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The Knowledge gardener

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Mais même en lieu d'exil, on a aimé la terre...
Yves Bonnefoy (Le chant de sauvegarde)

1

Mathematical research has never been easy to justify, especially when it is financed, as has almost always been the case over the last two centuries, by state resources. The need to establish its legitimacy becomes more pressing in difficult, if not dark, times such as ours, when a growing proportion of the population feels excluded from the sharing of wealth, when the main dynamics of societies are driven by forces of fear, and when the notion of what is useful tends to be more and more difficult to distinguish from that of what is indispensable.

Today, we must, or at least we should, convince people that the research work of mathematicians makes a significant contribution to alleviating the anxieties that undermine the society in which they practise their profession. These anxieties are cultural. The anxieties that prevail in Western Europe as I write this text, at least in that part of the middle and upper classes that do not have to worry about food and shelter, are anxieties about disease, energy, ecology and insecurity. Thus, society expects mathematicians to contribute, for example, to the fight against cancer, to obtaining clean, cheap and inexhaustible energy, to safeguarding the planet, to the survival of humanity, but also, more grimly, to identifying terrorists, separatists, contagious people, the enemies of today and predicting those of tomorrow. Since this main objective is (very) fortunately not easy to achieve, we can still hope to justify ourselves by arguing that our new mathematics is being used, if not immediately then at least before the next elections, to develop the tools that will enable us to cope with our anxieties or even just to distance ourselves from them, possibly by turning away : Data networks, the machines that process, use and transmit them, the algorithms that lock Internet users in the safety of virtual worlds or optimally exploit the energy of fears in the interests of economic or political powers. In addition to these new themes, I should perhaps add one of the oldest, that of armaments, which the academic world readily reviled just a few decades ago.

2

The problem is arduous. I'm not talking about that of saving humankind, which for millennia has been in the hands of very strong and visibly very efficient minds, but about justifying mathematical research. It is even so difficult that when mathematicians, myself included, are asked by friends, friends of friends, high-school pupils, students, journalists, teachers, members of parliament, the curious : what is the point of mathematical research, they answer the considerably simpler question — and this is a proven habit, since our solutions stem almost always from some sort of simplification — which is: what is the point of mathematics ?

However, in this case, we are transforming an arduous problem into a trivial one. Mathematics is everywhere, or to be more precise, it is wherever numbers are, and numbers are everywhere. Numbers are even so convenient, so easy to handle, so simple that, like the Asian hornet in Europe, they have colonised foreign territories by eliminating more useful species, that is to say by taking the place that words, ideas, theories and doctrines should have kept, as in whole areas of social science, economics, psychology, sociology and linguistics. The fact remains that there is indeed an extraordinarily vast territory where numbers and mathematics are at home, and they are there by necessity. They are the fundamental tools of technological engineering. From then on, defending the use of mathematics becomes child's play, and can even be transformed into a cabaret performance.

At a meeting to prepare a campaign to promote mathematics, at which I was the only other mathematician, I saw an experienced colleague, a great communicator, administrator and excellent politician, take off his glasses, put his index finger on them and declaim: "there's mathematics in there", take off his watch, show it to the audience: "there's mathematics in there", pulling a credit card out of his wallet: "there's mathematics in there", waving his phone: "there's mathematics in there" and, after some effort, showing off the soles of his tennis shoes : "there's mathematics in there". He could have pointed to his dental implants. He could have done a striptease, cooing "there's mathematics in there" every time he pulled out a layer. He would not have missed the truth, but, being a great communicator, he knew it would not have been for the good of mathematics. However, mathematicians who have taken an interest in the mathematics used in industry are well aware that the illustrious colleague only

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carried with him mathematics whose core had been known for at least a century. He could have created an illusion by invoking, for example, the Bézier splines, slightly more recent, but very old and very elementary in terms of tools, which delimited the contours of his luxurious fountain pen, but the illusion would have been a blow against the entire profession since the Bézier splines are a jewel that industrialists have given to mathematicians and not the other way round. Clearly, this simplification is a dead end. Not only does it lead to triviality, but to a dangerous triviality: would not the relationship of utility between mathematical research and industry flow exclusively from the latter to the former?

3

There is a second common way of simplifying the question, which, although abusive, deserves a mention, and that is to limit the group of those who pose the question to mathematicians themselves. This would seem to be a fruitful path to follow, because, as far as I know, a significant proportion of the social activities of many mathematicians consist of suspecting, implying and asserting that the work of their colleagues, neighbours in the corridor, department, seminar, colloquium, theme or journal is useless, indicating in this elegant way that they have given some thought to the question of the usefulness of research. However, this reflection is not necessarily judicious. More than once, uninspired colleagues, but in their usual state of lucidity, have argued to me — let's rejoice that naivety is also present in the academic world — that if their theorems about obscure function spaces were of no use to today's society, they might well be used by an engineer in a century or two. It is not their work that I remembered.

The best description of the ambitions of an honest mathematician I ever heard came from the mouth of Gérard Letac, a probabilist from Toulouse, when I was his student in my third year at university. I don't remember exactly the wording, but I don't think I'm betraying his thoughts (at the time), which most mathematicians should also recognise. He said that the highest ambition that a mathematician could have was to contribute to the permanent edifice of mathematics, in other words to ensure that his or her work became a milestone, a stone on the path to the next milestone, in the path of others, a node in the network of exchanges and interactions between theorems and theories that live and die. This is more subtle than saying that the aim is to be a highly cited author, but whoever reads me and is unable to put a different meaning to it can still stay with it, provided that they replace 'highly' with 'at length'. I remember that, at the time, Letac's ambition left me wanting more. It seemed more modest than building a house. Today, I know that it was excessive, that not one mathematician in ten thousand can reasonably hope to have contributed to the permanent edifice of mathematics. But, excessive as it was, the objective pursued did not answer the question. If it offered a perspective, let us say a metaphysical one, to the work of the mathematician, it left aside the question of the usefulness of mathematical research, outside the edifice, within society.

4

There is no point in seeking support from the closest sciencesÂ : mathematical research serves physics and mechanics only at the point where they become a part of it. It is more fruitful to seek help from fellow modellers. Modelling has always been a fundamental theme in mathematics; a large part of the public only knows mathematics through modelling, and those who, a few years ago, when it was not yet an imprudent opinion, scoffed at studies in modelling, simply meant that they were ignorant of the science whose finest achievements they claimed to bear. Today, in France at least, it is scarcely possible to find an end-of-secondary-school mathematics exercise that is not motivated by the modelling of a (seemingly) concrete problem and, naturally, entirely manufactured by the teacher. There is a modelling test in the *agrégation* examination in mathematics, based on the same principle. So it is natural to think that modelling is the right channel for linking mathematical research to society as a whole. Alas, once the communication effects have worn off, it appears that the model that has just been published is essentially as far removed from usefulness as category theory can be.

I will only illustrate this idea using a model that I co-invented, which will allow others to condemn my opinion as the result of my own incompetence. In collaboration with Livio Tilatti, a specialist in forest management, I had proposed a relatively simple model (which, incidentally, was built on mathematics that was almost a century old) a model for measuring forest structures (from the point of view of forest exploitation) based on a limited number of observations. It turned out that, far more difficult than creating the model, was explaining it to people who knew mathematics only through fractional calculus and the automatic practice of statistical tests used with the same lucidity as Diafoirus did with his purgatives. Inevitably, we would have had to explain the new with the old, convince with the methods of marketing and political communication, provide calculation software and a graphical interface that was as unobtrusive as possible: it was a long, contorting and infinitely patient task, which no mathematician would wish to sacrifice his or her hours of reflection to, especially as the question dealt with was marginal and offered no hope of promotion, a medal, an article in the mainstream press, or even enrichment. The difficulty I faced, that of the wall between the researcher and the engineer, is well known. Most universities, strongly encouraged by their supervisory bodies, have set up research transfer units. A happy invention, it goes without saying, which offers the undoubted advantage of simplifying the climb by doubling the length of the path and creating a proof, by the absurd and which I will be allowed to disregard, of the applicability of our research by counting the number of staff supporting its exploitation. Tilatti was the only one

to apply our model, and only until he changed his job, but I can proudly claim to have been one of the very few mathematicians who ever saw one of his models applied in the day-to-day work of a (single) engineer, even if it was only for a couple of years.

In fact, my work with Tilatti was practically that of an engineer, because as soon as I was presented with the problem, I was convinced that I could come up with an answer, and all that remained was the obstacle of finding the best one. This is the fundamental difference between the work of a researcher and the work of an engineer. In research work, you know the starting point and you never know in advance the precise place where each advance will take you. In engineering work, you start where you wish, or where you can, but you are not allowed to arrive at anything other than the specifications, and the less sure you are of yourself, the greater your safety margins. The universes are disjointed. There is no continuous path between one and the other. This does not mean that mathematical researchers cannot be useful to engineers. On the contrary, their interventions would greatly improve the sub-optimal, often clumsy, sometimes sclerotic use that engineers make of long-established mathematics, and I am convinced that the day will come when engineering assistance will take its place alongside teaching among the activities expected of mathematicians, alongside their research. The fact remains that the route via modelling leaves open the question of the usefulness of research in mathematics.

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At this point, all that remains of the easily intelligible arguments are variants of swagger and provocation. Swagger and provocation are to academic life what peanuts are to Indonesian cuisine, although the result may be less convincing. Let us start with the worst that can be found in newspapers (not necessarily sensationalist ones). You can expect to read that this or that mathematical discovery, particularly when a Fields Medal is announced, will help to unravel the mystery of the origins of life, the universe, possibly the existence of God, or at least the miracle of the multiplication of the loaves. Why not write, as Hardy had the courage to do, about his own work, in his famous *A Mathematician Apology*, that nothing mathematicians discover has the slightest influence on the course of the world, none of the utilities I have listed above. Hardy is also credited, perhaps unfairly, with an idea that in one form or another has always existed in the academic world: governments have an interest in funding mathematicians by attaching to harmless problems brains that could otherwise be used to pursue anti-social activities. Hardy was provoking, of course. But which mathematician who was given uselessness by an arrogant self-important person was not tempted to reply: "Yes, what I'm doing is useless and go to hell!" The fact is though that Hardy was an extraordinary pedagogue — his lectures and teaching books had perhaps a greater influence on mathematics than his research work — and he had well earned the time he devoted to his research by his service to students. Hardy's position, in any case, was clearly that of an individualist, the defender of a kind of intellectual aristocracy, an attitude typical of pre-war English society, and his views, when presented to a class of high school students today, provoke a rumble of scandal. The same idea, in fact, can also be expressed in a point of view that is, say, humanist rather than individualist. I can think of no better example of this than Jean Dieudonné's postulate when he entitled his popularisation text *Pour l'honneur de l'esprit humain* (For the Honour of the Human Spirit). But that was in 1987. Honour had long since become an obsolete notion, its vague meaning a subject of comedies of manners; it drew smiles from those, the vast majority, who had already converted to the only moral principle of our time, the measure to which we defer all that is human: money. In fact, if you talk to high school students today about financing the honour of the human spirit, they will tell you that humanity is suffering from a thousand afflictions that need to be treated as a priority. Outside the academic world, swagger and provocation must be used with caution.

6

There is indeed a useful role for research in mathematics, even in the most abstract of fields. The mathematician is a gardener. Every work, every invention, however tiny, as in the work of the ant, is a turning over, almost always on the surface, sometimes in the depths, an aeration, a deposit of re-generative substances in the soil of mathematics, organic food for future plants, a transformation of waste into resources. Even the most unfortunate of mathematicians, those who have lost their way in the misery of the race to publications, the impact factor, the citation, the search for funding, those who have become the clients of the behemoths of scientific publishing, or the devotees of the latest whim of ministry officials in long need of light, even these continue to garden knowledge, to make their contribution to preserving the soil of mathematics. But the work of the gardeners of knowledge is invisible; they cannot offer the passer-by either the colour or the fragrance of the flowers. Even in the eyes of mathematicians, the service that one of them provides to the land will, in most cases, remain invisible. And societies only realise the misfortune of having lost the knowledge gardeners when the land has become barren. So if, by some catastrophe, mathematical research were to cease in all universities, for example by increasing teaching and administrative services to the point of making it impossible, the first dramatic effects on society, in terms of both the useful and the indispensable, and which would begin with a loss of momentum in technological progress, would not be felt until a generation, probably two or three, after the land had been abandoned.

7

I can think of no more commendable example of a knowledge gardener than Leonard Peter Bos. There is no point in trying to describe the whole of mathematical activity in a few lines, but for the purposes of this article I can, without loss of generality, as we like to say, classify it into three main components: the creation of objects, the transformation of objects, the identification of objects and, more generally, the study of their relationships. Each of these activities lives only through the links it develops with the other two. New objects are almost always created to assist in the transformation or identification of existing objects. New objects are often modifications of existing objects, born of a new point of view through which an inventive mind is able to look at it. The land that Bos cultivates is in the territory of the transformation of objects, in the place where it has its fundamental source: calculation. In the alleys of calculation, Bos is a majestic gardener. He manipulates to perfection the three fundamental tools of his mind: the blackboard, the formal calculation software and the scientific calculation software. He always knows which season to use one rather than the other for the best effect on his land. No serious mathematician who has crossed his path, or rather, his blackboard, can fail to have been dazzled by the strength and virtuosity of his talent. There were occasions when I accompanied him for a long time into tangles where, unable to see the beginnings of the glimmer of a clearing, I had to give up following him. On those occasions, he turned back to me, I was sitting with my cheeks puffed out on the path that our billhooks had just cut, he asked me candidly if I was thinking of forking off: I didn't dare reply: "Damn it, Len, neither this way nor that, I'm not taking another step in this jungle!", I remained silent, perplexed, thinking: "How is this guy's brain made?" Because he had the strength to transform a jungle of calculations into gardens of Babylon.

And yet I was far from holding that opinion on the distant day I met him in Toronto, where we were both guests of Thomas Bloom. I was still a pure product of French university: calculation was the ugly side of mathematics there, the one that was never shown to visitors; I myself had always hated calculation exercises; my professors, including those of numerical analysis, were bent on corrupting, at least in my mind, the notion of algorithm; my masters would have fainted at the idea of using a computer to clear their land. There are still traces of this. I still know mathematicians who believe that the only worthwhile works are those that can be written without the use of Tex, without the need for any calculations or complex symbolic constructions. This explains why, the first time he took up a piece of chalk in front of my blackboard in Bloom's guest office, I quickly got the distinct impression that I was dealing with a madman. I was thinking about how to avoid calculating, he was thinking by calculating. However, as I have come to realise with life experience, as was the case that day, the impression of madness is common when you cross the frontier of consciousness. And today, decades and dozens of sticks of chalk later, I believe that algorithms are the beginning and the end of mathematics, that the very essence of mathematics lies in calculations, that those who hide them are Tartuffes and those who avoid them are illusion-breathers. How many calculations have we carried out together, sometimes in the squalid basements that the Institut de Mathématiques de Toulouse was kind enough to make available to us? How many magnificent expressions on our blackboards have died, copied onto loose sheets of paper or captured on lost pictures rather than in the pages of a defunct journal? From those formulas, I have nothing left but the feeling of happiness they brought. But what is more important is that we were useful, alone in front of our blackboard, useful to everyone. Because we were gardening.