three distinct constraints, given by the time-averaged power. The continuous black line corresponds to $\overline{P} = 300\,\mathrm{W}$, used in subsequent examples. The dashed lines correspond to $\overline{P} = 200\,\mathrm{W}$ and $\overline{P} = 400\,\mathrm{W}$. The level curves of ascent time are white. The greater the power the shorter the ascent time. For each constraint, the minimum ascent times is achieved with $V_1 = V_2$, which correspond to black dots: for $\overline{P} = 200\,\mathrm{W}$, $V_1 = V_2 = 2.6842\,\mathrm{m/s}$ and $T = 374.5267\,\mathrm{s}$; for $\overline{P} = 300\,\mathrm{W}$, $V_1 = V_2 = 3.9450\,\mathrm{m/s}$ and $T = 254.8288\,\mathrm{s}$; for $\overline{P} = 400\,\mathrm{W}$, $V_1 = V_2 = 5.1272\,\mathrm{m/s}$ and $T = 196.0724\,\mathrm{s}$.

4 Application to arbitrary ascent profiles

4.1 Ascent profiles

As stated in Section 3, the method presented therein can be applied to any uphill by discretizing it into N straight-line segments. In this section, we apply the method to various ascent profiles. To compare the optimal strategy of constant speed and variable power with another strategy, we include the ascent times under the assumption of constant power and variable speed.

To generate continuous ascent profiles, which can be sampled with arbitrary resolution, we consider $(n+1)$ nodes in $x\,G$-plane, where x corresponds to the horizontal distance and G is the grade, which is the tangent of the slope angle. We subdivide the distance with strictly increasing coordinates, $x_0 < x_1 < \cdots < x_n$. Along the ith segment, we define the grade as a piecewise linear

(a) Zero concavity

(b) Positive concavity

(c) Negative concavity

(d) Positive then negative concavity
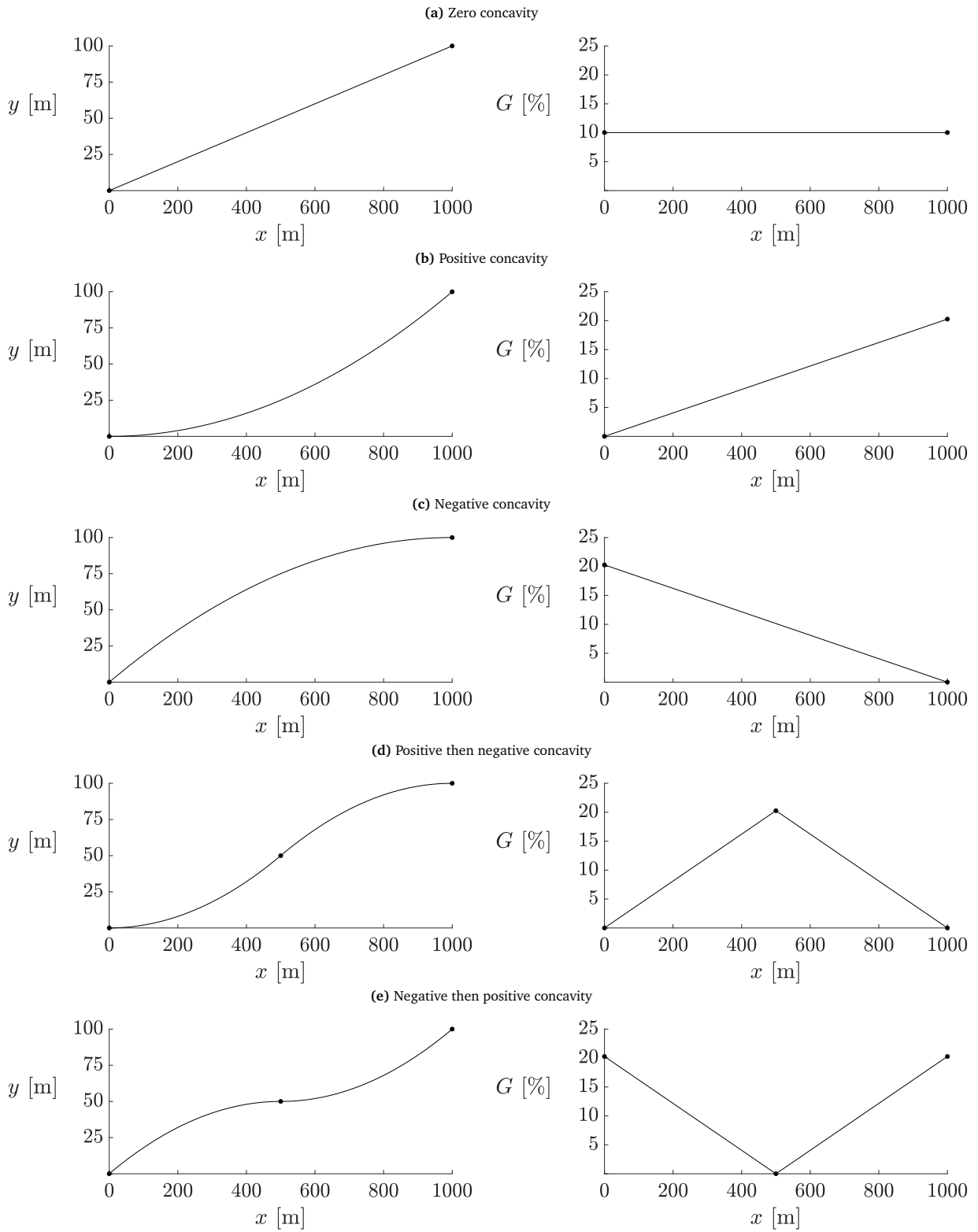
(e) Negative then positive concavity

Figure 4: Ascent and grade profiles of varying concavity

interpolation of its endpoints,

$$\mathcal{G}_i(X) = G_{i-1} + \left(\frac{G_i - G_{i-1}}{x_i - x_{i-1}} \right)(X - x_{i-1}), \qquad x_{i-1} \leqslant X \leqslant x_i. \tag{20}$$

Through integration of the grade, we obtain the vertical coordinates of the ascent in the xy-plane, where y is the elevation. Thus, the change in elevation along the ith segment is

$$\Delta y_i(X) = \int_{x_{i-1}}^{X} \mathcal{G}_i(\xi) \, d\xi = G_{i-1}(X - x_{i-1}) + \frac{1}{2}\left(\frac{G_i - G_{i-1}}{x_i - x_{i-1}} \right)(X - x_{i-1})^2, \qquad x_{i-1} \leqslant X \leqslant x_i, \tag{21}$$

which, at the endpoint, is

$$\Delta y_i(x_i) = \left(\frac{G_{i-1} + G_i}{2} \right)(x_i - x_{i-1}). \tag{22}$$

Therefore, the ith vertical ascent coordinate is $y_i = y_{i-1} + \Delta y_i(x_i)$ and, given initial coordinates x_0 and y_0, the final vertical coordinate is

$$y_n = y_0 + \sum_{i=1}^{n} \Delta y_i(x_i). \tag{23}$$

The average grade percentage for the entire ascent is

$$\overline{\mathcal{G}} = \left(\frac{y_n - y_0}{x_n - x_0} \right) \cdot 100\% \tag{24}$$

and the ascent distance is

$$L = \sum_{i=1}^{n} \left(\int_{x_{i-1}}^{x_i} \sqrt{1 + \mathcal{G}_i(X)^2} \, dX \right). \tag{25}$$

For the numerical examples, we consider five ascents with a common average grade of $\overline{\mathcal{G}} = 10\%$, with the same initial and final coordinates as specified in Section 3.4. The ascents differ by their concavity, shown in Figure 4.

The first ascent has zero concavity. To calculate the required grade, we use $n = 1$, set $G_0 = G_n$ in expression (23) and solve for

$$y_1 = y_0 + \left(\frac{G_0 + G_0}{2} \right)(x_1 - x_0) \implies G_0 = \frac{y_1 - y_0}{x_1 - x_0} = 0.1. \tag{26}$$

According to expression (25), the ascent length is

$$L = \int_{0}^{1000} \sqrt{1 + 0.1^2} \, dx = \sqrt{1.01}\,(1000 - 0) = 1004.9876 \, \text{m}. \tag{27}$$

The second ascent has positive concavity. In a similar manner, we use $n = 1$, but set $G_0 = 0$ in expression (23), and solve for $G_1 = 2(y_1 - y_0)/(x_1 - x_0) = 0.2$. For the third ascent, which has negative concavity, we set $n = 1$ and $G_1 = 0$, and solve for $G_0 = 0.2$. For fourth ascent, which has positive then negative concavity, we set $n = 2$, $x_1 = (x_2 - x_0)/2 = 500$ m, and $G_0 = G_2 = 0$, and solve for $G_1 = 0.2$. For the fifth ascent, which has negative then positive concavity, we set $n = 2$, $x_1 = 500$ m, and $G_1 = 0$, and solve for $G_0 = G_2 = 0.2$. Using expression (25), the ascent distance for the latter four ascents is 1006.6272 m.

4.2 Speed

As demonstrated in Section 3, the shortest ascent time is achieved by constant speed. In this case, constraint (6) — using time (3), work (5) and $V_j = V$ for $j = 1, \ldots, N$ — is

$$0 = \overline{P} - P_0 = \frac{\sum_{j=1}^{N} P_j L_j}{\sum_{j=1}^{N} L_j} - P_0. \tag{28}$$

Using model (2), the constraint becomes

$$0 = \frac{\sum_{j=1}^{N} (\alpha_j + \beta V^2) V L_j}{\sum_{j=1}^{N} L_j} - P_0, \tag{29}$$

where

$$\alpha_j = \frac{m g \left(C_{rr} \cos \theta_j + \sin \theta_j \right)}{1 - \lambda} \quad \text{and} \quad \beta = \frac{\frac{1}{2} C_d A \rho}{1 - \lambda}. \tag{30}$$

Through algebraic manipulation, we write the constraint as a depressed cubic equation

$$0 = V^3 + pV + q,\tag{31}$$

where

$$p = \frac{\displaystyle\sum_{j=1}^{N}\alpha_j L_j}{\beta \displaystyle\sum_{j=1}^{N} L_j} \quad\text{and}\quad q = -\frac{P_0}{\beta}.\tag{32}$$

Using Cardano's formula (e.g., Tanton, 2005, pp. 112–113), the solution of the cubic is

$$V = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.\tag{33}$$

Since $p > 0$ for $\theta_j \in [\,0, \pi/2\,)$, the discriminant of the cubic is strictly positive, which means that the equation has one real and two complex roots. In other words, the constraint is the intersection of a cubic monomial and a line of negative slope $V^3 = -pV - q$, which only has one real solution. Moreover, since the vertical intercept of the line is $-q = P_0/\beta > 0$, the real solution is positive, consistent with its physical interpretation as a speed.

Ascent	Constant-speed case			Constant-power case ($P_j = 300\,\text{W}$)		
	T [s]	V [m/s]	$\max\{P_j\}$ [W]	T [s]	$\overline{V}_j$ [m/s]	$\max\{V_j\}$ [m/s]
(a)	254.8206	3.9439	300.0000	254.8206	3.9439	3.9439
(b)	254.8642	3.9497	567.6278	263.7539	3.8165	11.2361
(c)	254.8642	3.9497	567.6278	263.7539	3.8165	11.2361
(d)	254.8642	3.9497	567.6018	263.7539	3.8165	11.2351
(e)	254.8642	3.9497	567.6018	263.7539	3.8165	11.2351

Table 1: Ascent times for five ascent profiles

In a similar manner, the constant-power case requires $P_j - P_0 = 0$ for $j = 1, \ldots, N$. Using model (2), the constraint on the jth segment is also a depressed cubic equation,

$$0 = V_j^3 + \frac{\alpha_j}{\beta} V_j - \frac{P_0}{\beta}.\tag{34}$$

Therefore, we calculate the jth speed in the constant-power case with expression (33), where we replace p with $p_j = \alpha_j/\beta$.

4.3 Times

In this section, to calculate the ascent times, we consider the bicycle-cyclist system with the parameter values as specified in Section 3.4. We specify the time-average power as $\overline{P} = 300\,\text{W}$.

For each ascent, we discretize the ascent coordinates using $N = 10\,000$ straight-line segments. For the constant-speed case, we calculate the speed, V, using expression (33) with constant-speed p and q coefficients of expressions (32). For the constant-power case, we calculate the speed on the jth segment, V_j using expression (33) with $p_j = \alpha_j/\beta$ in place of p.

In both cases, we calculate the ascent time, T, using expression (3) and tabulate the results in Table 1: T is the shortest for ascent (a) because the distance travelled is the shortest; however, T is longer for ascents (b)–(e) because their distances are longer. T is the same for ascents (b)–(e) because their distances are equal, as stated in Section 4.1. Also, we present the power and speed along each ascent for both cases in Figure 5.

5 Discussion and conclusions

The ascent time is the shortest for the constant-speed case. For a constant slope, shown in Figure 4a, constant speed is tantamount to constant power; hence, as shown in the first row of Table 1, the ascent times are equal to one another for both cases. For the remaining rows of this table, the constant-speed case results in a time shorter by about 3.5 %.

The difference between the two strategies increases with the steepness of a portion of an ascent. Let us consider a three-segment ascent whose overall grade is 10 %, with individual segments of 2.5 %, 25 % and 2.5 %. Letting $\overline{P} = 300\,\text{W}$, for a constant-speed strategy, we get $V = 3.9631\,\text{m/s}$ and $T = 254.9668\,\text{s}$, with power of individual segments of 94.7177 W, 698.4306 W and 94.7177 W. For a constant-power strategy, $T = 276.3511\,\text{s}$, with speed of individual segments of 8.6395 m/s, 1.7252 m/s and 8.6395 m/s; since the speed is the same at the bottom and on the top, there is no change in kinetic energy. In this example, the constant-power strategy is slower by about 8 %. For either strategy, variations between the lowest and highest
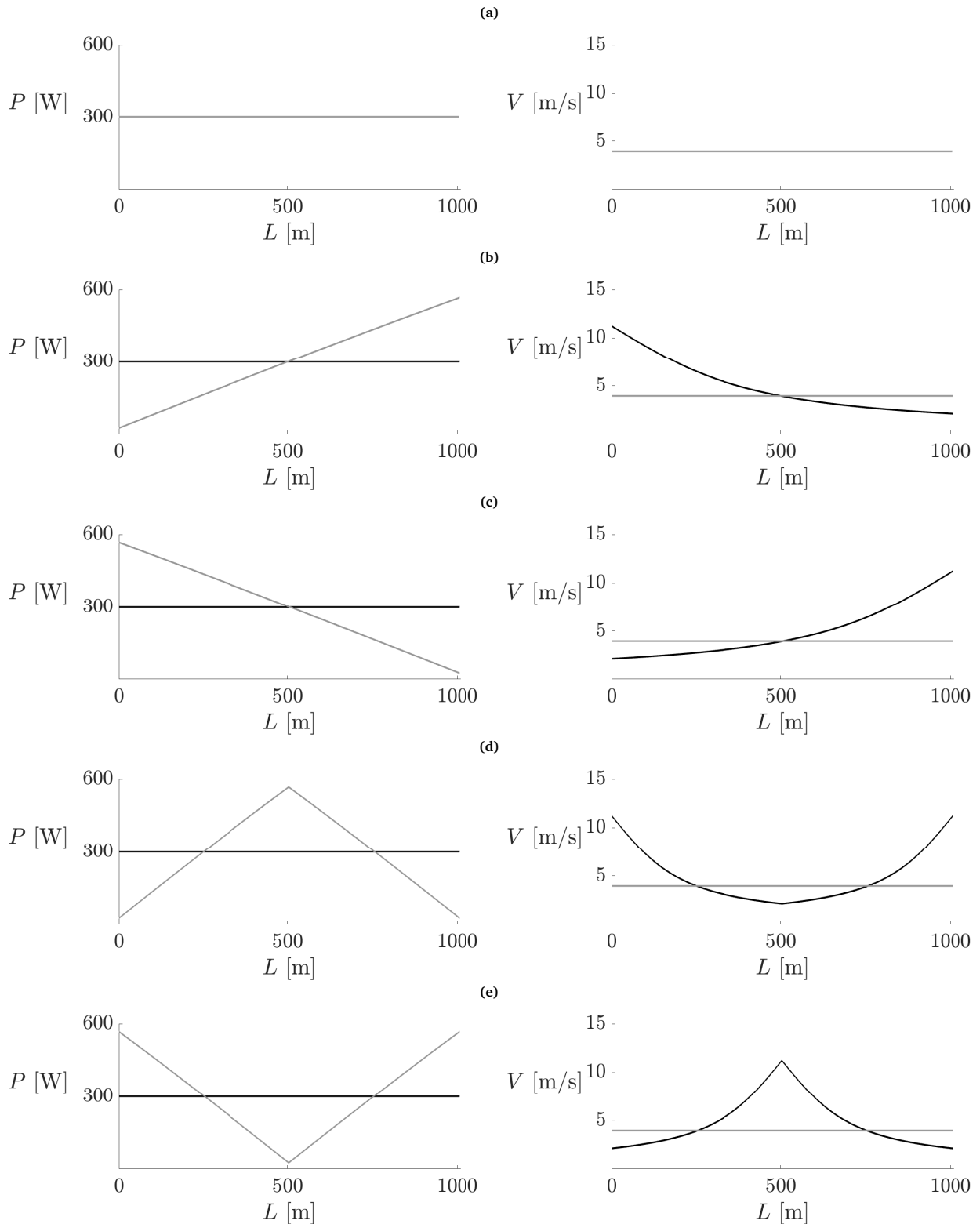
Figure 5: Power and speed corresponding to the five ascent profiles in Figures (4a) – (4e). Black lines correspond to the constant-power solution and grey lines to the constant-speed solution; for profile (4a), both have the same solution. In contrast to Figure 4, the horizontal axis is the distance travelled, not its horizontal component.
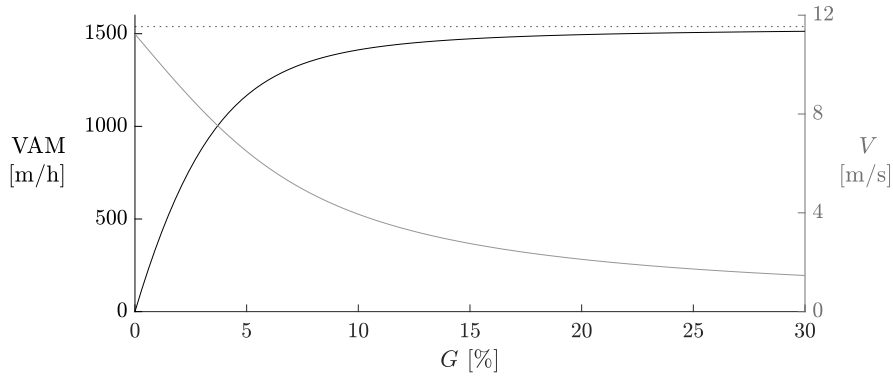
Figure 6: VAM and corresponding ground speed as a function of average power

power or speed might be exceedingly large for pragmatic considerations but this example allows us to compare the two strategies and to gain an insight into adjusting them for a given course.

Commonly, the two strategies — albeit distinct in requirements imposed upon a rider — result in similar ascent times, which might allow one to choose the approach preferable for particular circumstances. Examining similarities of time, however, we need to comment on the assumption discussed in Appendix B. Not including, in a given constant power, a portion expended to increase kinetic energy — as is the case for ascent (4c) — ignores the fact that this portion is not available for speed. Hence, the resulting constant-power ascent time is an underestimate; without this assumption, the difference between the two strategies is expected to be greater. The opposite is true for ascent (4b), where the constant-power ascent time is an overestimate. Including a consideration of kinetic energy is a part of our subsequent work.

Minimizing the ascent time is equivalent to maximizing — for a given uphill — the corresponding mean ascent velocity (VAM: *velocità ascensionale media*), which has typically units of vertical metres per hour. Our approach is valid, in general, given an arbitrary uphill. For a VAM problem, however, we would also look for a specific ascent profile to maximize the speed of gaining altitude, as discussed by Bos et al. (2021).

Herein we gain certain insights into the VAM problem. We consider a straight-line ascent with a vertical gain of 100 metres akin to ascent (a), but as a function of grade percentage such that $0 < G < 30$ %. As shown in Figure 6, under the constraint of a given average power, $\overline{P}$, there is an upper bound for the vertical speed, VAM, since — as the grade increases and the speed decreases — the expended power goes into climbing and overcoming the air and drivetrain resistances, not to doing work against rolling resistance. In accordance with expression (2), for $G \to \infty$ and, hence, $\theta \to \pi/2$,

$$P = \frac{m\, g\, \frac{\text{VAM}}{3600} + \frac{1}{2}\, C_d A \rho\, V^3}{1 - \lambda}\,, \tag{35}$$

where

$$\text{VAM} := \frac{3600\,\text{s}}{\text{h}}\, V \sin\theta. \tag{36}$$

Hence, with $\lambda = 0.02$, $m = 70\,\text{kg}$, $g = 9.81\,\text{m/s}^2$ and $\overline{P} = 300\,\text{W}$,

$$\text{VAM} = 3600\, \frac{(1-\lambda)\overline{P} - \frac{1}{2}\, C_d A \rho\, V^3}{m\, g} = 1541.2104\,\text{m/h}\,, \tag{37}$$

which is the sought upper bound, shown as the dotted line in Figure 6.

Furthermore, as shown in Figure 6 — for a given power — VAM increases monotonically with the grade, which is consistent with Bos et al. (2021, Theorem 1). The increase is not linear: VAM increases quickly for $0 < G < 10$ %, but the gains in VAM are marginal for $G > 10$ %. Also, as shown in Table 1, the shortest time corresponds to ascent (4a), which is a straight line; again, this is consistent with Bos et al. (2021, Theorem 2), as well as illustrated in Figure 1, on page 6.

For the proof presented in Section 3, with its results illustrated in Section 4, the only constraint of the sought minimum is the average power. Within this sole constraint, the optimal constant speed might require excessive power on steeper portions of the ascent. As shown on the left-hand plots of Figures (5b) – (5e), the maximum power reaches almost 570 W, as specified in Table 1, which might be more than a rider can generate given that $\overline{P} = 300$ W. The effect of including an upper power limit as another constraint is examined in Appendix C, and its numerical implementation is discussed in Appendix D.

Following recent measurements, in Appendix E, we examine briefly the empirical adequacy of our model. We do so for the constant-power strategy, which is easier to keep for a rider since it corresponds to steady effort.
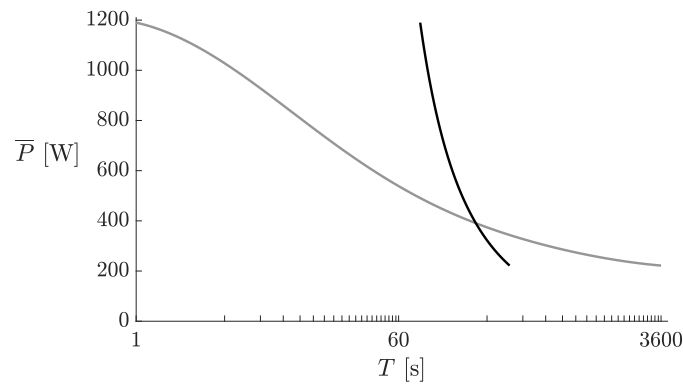
Figure A.1: Cyclist's power profile, in grey, and $(T, \overline{P})$ pairs for minimum ascent times with power-average constraints, $200\,\mathrm{W} \leqslant \overline{P} \leqslant 1200\,\mathrm{W}$; a logarithmic scale is used on the time axis.

Acknowledgements

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A   Obtaining average-power constraint from power profile

In Section 4.3, we choose the constraint, given by the time-average power, to be $\overline{P} = 300\,\mathrm{W}$. In this appendix, we show a method of estimating $\overline{P}$ using a power profile, which quantifies a cyclist's power-duration relationship, and is generated from power-meter measurements collected over a period of a ride, a week, and a season.

The grey curve in Figure A.1 is a generic power profile (e.g., Leo et al., 2022, Figure 2). The black curve is composed of $(T, \overline{P})$ pairs calculated for ascent (4a), with power-average constraints of $200\,\mathrm{W} \leqslant \overline{P} \leqslant 1200\,\mathrm{W}$ and $m = 70\,\mathrm{kg}$, $g = 9.81\,\mathrm{m/s^2}$, $\rho = 1.2\,\mathrm{kg/m^3}$, $\mathrm{C_dA} = 0.3\,\mathrm{m^2}$, $\mathrm{C_{rr}} = 0.005$, $\lambda = 0.02$, by the method introduced in Section 3.

For each power constraint, $\overline{P}$, we calculate the ground speed, V, and ascent time, T, using expressions (33) and (3), respectively. Since the grade of ascent (4a) is constant, the coefficients of equation (31) are the same for the constant-power and constant-speed strategies, which results in the same V.

The highest average-power values — to be used as the constraint for a given ascent time — are along the grey curve. The shortest-ascent times — corresponding to a given average power — are along the black line. Hence, the intersection of these curves corresponds to the best performance possible for a given cyclist. If, for a given ascent time, $\overline{P}$ is below the grey curve, the rider could expend more power; if it is above, it is beyond the rider's capacity.

The sought intersection is calculated numerically by an iterative process. Herein, its coordinates are $T = 200.2343\,\mathrm{s}$ and $\overline{P} = 390.6689\,\mathrm{W}$. In accordance with expressions (33) and (36), these values correspond to $V = 5.0191\,\mathrm{m/s}$ and $\mathrm{VAM} = 1797.9005\,\mathrm{m/h}$, respectively.

B Power estimate for changes in kinetic energy

As stated in Section 3.1, in this appendix we wish to gain an insight — albeit indirect and *a posteriori* — into the justification of neglecting the power associated with changes in kinetic energy. The insight is indirect because to estimate changes in kinetic energy we consider constant-power, not constant-speed cases, which are the sought solution; this solution necessarily results in $P_K \equiv 0$. It is *a posteriori* because we use the formulation wherein changes in kinetic energy are neglected *a priori*. Yet, we claim that our reasoning is not circular; it remains a reasonable estimate of the error committed.

To gain an insight into the power expended to increase kinetic energy, we use the fact that — for conservative forces — we need to consider only the initial and final states. As we see in the right-hand side column of Figure 5 — and considering the cases of constant power — the same speed at the bottom and at the top of the ascent corresponds to profiles (a), (d), (e) of Figure 4. As shown in the right-hand side column of Figure 4, the same speed — for constant power — is tantamount to the same slope. For these profiles, the change of kinetic energy is zero, since

$$\Delta K = \frac{m}{2}\left(V_N^2 - V_0^2\right) = 0\,\text{J}, \tag{B.1}$$

as expected.

Let us consider profile (c) in Figure 4, which is steeper at the bottom than at the top. In the constant-power case, the speed at the bottom is $V_0 = 2.1175\,\text{m/s}$ and the speed at the top is $V_N = 11.2361\,\text{m/s}$. Hence, the change in kinetic energy is

$$\Delta K = \frac{m}{2}\left(V_N^2 - V_0^2\right) = 4261.8467\,\text{J} \tag{B.2}$$

and — given $T = 263.7539\,\text{s}$, in Table 1 — the power required for this increase is

$$P_K = \frac{1}{1-\lambda}\frac{\Delta K}{T} = 16.4882\,\text{W}. \tag{B.3}$$

Thus, P_K is an order of magnitude less than the specified average power of $\overline{P} = 300\,\text{W}$.

Examining expression (B.3), we see that P_K is inversely proportional to the ascent time. Hence, neglecting changes in kinetic energy becomes more justifiable for longer ascents.

To quantify this statement, let us return to profile (c) and scale the horizontal and vertical distances by $1/2$; in other words, we specify $x_n = 500\,\text{m}$ and $y_n = 50\,\text{m}$, which maintains the overall grade percentage of $\overline{\mathcal{G}} = 10\%$. Also, the speeds at the bottom and top do not change. However, the ascent time is halved to $T = 131.8770\,\text{s}$ and, hence, the P_K is doubled to $P_K = 32.9764\,\text{W}$. If we double the horizontal and vertical distances, specifying $x_n = 2000\,\text{m}$ and $y_n = 200\,\text{m}$, which also maintains the overall grade percentage, the speeds at the bottom and top do not change but the ascent time is doubled to $T = 527.5078\,\text{s}$ and, hence, the P_K is halved to $P_K = 8.2441\,\text{W}$, which is two orders of magnitude less than the specified average power.

C Average-power constraint subject to maximum power

As stated in Section 3.3, a constant speed is a global minimum of the ascent time, if the only constraint is the average power. However, this solution can require power that might exceed an athlete's capacity. In this appendix, we complement the average-power constraint, $\overline{P} = P_0$, with a maximum power constraint $P_j \leqslant P_{\max}$, for $j = 1, \ldots, N$.

Let us call a segment extremal if, in the original problem with only the average power constraint, $P_j > P_{\max}$. If so, then for that segment we set $P_j = P_{\max}$. As such, the region over which we minimize the ascent time is the space of segment speeds for which $P_j \leqslant P_{\max}$; consequently, the segments with $P_j > P_{\max}$ lie outside of the minimization region. With the additional constraint of maximum power, the subset of extremal segments becomes

$$\mathcal{J} = \{\, j \mid P_j = P_{\max}\}, \qquad j = 1, \ldots, N. \tag{C.1}$$

In other words, on extremal segments, the speeds, $V_{\mathcal{J}}$, are fixed; their magnitudes are determined using equation (34), in the same manner as the constant-power case, except that P_0 is replaced with $P_{\max}$. Consequently, the time to traverse, as well as the total work done over, all extremal segments is fixed; the former is

$$T_{P_{\max}} = \sum_{\substack{j=1 \\ j \in \mathcal{J}}}^{N} \frac{L_j}{V_j}, \tag{C.2}$$

and the latter is

$$W_{P_{\max}} = \sum_{\substack{j=1 \\ j \in \mathcal{J}}}^{N} P_{\max}\,\frac{L_j}{V_j} = P_{\max}\,T_{P_{\max}}. \tag{C.3}$$
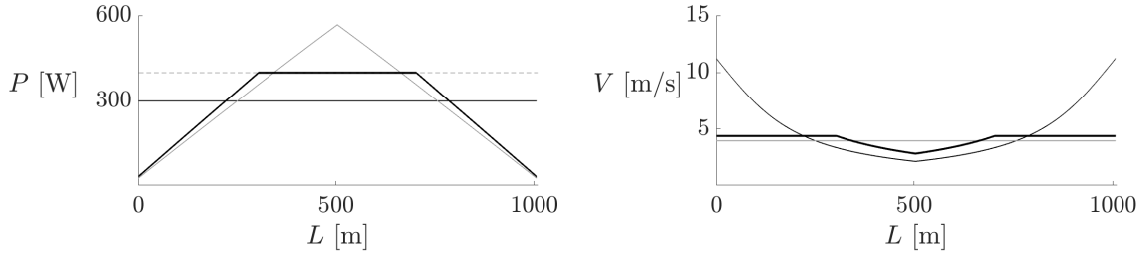
Figure C.1: Power and speed corresponding to ascent (4d): Thin black lines correspond to the constant-power solution, $P_j = 300\,\text{W}$, grey lines to the constant-speed solution, subject to $\overline{P} = 300\,\text{W}$, thick black lines to the constant-speed solution subject to both $\overline{P} = 300\,\text{W}$ and $P_{\max} = 400\,\text{W}$.

With time (C.2) and work (C.3), expression (4) becomes

$$\overline{P} = \frac{W}{T} = \left(\sum_{\substack{j=1 \\ j \notin \mathcal{J}}}^{N} \frac{L_j}{V_j} + T_{P_{\max}} \right)^{-1} \left(\sum_{\substack{j=1 \\ j \notin \mathcal{J}}}^{N} P_j \frac{L_j}{V_j} + W_{P_{\max}} \right), \tag{C.4}$$

which is the corresponding time-average power for the entire ascent. For the submaximal segments, the speeds are determined by constraint (6) and, hence, have to be constant. Thus, following the procedure presented in Section 4.2, we rewrite the power constraint—using $V_j = V$ for $j = 1, \dots, N$ such that $j \notin \mathcal{J}$—as

$$0 = V^3 + p\,V + q\,, \tag{C.5}$$

which is equation (31) with

$$p = \left(\beta \sum_{\substack{j=1 \\ j \notin \mathcal{J}}}^{N} L_j \right)^{-1} \left(\sum_{\substack{j=1 \\ j \notin \mathcal{J}}}^{N} \alpha_j L_j + W_{P_{\max}} + P_0\,T_{P_{\max}} \right) \tag{C.6}$$

and q the same as in expression (32).

Let us revisit ascent (4d), for which we calculate ascent times in Section 4.3 under the constraint of $\overline{P} = 300\,\text{W}$ and using a discretization of $N = 10\,000$ straight-line segments. As stated in Table 1, the ascent time of $T = 254.8642\,\text{s}$ is achieved with a constant speed of $V = 3.9497\,\text{m/s}$ and a maximum power of $\max\{P_j\} = 567.6278\,\text{W}$.

To gain insight to the effect of $P_{\max}$, let us restrict the power to $P_{\max} = 400\,\text{W}$. In accordance with expression (C.4), there are 3932 extremal segments, along which $T_{P_{\max}} = 116.3710\,\text{s}$ and $W_{P_{\max}} = 46548.4088\,\text{J}$. For the submaximal segments, the constant speed is $V = 4.3758\,\text{m/s}$, which results in the ascent time of $T = 255.3814\,\text{s}$. In this example, which we illustrate in Figure C.1, the effect of $P_{\max}$ is to increase the ascent time due to extremal segments. If $P_{\max} > \max\{P_j\}$, there are no extremal segments and the ascent time is the same as that of the constant-speed solution. If $P_{\max} \to \overline{P}$, all segments become extremal and the ascent time is the same as for the constant-power solution. Thus—even with the $P_{\max}$ constraint—the ascent time is the shortest with, albeit restricted, constant-speed strategy.

Examining Figures (4d) and (5d), we see that the power along the less steep parts of the ascent is much lower than one might expect. However, constraining the power from below on such segments lowers the work budget for the remaining segments and results in an increased time, which is an intrinsic consequence of the average-power constraint.

D   Numerical optimization considerations

Throughout this paper, we use numerical methods to obtain segment speeds that minimize the ascent time subject to nonlinear constraints. These speeds can be obtained numerically using any constrained numerical optimization program, such as provided by Matlab. However, this approach might be inefficient for ascent discretizations with large N. As such, the purpose of this appendix is to demonstrate the efficiency of our formulations, as opposed to standard numerical procedures, for the problem at hand.

Let us turn our attention to the numerical example of Appendix C, which pertains to ascent (4d) using a discretization of $N = 10\,000$ straight-line segments. For the solution to the minimum ascent time, where the only constraint is average power $\overline{P} = P_0 = 300\,\text{W}$, we use Matlab's `fmincon` function (MATLAB, 2022), which finds the minimum of a constrained nonlinear multivariable function. Starting with segment speeds sampled from the uniform distribution, $V_j \sim U(1\,\text{m/s}, 10\,\text{m/s})$, `fmincon` obtains the solution with 237 iterations, requiring $2\,380\,830$ function evaluations within the optimization algorithm. By contrast, we obtain the global minimum of the ascent time with a direct calculation using coefficients (32) in solution (33).

For the solution to the restricted-power problem, we include the $P_{\max} = 400\,\text{W}$ restriction. Starting with the constant segment speeds that minimize the previous problem, `fmincon` obtains the solution with 870 iterations, requiring $8\,711\,250$ function

evaluations within the optimization algorithm. As an alternative, we propose the following iterative procedure that obtains — for the problem at hand — the same solution, but typically with fewer than five iterations.

P1. [Initialize.] Set segment speeds to the constant speed, $V_j = V$, that corresponds to the global minimum of the ascent time, where the only constraint is the average power, using coefficients (32) in solution (33), set $\mathcal{J} = \emptyset$, and initialize a counter $i = 1$.

P2. [Check extremal segments.] If all segment powers are $P_j \leqslant P_{max}$, terminate procedure successfully. Otherwise, identify the segments that exceed P_{max} and fix V_j such that $P_j = P_{max}$, using coefficients $p_j = \alpha_j/\beta$ and $q = -P_{max}/\beta$ in solution (33); assign these segments as extremal and concatenate to subset $\mathcal{J}$.

P3. [Update constant speed.] Recalculate $V_j = V$ for $j \notin \mathcal{J}$, using p from expression (C.6) and q from expression (32), iterate the counter $i = i + 1$, and return to P2.

Using this procedure, in contrast to `fmincon`, we obtain the same solution within four iterations only. The program terminates with 3932 extremal segments: on the first iteration, 3162 segments are identified as extremal; on the second, 726 become extremal due to the updated constant speed; on the third, the remaining 44 become extremal.

To summarize the procedure, on the first iteration, the segment speeds are set to the value that minimizes the ascent time, under the average-power constraint only. In such a case, 3 162 segments exceed $P_{max} = 400$ W. For these segments, which are extremal, we fix the speeds such that they correspond to P_{max}. For the remaining $10\,000 - 3\,162 = 6\,838$ segments, we recalculate the speeds subject to the maximum-power constraint, which is the end of the first iteration.

On the second iteration, we find that 726 of the 6 838 segments exceed P_{max}. We fix the speeds such that they correspond to P_{max} and recalculate the segment speeds for the remaining $6\,838 - 726 = 6\,112$ segments, which is the end of the second iteration.

On the third iteration, we find that 44 of the 6 112 segments exceed P_{max}. We repeat the fixing and recalculating of segment speeds, which is the end of the third iteration.

On the fourth iteration, we find that all of the $6\,112 - 44 = 6\,068$ segments comply with $P_j \leqslant P_{max}$. Thus, in the end, the program terminates with $3162 + 726 + 44 = 3932$ extremal segments.

This procedure always terminates in, at most, N iterations as the number of extremal segments is nondecreasing and there are only N segments in total. In our experience, the procedure converges, in practice, after at most five iterations, but we have no proof of this. We also cannot guarantee that the computed solution is indeed the global constrained minimum subject to both the average and maximum power constraints, although it seems to be for all the examples we have experimented with. A proof of this would be interesting.

E   On empirical adequacy

The empirical adequacy of a phenomenological model needs to be examined by a comparison with measurements. For this process, we use the ascent — referred to as the San Bernardo segment — whose elevation gain is 162 m. As shown in Figure E.1, the slope is variable; the average and maximum grades are 4.8 % and 8.4 %, respectively. A recent ascent — performed specifically for our study — resulted in $T = 543$ s, which is 9 minutes and 3 seconds.
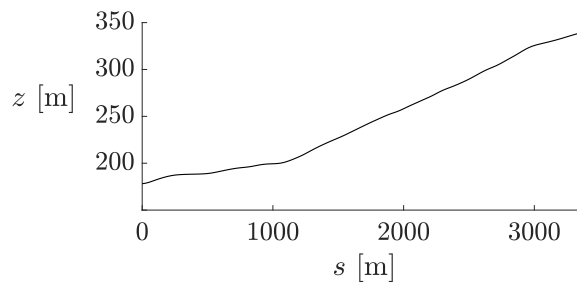

Figure E.1: San Bernardo segment, which is a 3rd category climb in the Italian foothills near Ovada; since the segment has three-dimensional coordinates, the horizontal axis is with respect to arclength, s.

The ascent was attempted with constant power, which is easier to maintain than constant speed, since it corresponds to a steady effort. As shown in Figure E.2, the measured power oscillates about its average of $\overline{P} = 322.0699$, in such a manner that its standard deviation is 33.9902 W. These oscillations are — at least in part — a consequence of cyclocomputer measurements which are sampled only once a second and, hence, occur necessarily at random instants of pedal revolutions.

To examine the model, we consider the following values in expression (2) : $m = 78.6$ kg, as reported by the rider, $C_dA = 0.35$ m^2, $C_{rr} = 0.005$, $\lambda = 0.02$, as estimated from standard measurements, $\rho = 1.1464$ kg/m^3, as calculated based on altitude and temperature, and $g = 9.81$ m/s^2. According to the constant-power strategy — with $P_0 = 322$ W — $T = 541.7489$ s, which is 9 minutes and 2 seconds; the ascent times agree to within one second.
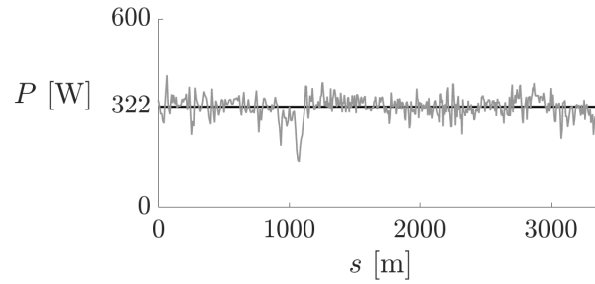
Figure E.2: Comparison between average (black) and instantaneous (grey) power

Another agreement between the model and measurements is the instantaneous speed, shown in Figure E.3. This agreement is symptomatic of the model's pertinence in examining certain subtleties, not only its capacity of predicting average or global quantities.
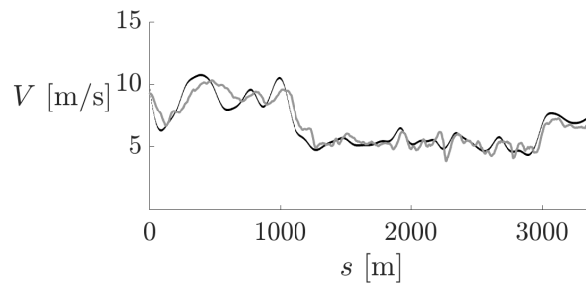

Figure E.3: Comparison between modelled (black) and measured (grey) instantaneous ground speed

This initial examination supports the empirical adequacy of the model. More rides are planned in the foreseeable future to be followed by a statistical analysis. Notably, in accordance with our formulation, a constant-speed ascent — with $\overline{P} = 322\,\text{W}$ — should result in $T = 527.4674\,\text{s}$, which is 8 minutes and 47 seconds, a quarter of a minute less than a constant-power ascent.