Dolomites Research Notes on Approximation
Volume 17 · 2024 · Pages 52–58

Iterates of positive linear operators and linear systems of equations

Vlad Paşca [a] · Andra Seserman [b] · Ancuţa Emilia Şteopoaie [c]

Dedicated to Professor Ioan Raşa on the occasion of his 70th birthday

Abstract

When studying the iterates of certain positive linear operators, systems of linear equations are naturally involved. The last step in investigating the limit of such iterates is represented by a special kind of system of equations. Problems of this type involving several classical operators are studied in the literature. In this paper, we investigate the iterates of new positive linear operators and the corresponding systems of equations. To solve the system we use an iterative algorithm. The approximate solution is used in order to approximate the limit of the iterates of operators.

1 Introduction

The EMML algorithm (see [4], [20]) was pioneered by L.A. Shepp and Y. Vardi in 1982 and independently by K. Lange and R. Carson in 1984, using the Expectation-Maximization (EM) algorithm in order to compute Maximum Likelihood (ML) estimates for the problem of tomography reconstruction (see [14]). The same algorithm was obtained by W.H. Richardson in 1972 and independently by L.B. Lucy in 1974, in the setting of restoration of astronomical images (see [14]).

EMML can be considered as a numerical procedure for calculating maximum likelihood estimates, or alternatively as an iterative procedure for solving a class of linear systems of equations (see [20], [14], [3], [5], [6], [19]).

The Image Space Reconstruction Algorithm (ISRA) was introduced by M. E. Daube-Witherspoon and G. Muehllehner in 1986, in the context of the Positron Emission Tomography problem. It serves to obtain Least-Squares estimates of the emission densities (see [14]). Alternatively, ISRA can be viewed as a procedure for solving linear systems (see [20], [14], [3], [5], [6], [19]).

In [4] the authors introduced an algorithm $A(p)$, depending on a real parameter p, such that:

(a) $A(1)$ coincides with EMML, and $A(-1)$ with a version of ISRA;

(b) $A(p)$ minimizes a suitable generalized Kullback-Leibler distance and solves a specific problem of convex optimization involving generalized log-likelihood functions and least-squares functions;

(c) $A(p)$ solves iteratively linear systems from a certain class and assigns generalized solutions to inconsistent systems.

Practical applications of $A(p)$ involving the Bernstein-Bézier representation of polynomials, B-spline interpolation, inverse problem for Markov chains and the problem of finding the stationary distribution of a Markov chain are presented in [4]. In this paper we present new applications of the algorithm $A(p)$, involving the iterates of certain Markov operators, i.e., positive linear operators which preserve the constant functions. Such iterates and their limits are studied in Ergodic Theory as well as in Approximation Theory. In many cases, the problem of finding the limit is reduced to solving a linear system of equations, see, e.g., [2], [9], [10] and the references therein. We will show that in such situations the algorithm $A(p)$ can be successfully applied.

The algorithm $A(p)$ is recalled in Section 2. In Section 3 we consider a family of Stancu operators, see [7], [8], and investigate the limit of the iterates of such an operator. Section 4 is devoted to a modification of the sequence of Bernstein operators B_n on $C[0,1]$, introduced by Schnabl [17] in order to investigate the global saturation of the sequence (B_n). We show that they can be obtained as a particular case of Stancu operators, and so the results from Section 3 can be applied. In Section 5 we present numerical examples. Section 6 is concerned with conclusions and further work.

Let Π_n be the space of polynomial functions of degree at most n. By $e_i(t)$ we denote the monomial of degree i.

2 The $A(p)$ algorithm

Consider the integers $m \geq 1$, $n \geq 1$, the matrix $A = (a_{ij})_{i=1,\dots,n; j=1,\dots,m}$ with $a_{ij} \geq 0$, $\sum_{i=1}^{n} a_{ij} > 0$, $\sum_{j=1}^{m} a_{ij} > 0$, $i = 1,\dots,n$, $j = 1,\dots,m$ and the vector $b = (b_1,\dots,b_n)^t$ with $b_i > 0$, $i = 1,\dots,n$.

We will study the system of linear equations

$$Ax = b, \tag{1}$$

where $x = (x_1,\dots,x_m)^t \in \mathbb{R}^m$.

[a] Lucian Blaga University of Sibiu, Department of Mathematics and Informatics, Str. Dr. I. Ratiu, No.5-7, RO-550012 Sibiu, Romania, e-mail: sergiu.pasca@ulbsibiu.ro
[b] Technical University of Cluj-Napoca, Faculty of Automation and Computer Science, Department of Mathematics, Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania e-mail: campan_aandra@yahoo.com
[c] Technical University of Cluj-Napoca, Faculty of Automation and Computer Science, Department of Mathematics, Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania e-mail: ancasteopoaie@yahoo.ro

Denote $K_m := \{x \in \mathbb{R}^m : x_j > 0, j = 1, \dots, m\}$.

To solve the system (1) we will use the following algorithm $A(p)$, $p \in \mathbb{R}$ (see [4]).

$$A(p): \ x_r^{(k+1)} = x_r^{(k)} \left(\frac{\sum_{i=1}^n a_{ir} \left(b_i/(Ax^{(k)})_i\right)^p}{\sum_{i=1}^n a_{ir}} \right)^{1/p}, \ k \ge 0, \tag{2}$$

for $p \neq 0$ and $r = 1, \dots, m$.

$$A(0): \ x_r^{(k+1)} = x_r^{(k)} \prod_{i=1}^n \left(\frac{b_i}{(Ax^{(k)})_i} \right)^{a_{ir}/\sum_{s=1}^n a_{sr}}, \ k \ge 0, \tag{3}$$

for $p = 0$ and $r = 1, \dots, m$.

Starting with $x^{(0)} := (x_1^{(0)}, x_2^{(0)}, \cdots, x_m^{(0)}) \in K_m$, we compute successively $x^{(1)}, x^{(2)}, \cdots$, using (2) or alternatively (3). So, we get the sequence $(x^{(k)})_{k \ge 0}$ in K_m.

The convergence of this sequence is governed by the general rules of the Expectation-Maximization Algorithm (see [14], [4]). If we denote

$$x^* := \lim_{k \to \infty} x^{(k)},$$

then x^* is a generalized solution of (1). For details and several examples see [14], [4]. Other examples will be provided in Section 5.

In particular, let T be a regular stochastic matrix,

$$T = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}.$$

It is well known that the system

$$\begin{cases} T^t x = x, \\ x_1 + \cdots + x_n = 1, \end{cases} \tag{4}$$

has a unique solution $x = (x_1, \dots, x_n)^t$.

It was remarked in [4] that (4) is equivalent to

$$\begin{cases} p_{11}x_1 + (p_{21}+1)x_2 + \cdots + (p_{n1}+1)x_n = 1, \\ (p_{12}+1)x_1 + p_{21}x_2 + \cdots + (p_{n2}+1)x_n = 1, \\ \cdots \\ (p_{1n}+1)x_1 + (p_{2n}+1)x_2 + \cdots + p_{nn}x_n = 1. \end{cases} \tag{5}$$

3 A family of Stancu operators

Let α, β, γ be some positive numbers with $\alpha \ge 0$ and $0 \le \beta \le \gamma$. Stancu [18] introduced and studied the following positive linear operator $S_n^{<\alpha,\beta,\gamma>} : C[0,1] \to \Pi_n$, where

$$S_n^{<\alpha,\beta,\gamma>}(f;x) := \sum_{i=0}^n w_{n,i}^{(\alpha)}(x) f\left(\frac{i+\beta}{n+\gamma}\right), \ x \in [0,1]. \tag{6}$$

Here

$$w_{n,i}^{(\alpha)}(x) := \binom{n}{i} \frac{x^{[i,-\alpha]}(1-x)^{[n-i,-\alpha]}}{1^{[n,-\alpha]}}, \ x \in [0,1], \ i = 0, \dots, n,$$

are the fundamental polynomials and

$$\begin{aligned} y^{[0,-\alpha]} &:= 1, \\ y^{[m,-\alpha]} &:= y(y+\alpha)\cdots(y+(m-1)\alpha), \ m \in \mathbb{N}. \end{aligned}$$

Mühlbach [15, 16] and Lupaş [12, 11] introduced the operators $\overline{\mathbb{B}}_\alpha : C[0,1] \to C[0,1]$, defined as

$$\overline{\mathbb{B}}_\alpha f(x) := \begin{cases} f(0), \ x = 0, \\ \dfrac{1}{B(\alpha x, \alpha(1-x))} \int_0^1 t^{\alpha x - 1} (1-t)^{\alpha(1-x)-1} f(t)\,dt \ (0 < x < 1), \\ f(1) \ (x = 1). \end{cases}$$

The Stancu operators have the following representation by means of the Bernstein operators (see [7])

$$S_n^{<\alpha,\beta,\gamma>}(f;x) = \overline{\mathbb{B}}_\alpha B_n\left(f\circ\left(\frac{n}{n+\gamma}e_1+\frac{\beta}{n+\gamma}e_0\right);x\right)$$
$$= \begin{cases} f\left(\dfrac{\beta}{n+\gamma}\right), \ x=0 \\ \dfrac{1}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}B_n\left(f\circ\left(\dfrac{n}{n+\gamma}e_1+\dfrac{\beta}{n+\gamma}e_0\right);t\right)dt, \ x\in(0,1), \\ f\left(\dfrac{n+\beta}{n+\gamma}\right), \ x=1, \end{cases}$$

where $B(p,q)$, $p,q>0$ is the Beta function.

Theorem 3.1. *[7] The eigenvalues of $S_n^{<\alpha,\beta,\gamma>}$ are $\lambda_{n,0}^\alpha=1$, $\lambda_{n,1}^\alpha=\dfrac{n}{n+\gamma}$ and*

$$\lambda_{n,j}^\alpha = \frac{n(n-1)\dots(n-j+1)}{(n+\gamma)^j}\frac{1}{(\alpha+1)(2\alpha+1)\dots(\alpha(j-1)+1)}, \ j=2,\dots,n.$$

The corresponding (normalized) eigenfunctions are $q_{n,0}=e_0$ and $q_{n,j}(x)=e_j(x)+a_{n,j-1}^{(j)}e_{j-1}(x)+\dots+a_{n,0}^{(j)}e_0(x)$, $j=1,\dots,n$, with uniquely determined coefficients.

Theorem 3.2. *[7] If $n\in\mathbb{N}$ is fixed and $\alpha>0$, $0\le\beta\le\gamma$, $\gamma>0$, then for all $f\in C[0,1]$,*

$$\lim_{m\to\infty} S_n^{<\alpha,\beta,\gamma>^m} f = b_0^\alpha e_0.$$

The scalar coefficient b_0^α of e_0 is a convex combination of "certain" values of the function f, namely,

$$b_0^\alpha = \sum_{j=0}^n d_j^\alpha f\left(\frac{j+\beta}{n+\gamma}\right),$$

with d_j^α, $0\le j\le n$, independent of f.

In order to prove Theorem 3.2, the authors used the eigenstructure of $S_n^{<\alpha,\beta,\gamma>}$, described in Theorem 3.1. In fact, consider the basis of Π_n formed with the eigenpolynomials $\{q_{n,0},q_{n,1},\dots,q_{n,n}\}$ and also the basis formed with the fundamental Stancu polynomials $\left\{w_{n,0}^{(\alpha)},w_{n,1}^{(\alpha)},\dots,w_{n,n}^{(\alpha)}\right\}$. The transition matrix between the two bases is defined by

$$w_{n,0}^\alpha = \theta_{0,0}\cdot q_{n,0}+\dots+\theta_{n,0}\cdot q_{n,n}$$
$$\dots$$
$$w_{n,n}^\alpha = \theta_{0,n}\cdot q_{n,0}+\dots+\theta_{n,n}\cdot q_{n,n}.$$

Then (see [7, p.122]) the coefficients d_j^α from Theorem 3.2 are given by $d_j^\alpha=\theta_{0,j}$, $j=0,1,\dots,n$.

If $0<\beta<\gamma$, an alternative method for finding the coefficients $d_0^\alpha, d_1^\alpha,\dots,d_n^\alpha$ is described in [7, Remark.1]. To present it, consider the matrix

$$T := \begin{pmatrix} w_{n,0}^{(\alpha)}(a_0) & w_{n,1}^{(\alpha)}(a_0) & \cdots & w_{n,n}^{(\alpha)}(a_0) \\ w_{n,0}^{(\alpha)}(a_1) & w_{n,1}^{(\alpha)}(a_1) & \cdots & w_{n,n}^{(\alpha)}(a_1) \\ \cdots & \cdots & \cdots & \cdots \\ w_{n,0}^{(\alpha)}(a_n) & w_{n,1}^{(\alpha)}(a_n) & \cdots & w_{n,n}^{(\alpha)}(a_n) \end{pmatrix}, \tag{7}$$

where $a_j := \dfrac{j+\beta}{n+\gamma}$, j=0,1,…,n.

It is a stochastic matrix and all entries are strictly positive. Consequently, the system of equations

$$\begin{cases} T^t\begin{pmatrix} d_0^\alpha \\ \vdots \\ d_n^\alpha \end{pmatrix} = \begin{pmatrix} d_0^\alpha \\ \vdots \\ d_n^\alpha \end{pmatrix}, \\ d_0^\alpha+\dots+d_n^\alpha=1, \end{cases} \tag{8}$$

has exactly one solution and this gives us the coefficients d_j^α from Theorem 3.2.

Remark 1. More general results can be found in [2].

4 A special case

Let $\varphi(x) = x(1-x)$. For $n \in \mathbb{N}$, $f \in C[0,1]$, and $0 < x < 1$, let us denote (see [17])

$$C_n f(x) = \frac{1}{\varphi(x)} B_n(\varphi f)(x), \tag{9}$$

where B_n is the classical Bernstein operator, i.e.,

$$B_n f(x) = \sum_{i=0}^{n} p_{n,i}(x) f\left(\frac{i}{n}\right), \quad p_{n,i}(x) := \binom{n}{i} x^i (1-x)^{n-i},$$

$i = 0, 1, \ldots, n$, $x \in [0,1]$.

By continuity, the function $C_n f$ can be extended to the interval $[0,1]$. The operators C_n were deeply investigated in [17]. Let us remark that for $n \geq 2$, and $0 < x < 1$,

$$\begin{aligned}
A_n f(x) := \frac{n}{n-1} C_n f(x) &= \frac{n}{n-1} \frac{1}{x(1-x)} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} . \frac{k}{n} \left(1 - \frac{k}{n}\right) f\left(\frac{k}{n}\right) \\
&= \frac{n}{n-1} \frac{1}{x(1-x)} \sum_{k=1}^{n-1} \binom{n}{k} \frac{k}{n} \frac{n-k}{n} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \\
&= \sum_{k=1}^{n-k} \binom{n-2}{k-1} x^{k-1} (1-x)^{n-1-k} f\left(\frac{k}{n}\right).
\end{aligned}$$

Consequently, we have the operators $A_n : C[0,1] \to C[0,1]$, $n \geq 2$, given by

$$A_n f(x) = \sum_{j=0}^{n-2} \binom{n-2}{j} x^j (1-x)^{n-2-j} f\left(\frac{j+1}{n}\right). \tag{10}$$

It is easy to verify that

$$A_n = S_{n-2}^{<0,1,2>}. \tag{11}$$

Therefore, the operators studied in [17] are strongly related with Stancu operators.

5 Numerical examples

Example 5.1. In (11) let us take $n = 5$. We get

$$\begin{aligned}
A_5 f(x) &= S_3^{<0,1,2>} f(x) \\
&= (1-x)^3 f\left(\frac{1}{5}\right) + 3x(1-x)^2 f\left(\frac{2}{5}\right) + 3x^2(1-x) f\left(\frac{3}{5}\right) + x^3 f\left(\frac{4}{5}\right).
\end{aligned}$$

In this specific case the matrix T^t from (4) is

$$T^t = \frac{1}{125} \begin{pmatrix} 64 & 27 & 8 & 1 \\ 48 & 54 & 36 & 12 \\ 12 & 36 & 54 & 48 \\ 1 & 8 & 27 & 64 \end{pmatrix}.$$

The system (5) becomes

$$\begin{cases} 64x_1 + 152x_2 + 133x_3 + 126x_4 = 125, \\ 173x_1 + 54x_2 + 161x_3 + 137x_4 = 125, \\ 137x_1 + 161x_2 + 54x_3 + 173x_4 = 125, \\ 126x_1 + 133x_2 + 152x_3 + 64x_4 = 125. \end{cases}$$

Table 1. Values of $x_1^{(k)}$, $x_2^{(k)}$, $x_3^{(k)}$, $x_4^{(k)}$ for $p = -1$

k	0	50	100	150	200
$x_1^{(k)}$	0.0714285714	0.1534326211	0.1782629219	0.1831218342	0.1840130160
$x_2^{(k)}$	0.4285714286	0.3462868724	0.3217265249	0.3168778113	0.3159869721
$x_3^{(k)}$	0.4285714286	0.3462868728	0.3217265257	0.3168778118	0.3159869722
$x_4^{(k)}$	0.0714285714	0.1534326212	0.1782629222	0.1831218349	0.1840130164

Table 2. Values of $x_1^{(k)}$, $x_2^{(k)}$, $x_3^{(k)}$, $x_4^{(k)}$ for $p = 0$

k	0	50	100	150	200
$x_1^{(k)}$	0.0714285714	0.1533971601	0.1782429620	0.1831176476	0.1840122386
$x_2^{(k)}$	0.4285714286	0.3464618006	0.3217517235	0.3168821746	0.3159877561
$x_3^{(k)}$	0.4285714286	0.3464618006	0.3217517231	0.3168821739	0.3159877559
$x_4^{(k)}$	0.0714285714	0.1533971604	0.1782429613	0.1831176470	0.1840122380

Table 3. Values of $x_1^{(k)}$, $x_2^{(k)}$, $x_3^{(k)}$, $x_4^{(k)}$ for $p=1$

k	0	50	100	150	200
$x_1^{(k)}$	0.0714285714	0.1532686696	0.1782022231	0.1831095912	0.1840107565
$x_2^{(k)}$	0.4285714286	0.3467313304	0.3217977769	0.3168904088	0.3159892436
$x_3^{(k)}$	0.4285714286	0.3467313304	0.3217977767	0.3168904086	0.3159892437
$x_4^{(k)}$	0.0714285714	0.1532686698	0.1782022234	0.1831095914	0.1840107566

According to Theorem 3.2,

$$\lim_{m\to\infty} S_3^{<0,1,2>^m} f \approx \left\{0.184 f\left(\frac{1}{5}\right) + 0.315 f\left(\frac{2}{5}\right) + 0.315 f\left(\frac{3}{5}\right) + 0.184 f\left(\frac{4}{5}\right)\right\} e_0.$$

Example 5.2. In (6) we consider $n=3$, $\alpha=1$, $\beta=2$, $\gamma=3$. Then, we have

$$S_3^{<1,2,3>} f(x) = \frac{(1-x)(2-x)(3-x)}{6} f\left(\frac{1}{3}\right) + \frac{x(1-x)(2-x)}{2} f\left(\frac{1}{2}\right) + \frac{x(x+1)(1-x)}{2} f\left(\frac{2}{3}\right) + \frac{x(x+1)(x+2)}{6} f\left(\frac{5}{6}\right).$$

In this case the matrix T^t from (4) is

$$T^t = \begin{pmatrix} 40/81 & 5/16 & 14/81 & 91/1296 \\ 5/27 & 3/16 & 4/27 & 35/432 \\ 4/27 & 3/16 & 5/27 & 55/432 \\ 14/81 & 5/16 & 40/81 & 935/1296 \end{pmatrix}.$$

The system (5) becomes

$$\begin{cases} 40/81 x_1 + 21/16 x_2 + 95/81 x_3 + 1387/1296 x_4 = 1, \\ 32/27 x_1 + 3/16 x_2 + 31/27 x_3 + 467/432 x_4 = 1, \\ 31/27 x_1 + 19/16 x_2 + 5/27 x_3 + 487/432 x_4 = 1, \\ 95/81 x_1 + 21/16 x_2 + 121/81 x_3 + 935/1296 x_4 = 1. \end{cases}$$

Table 4. Values of $x_1^{(k)}$, $x_2^{(k)}$, $x_3^{(k)}$, $x_4^{(k)}$ for $p=0$

k	0	50	100	150	200
$x_1^{(k)}$	0.2407	0.2008878908	0.2007358646	0.2007069010	0.2007047845
$x_2^{(k)}$	0.1551	0.1252139744	0.1251678719	0.1251592125	0.1251585815
$x_3^{(k)}$	0.1275	0.1475247062	0.1475612683	0.1475681899	0.1475686951
$x_4^{(k)}$	0.5065	0.5263734260	0.5265349946	0.5265656967	0.5265679387

According to Theorem 3.2,

$$\lim_{m\to\infty} S_3^{<1,2,3>^m} f \approx \left\{0.2007 f\left(\frac{1}{3}\right) + 0.1251 f\left(\frac{1}{2}\right) + 0.1475 f\left(\frac{2}{3}\right) + 0.5265 f\left(\frac{5}{6}\right)\right\} e_0.$$

Example 5.3. Let $c\in[1,\infty)$, $n\geq 1$, $a,b>-1$,

$$\varphi_{n,i}(t) := \frac{t^{ci+a}(1-t)^{c(n-i)+b}}{B(ci+a+1, c(n-i)+b+1)}, \quad t\in[0,1],\ i=0,1,\ldots,n.$$

The Mache-Zhou operator is defined as (see [13]):

$$P_n f(x) := \sum_{i=0}^{n} p_{n,i}(x) \int_0^1 f(t)\varphi_{n,i}(t)dt.$$

Th limit of the iterates of P_n was determined in [1] (see also [2]), up to solving a linear system of equations.

For $n=3$, $c=1$, $a=1$, $b=2$ that system is equivalent to (see (5))

$$\begin{cases} 336x_1 + 930x_2 + 840x_3 + 780x_4 = 720, \\ 972x_1 + 270x_2 + 960x_3 + 900x_4 = 720, \\ 828x_1 + 900x_2 + 240x_3 + 990x_4 = 720, \\ 744x_1 + 780x_2 + 840x_3 + 210x_4 = 720. \end{cases}$$

Table 5. Values of $x_1^{(k)}$, $x_2^{(k)}$, $x_3^{(k)}$, $x_4^{(k)}$ for $p=-2$

k	0	50	100	150	200
$x_1^{(k)}$	0.2657	0.2847443610	0.2847828587	0.2843701646	0.2841754795
$x_2^{(k)}$	0.3638	0.3431329449	0.3418709107	0.3416953871	0.3416400194
$x_3^{(k)}$	0.2781	0.2627557436	0.2609266164	0.2603800485	0.2601639800
$x_4^{(k)}$	0.0952	0.1078555881	0.1112018255	0.1124999328	0.1130359147

Table 6. Values of $x_1^{(k)}$, $x_2^{(k)}$, $x_3^{(k)}$, $x_4^{(k)}$ for $p = 0$

k	0	50	100	150	200
$x_1^{(k)}$	0.2657	0.2869649702	0.2870756173	0.2863686541	0.2859984357
$x_2^{(k)}$	0.3638	0.3451406564	0.3435428082	0.3431263754	0.3429687678
$x_3^{(k)}$	0.2781	0.2587648338	0.2569442874	0.2569657113	0.2570584815
$x_4^{(k)}$	0.0952	0.1091262668	0.1124369469	0.1135392006	0.1139743049

Table 7. Values of $x_1^{(k)}$, $x_2^{(k)}$, $x_3^{(k)}$, $x_4^{(k)}$ for $p = 2$

k	0	50	100	150	200
$x_1^{(k)}$	0.2657	0.2847780080	0.2847990630	0.2843788150	0.2841805796
$x_2^{(k)}$	0.3638	0.3431409560	0.3418775488	0.3416992042	0.3416425785
$x_3^{(k)}$	0.2781	0.2627823074	0.2609393522	0.2603834089	0.2601630650
$x_4^{(k)}$	0.0952	0.1077900627	0.1111639649	0.1124833683	0.1130295306

According to Theorem 3.2,

$$\lim_{m\to\infty} P_3{}^m f \approx \left\{ 11.97 \int_0^1 t(1-t)^5 f(t)dt + 35.91 \int_0^1 t^2(1-t)^4 f(t)dt \right.$$
$$\left. + 35.98 \int_0^1 t^3(1-t)^3 f(t)dt + 11.865 \int_0^1 t^4(1-t)^2 f(t)dt \right\} e_0.$$

We conclude this section by modifying the operators from Example 5.1 and Example 5.2 in the sense described in [2, Sect. 4.5]. We will see that the limits of the iterates will change dramatically.

Example 5.4. Let $p_0(x) = (1-x)^3$, $p_1(x) = x^3$, $p_2(x) = 3(1-x)^2x$, $p_3(x) = 3(1-x)x^2$, $x \in [0,1]$. Instead of the operator from Example 5.1, consider the operator $S : C[0,1] \to C[0,1]$,

$$Sf(x) := p_0(x)f(0) + p_1(x)f(1) + p_2(x)f\left(\frac{2}{5}\right) + p_3(x)f\left(\frac{3}{5}\right).$$

To find $\lim\limits_{m\to\infty} S^m f$ we apply the general results from [2]. Namely, consider the matrix

$$M := \begin{pmatrix} p_0(0) & p_1(0) & p_2(0) & p_3(0) \\ p_0(1) & p_1(1) & p_2(1) & p_3(1) \\ p_0\left(\frac{2}{5}\right) & p_1\left(\frac{2}{5}\right) & p_2\left(\frac{2}{5}\right) & p_3\left(\frac{2}{5}\right) \\ p_0\left(\frac{3}{5}\right) & p_1\left(\frac{3}{5}\right) & p_2\left(\frac{3}{5}\right) & p_3\left(\frac{3}{5}\right) \end{pmatrix}$$

According to [2, Th.2.2],

$$\lim_{m\to\infty} S^m f(x) = p_0(x)f(0) + p_1(x)f(1) + p_2(x)v_2(f) + p_3(x)v_3(f),$$

where

$$v(f) := (f(0), f(1), v_2(f), v_3(f))^t$$

is the unique solution to $Mv(f) = v(f)$. We get

$$v_2(f) = \frac{1}{107}(63f(0) + 44f(1)),\ v_3(f) = \frac{1}{107}(44f(0) + 63f(1)),$$

so that

$$\lim_{m\to\infty} S^m f(x) = (1-x)^3 f(0) + x^3 f(1) + \frac{3}{107}(1-x)^2 x(63f(0) + 44f(1))$$
$$+ \frac{3}{107}(1-x)x^2(44f(0) + 63f(1)),$$

for $f \in C[0,1]$, uniformly with respect to $x \in [0,1]$.

Example 5.5. Let $p_0(x) = \dfrac{(1-x)(2-x)(3-x)}{6}$, $p_1(x) = \dfrac{x(x+1)(x+2)}{6}$, $p_2(x) = \dfrac{x(1-x)(2-x)}{2}$, $p_3(x) = \dfrac{x(x+1)(1-x)}{2}$. Instead of the operator from Example 5.2, let us consider $S : C[0,1] \to C[0,1]$,

$$Sf(x) := p_0(x)f(0) + p_1(x)f(1) + p_2(x)f\left(\frac{1}{2}\right) + p_3(x)f\left(\frac{2}{3}\right).$$

Using the same arguments as in Example 5.4 we find

$$\lim_{m\to\infty} S^m f(x) = \frac{(1-x)(2-x)(3-x)}{6} f(0) + \frac{x(1+x)(2+x)}{6} f(1)$$
$$+ \frac{x(1-x)(2-x)}{274}(62f(0) + 75f(1)) + \frac{x(x+1)(1-x)}{822}(121f(0) + 290f(1)).$$

6 Conclusions and further work

The EMML and the ISRA algorithms can be used in particular for solving certain classes of systems of linear equations. The algorithm $A(p)$ introduced in [4] serves to a similar goal. In fact $A(-1)$ coincides with a version of ISRA, while $A(1)$ coincides with EMML. The systems of linear equations mentioned above can be solved iteratively with $A(p)$. They appear in the study of limits of iterates of positive linear operators. Results of this type were established in many papers (see [2], [9], [10] and the references therein). In this paper we consider the iterates of operators of Stancu type (see [7]). The corresponding system of linear equations was determined in [7]. We use the algorithm $A(p)$ in order to solve iteratively such a system. An interesting modification of the classical Bernstein operators B_n was introduced by R. Schnabl in [17] in order to investigate the global saturation of the sequence $(B_n)_{n\geq 1}$. We prove that the operators introduced by R. Schnabl are Stancu type operators from the family mentioned above. For the study of their iterates we use again the algorithm $A(p)$. Similar problems are investigated for the Mache-Zhou operators. Numerical experiments illustrate the general results.

As further work we propose to extend the results in order to cover other classes of operators. Moreover, we intend to compare the algorithm $A(p)$ with other algorithms and to improve its performances.

Acknowledgement. The work was supported by the project "Mathematical Methods and Models for Biomedical Applications" financed by National Recovery and Resilience Plan PNRR-III-C9-2022-I8.

References

[1] A.M. Acu, M. Heilmann, I. Rasa, Eigenstructure and iterates for uniquely ergodic Kantorovich modifications of operators II, Positivity 25, 1585-1599 (2021).

[2] A.M. Acu, I. Raşa, Iterates and invariant measures for Markov operators, Results Math 76, 218 (2021).

[3] G. E. B. Archer, D. M. Titterington, The Iterative Image Space Reconstruction Algorithm (ISRA) as an Alternative to the EM Algorithm for Solving Positive Linear Inverse Problems, Statistica Sinica, 5, 77-96 (1995).

[4] A. Ciurte, S. Nedevschi, I. Rasa, A Generalization of the EMML and ISRA Algorithms for Solving Linear Systems, J. Comput. Anal. Appl., 12(2010), 799-816

[5] A. R. De Pierro, On the Convergence of the Iterative Image Space Reconstruction Algorithm for Volume ECT, IEEE Trans. Med. Imaging, 6, 174-175 (1987).

[6] A. R. De Pierro, On the Relation Between the ISRA and the EM Algorithm for Positron Emission Tomography, IEEE Trans. Med. Imaging, 12, 328-333 (1993).

[7] H. Gonska, P. Piţul, I. Raşa, Over-iterates of Bernstein-Stancu operators, Calcolo, 44 (2007), 117-125.

[8] H.H. Gonska, J. Meier, Quantitative theorems on approximation by Bernstein-Stancu operators, Calcolo 21 (1984), 317-335.

[9] M. Heilmann, I. Raşa, Eigenstructure and iterates for uniquely ergodic Kantorovich modifications of operators, Positivity (2017) 21:897-910.

[10] M. Heilmann, I. Raşa, C_0-semigroups associated with uniquely ergodic Kantorovich modifications of operators, Positivity (2018) 22:829-835.

[11] A. Lupaş, Die Folge der Beta-Operatoren, Dissertation Universität Stuttgart, 1972.

[12] A. Lupaş, L. Lupaş, Polynomials of binomial type and approximation operators, Stud. Univ. Babeş-Bolyai Math. 32 (4) (1987), 61–69.

[13] D. Mache, D.X. Zhou, Characterization theorems for the approximation by a family of operators, J Approx Theory 84, (1996), 145-161.

[14] G. J. McLachlan and Th. Krishnan, The EM Algorithm and Extensions, Wiley, 2008.

[15] G. Mühlbach, Verallgemeinerungen der Bernstein- und der Lagrangepolynome, Bemerkungen zu einer Klasse linearer Polynomoperatoren von D.D. Stancu, Rev. Roumaine Math. Pure Appl. 15 (1970), 1235–1252.

[16] G. Mühlbach, Rekursionsformeln für die zentralen Momente der Polya- und der Beta-Verteilung, Metrika 19 (1972), 171–177.

[17] R. Schnabl, Zum globalen Saturationsproblem der Folge der Bernsteinoperatoren, Acta Sci. Math. 31(3-4) (1970), 351-358.

[18] D.D. Stancu, Approximation of functions by means of some new classes of positive linear operators. In: Collatz, L., Meinardus, G. (eds.): Numerische Methoden der Approximationstheorie. Band 1. Basel: BirkhÃ¤user 1972 pp. 187-203

[19] D. M. Titterington, On the Iterative Image Space Reconstruction Algorithm for ECT, IEEE Trans. Med. Imaging, 6, 52-56 (1987).

[20] M. Velez-Reyes, A. Puetz, P. Hoke, R. B. Lockwood, S. Rosario, Iterative Algoritms for Unmixing of Hyperspectral Imagery, in Algorithms and Technologies for Multispectral, Hyperspectral and Ultraspectral Imagery IX (S. S. Shen and P. E. Lewis, eds.), Proc. SPIE 5093, Bellingham , WA, 2003, pp. 418-429.