A Stancu type generalization of the Balázs operator

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Dedicated to Professor Ioan Rașa on the occasion of his 70\textsuperscript{th} birthday

Abstract

In this paper we investigate certain properties of Stancu type generalization of the Balázs operator.

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1 The Balázs-Stancu operators

For \( f \in C[0, \infty) \), the Balázs operators \([5]\) are defined by

\[
(R_n f)(x) = \frac{1}{(1 + a_n x)^n} \sum_{j=0}^{n} \binom{n}{j} (a_n x)^j \left( \frac{j}{b_n} \right)
\]

and the Balázs-Stancu operators, defined as follows:

\[
(S_{n,f})(x) = \sum_{j=0}^{n-rs} p_{n-rs}(x) \sum_{i=0}^{s} p_{s}(x) f \left( \frac{j + ir}{n} \right),
\]

where

\[
p_{n,j}(x) = \binom{n}{j} (1 - x)^{n-j}, x \geq 0,
\]

and \((a_n), (b_n)\) are two sequences of positive real numbers suitably chosen.

These operators have been studied and generalized in many directions \([6, 11, 7, 1, 2, 3, 9]\).

In this paper we consider a generalization of Balázs operators in the manner of the generalization of Bernstein operators introduced by D. D. Stancu in \([10]\)

\[
(S_{n,f})(x) = \sum_{j=0}^{n-rs} p_{n-rs}(x) \sum_{i=0}^{s} p_{s}(x) f \left( \frac{j + ir}{n} \right),
\]

\[
(f \in C[0, 1], x \in [0, 1], n, s, N) \text{ are fixed such that } rs < n. \text{ Bernstein’s operators are obtained for } s = 0 \text{ or } s = 1, r = 0 \text{ or } s = 1, r = 1.
\]

We consider the Balázs-Stancu operators, defined as follows:

\[
(R_{n,f})(x) = \sum_{j=0}^{n-rs} p_{n-rs}(x) \sum_{i=0}^{s} p_{s}(x) f \left( \frac{j + ir}{n} \right),
\]

\[
f \in C[0, \infty), x \geq 0, \text{ where } n \in N, r, s, N \in [0, n] \text{ such that } rs < n, (a_n)_n \text{ being a sequence of positive real numbers.}
\]

If \(a_n = 1, (y)n \in N\), we have \((R_{n,f})(x) = (S_{n,f})(x - 1 + x)\)

2 Convergence properties

Lemma 2.1. The operator \(S_{n,f}\) satisfies the following relations:

(i) \((S_{n,f} \epsilon_0)(x) = 1;\)

(ii) \((S_{n,f} \epsilon_1)(x) = x;\)

(iii) \((S_{n,f} \epsilon_2)(x) = x^2 + \left(1 + \frac{rs(r-1)}{n}\right)x(1-x)\)

where \(x \in [0, \infty)\) and \(\epsilon_i(y) = y^i, i = 0, 1, 2.\)

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Proof. For \( p \in \mathbb{N}_0 \), we have
\[
(S_{n,r,s}e_{p+1})(x) = \sum_{j=0}^{n-r} p_{n-r,s}(x) \sum_{i=0}^{j} p_{s,i}(x) \left( \frac{j + ir}{n} \right)^{p+1}
\]
\[
= \sum_{j=0}^{n-r} p_{n-r,s}(x) \sum_{i=0}^{j} \binom{j}{i} \left( \frac{p+1}{k} \right) \left( ir - j \right)^{p+1} k
\]
\[
= \sum_{k=0}^{n-r} \binom{p+1}{k} \left( rs \right)^{k} \left( n - rs \right)^{p+1} k
\]
\[
\cdot \sum_{j=0}^{n-r} p_{n-r,s}(x) \left( \frac{j}{n} \right)^{p+1} \sum_{i=0}^{j} p_{s,i}(x) \left( \frac{j}{n} \right)^{i}
\]
\[
= \sum_{k=0}^{n-r} \binom{p+1}{k} \left( rs \right)^{k} \left( 1 - \frac{rs}{n} \right)^{p+1} \left( B_{p+1}(x) \right) \left( B_{n-r,s}e_{p+1-k}(x) \right),
\]
where \((B_{p,f}(x))\) are the Bernstein operators.

From the above relation, one has:

(i) \( (S_{n,r,s}e_0)(x) = \sum_{j=0}^{n-r} p_{n-r,s}(x) \sum_{i=0}^{j} p_{s,i}(x) = 1 \).

(ii) \( (S_{n,r,s}e_1)(x) = \left( 1 - \frac{rs}{n} \right) (B_{p+1}(x)(B_{n-r,s}e_1)(x) + \frac{rs}{n} (B_{p+1}(x)(B_{n-r,s}e_0)(x)) \)

\[
= \left( 1 - \frac{rs}{n} \right) x + \frac{rs}{n} x = x.
\]

(iii) \( (S_{n,r,s}e_2)(x) = \left( 1 - \frac{rs}{n} \right)^2 (B_{p+1}(x)(B_{n-r,s}e_1)(x) \)

\[
+ 2 \frac{rs}{n} \left( 1 - \frac{rs}{n} \right) (B_{p+1}(x)(B_{n-r,s}e_1)(x) \)

\[
+ \left( \frac{rs}{n} \right)^2 (B_{p+1}(x)(B_{n-r,s}e_0)(x) \)
\]
\[
= \left( 1 - \frac{rs}{n} \right)^2 \left( x^2 + \frac{x(1-x)}{n-rs} \right) + 2 \frac{rs}{n} \left( 1 - \frac{rs}{n} \right) x^2 \]
\[
+ \left( \frac{rs}{n} \right)^2 \left( x^2 + \frac{x(1-x)}{s} \right) \]
\[
= x^2 \left( 1 + \frac{rs(r-1)}{n} \right), \frac{x(1-x)}{n}.
\]
(ii) It is clear that
\[
(R_{n,r} e_0)(x) = (S_{n,r} e_0) \left( \frac{a_n x}{1 + a_n x} \right) = 1.
\]

(iii) From \((S_{n,r} e_1)(x) = x\) it is obtained
\[
(R_{n,r} e_1)(x) = \frac{1}{a_n} (S_{n,r} e_1) \left( \frac{a_n x}{1 + a_n x} \right) = \frac{x}{1 + a_n x}.
\]

(iv) From \((S_{n,r} e_2)(x) = x^2 + \left( 1 + \frac{r s (r-1)}{n} \right) \frac{x(1-x)}{n}\) it is obtained
\[
(R_{n,r} e_2)(x) = \frac{1}{a_n^2} (S_{n,r} e_2) \left( \frac{a_n x}{1 + a_n x} \right) = \frac{x^2}{1 + a_n x} + \left( 1 + \frac{r s (r-1)}{n} \right) \frac{x}{a_n(1 + a_n x)^2}.
\]

**Lemma 2.3.** Let the m-th order moment for the operator be denoted as follows:
\[
(R_{n,r} (e_1 - x e_0)^m)(x) = \sum_{j=0}^{m+1} p_{b-rs} \left( \frac{a_n x}{1 + a_n x} \right) \sum_{j=0}^{m} p_{x} \left( \frac{a_n x}{1 + a_n x} \right) \left( \frac{r + ir}{n a_n} - x \right)^m, m = 1, 2, \ldots.
\]

Then we have

(i) \[
(R_{n,r} (e_1 - x e_0))(x) = - \frac{a_n x^2}{1 + a_n x};
\]

(ii) \[
(R_{n,r} (e_1 - x e_0)^2)(x) = r \frac{a_n^2 x^4}{(1 + a_n x)^2} + \left( 1 + \frac{r s (r-1)}{n} \right) \frac{x}{a_n(1 + a_n x)^2}.
\]

**Proof.**

(i) \[
(R_{n,r} (e_1 - x e_0))(x) = \sum_{j=0}^{m+1} p_{b-rs} \left( \frac{a_n x}{1 + a_n x} \right) \sum_{j=0}^{m} p_{x} \left( \frac{a_n x}{1 + a_n x} \right) \left( \frac{r + ir}{n a_n} - x \right) = (R_{n,r} e_1)(x) - x = - \frac{a_n x^2}{1 + a_n x};
\]

(ii) \[
(R_{n,r} (e_1 - x e_0)^2)(x) = \sum_{j=0}^{m+1} p_{b-rs} \left( \frac{a_n x}{1 + a_n x} \right) \sum_{j=0}^{m} p_{x} \left( \frac{a_n x}{1 + a_n x} \right) \left( \frac{r + ir}{n a_n} - x \right)^2 = (R_{n,r} e_2)(x) - 2x(R_{n,r} e_1)(x) + x^2 = \frac{a_n^2 x^4}{(1 + a_n x)^2} + \left( 1 + \frac{r s (r-1)}{n} \right) \frac{x}{a_n(1 + a_n x)^2}.
\]

**Theorem 2.4.** If \(\lim_{n \to \infty} a_n = 0\) and \(\lim_{n \to \infty} n a_n = \infty\), then for a bounded function \(f \in C[0, \infty)\) it follows
\[
\lim_{n \to \infty} R_{n,r} f = f
\]
uniformly on any compact interval \(K \subset [0, \infty)\).

**Proof.** Let \(K \subset [0, \infty)\) be a compact interval, \(K = [m, M], 0 \leq m < M < \infty\).

It is obvious that
\[
\lim_{n \to \infty} \| R_{n,r} e_0 - e_0 \|_{[m,M]} = 0.
\]

Since
\[
| (R_{n,r} e_1)(x) - e_1(x) | = \frac{a_n x^2}{1 + a_n x} \leq a_n M^2, (\forall) x \in [m, M]
\]
and \( a_n M^2 \to 0 \), result

\[ \lim_{n \to \infty} \| R_{n,s} e_1 - e_1 \|_{[0,M]} = 0. \]

Since

\[ |(R_{n,s} e_2)(x) - e_2(x)| = \left| \frac{a_n x^3 (2 + a_n)}{1 + a_n x^2} + \frac{r s(r-1)}{n} \frac{x}{n a_n (1 + a_n x^2)} \right| \leq a_n M^3 (2 + a_n M) + \frac{r s(r-1)}{n} M \]

and \( a_n M^3 (2 + a_n M) + \frac{r s(r-1)}{n} M \to 0 \) as \( n \to \infty \), result

\[ \lim_{n \to \infty} \| R_{n,s} e_2 - e_2 \|_{[0,M]} = 0. \]

Finally, Theorem 2.4 results by applying [4]-Theorem 4.1.

The modulus of continuity of a continuous function \( f \) on \([0, \infty)\), is defined by

\[ \omega(f, t) = \sup \{|f(y) - f(x)| : x, y \in [0, \infty), |y - x| \leq t\}, t > 0. \]

**Theorem 2.5.** For any function \( f \in C[0, \infty) \) such that \( \omega(f, t) < \infty \), \( (\forall) t > 0 \), the following inequality holds

\[ |(R_{n,s} f)(x) - f(x)| \leq 2 \omega(f, \theta_{n,s,x}), \]

where

\[ \theta_{n,s,x} = \sqrt[3]{\frac{a_n^2 x^4}{(1 + a_n x^2)} + \frac{r s(r-1)}{n} \frac{x}{n a_n (1 + a_n x^2)}}. \]

**Proof.** Since

\[ |f(y) - f(x)| \leq \omega(f, |y - x|) \leq \left( 1 + \frac{(y-x)^2}{\theta^2} \right) \omega(f, \theta) \]

turns out that

\[ |(R_{n,s} f)(x) - f(x)| \leq \left( \frac{r s(r-1)}{\theta^2} (R_{n,s} (e_1 - x e_0^2))(x) \right) \omega(f, \theta). \]

The result is obtained by choosing

\[ \theta = \theta_{n,s,x} = \sqrt{(R_{n,s} (e_1 - x e_0^2))(x)} \]

\[ = \sqrt[3]{\frac{a_n^2 x^4}{(1 + a_n x^2)} + \frac{r s(r-1)}{n} \frac{x}{n a_n (1 + a_n x^2)}}. \]

\[ \square \]

**Remark 1.** For \( f \in C[0, \infty) \) and \( M > 0 \) we have

\[ ||R_{n,s} f - f||_{[0,M]} \leq 2 \omega(f, \sqrt[3]{a_n^2 M^4 + \frac{r s(r-1)}{n} M \frac{M}{n a_n}}). \]

**Corollary 2.6.** If \( f \) is a function which is uniformly continuous on \([0, \infty)\), then \( f \) can be uniformly approximated on any compact interval \( K \subset [0, \infty) \).

### 3 Some preservation properties

**Lemma 3.1.** For \( f \in C[0, \infty) \), \( 0 \leq x < y \), \( \lambda \in [0,1] \) we have

\[ (R_{n,s} f)((1-\lambda)x + \lambda y) = \sum_{k_1, k_2=0}^{n-1} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} P_{k_1, k_2} \left( \frac{a_n x}{1 + a_n x} \right) \frac{a_n (y-x)}{(1 + a_n x)(1 + a_n y)}, \]

\[ \cdot \left( \sum_{m_1=1}^{l_1} \sum_{m_2=0}^{l_2} P_{m_1, m_2} \left( \frac{\lambda (1 + a_n y)}{\lambda (1 + a_n y) + (1-\lambda)(1 + a_n x)} \right) \right), \]

\[ \cdot \left( \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} P_{m_1, m_2} \left( \frac{\lambda (1 + a_n y)}{\lambda (1 + a_n y) + (1-\lambda)(1 + a_n x)} \right) \right), \]

\[ \cdot \left( \sum_{m_1=1}^{l_1} \sum_{m_2=0}^{l_2} P_{m_1, m_2} \left( \frac{\lambda (1 + a_n y)}{\lambda (1 + a_n y) + (1-\lambda)(1 + a_n x)} \right) \right), \]

\[ \cdot \left( \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} P_{m_1, m_2} \left( \frac{\lambda (1 + a_n y)}{\lambda (1 + a_n y) + (1-\lambda)(1 + a_n x)} \right) \right). \]
where \( p_{m,k,l}(u,v) = \frac{m!}{k!((m-k-l)!)!} u^k v^l (1-u-v)^{m-k-l} \) is the two-variable Bernstein basis.

**Proof.** Let \( f \in C[0, \infty) \), \( 0 \leq x < y \), \( \lambda \in [0,1] \).

We denote by \( \alpha_n(x) = \frac{\alpha_n x}{1+\alpha_n x} \), \( n \in \mathbb{N} \) and note that \( \alpha_n(x) < \alpha_n(y) \).

If \( F_{n,j} : C[0, \infty) \rightarrow \mathbb{R} \), \( i = 0,\ldots,n \), are linear positive functionals, proceeding similarly as in [8], we obtain:

\[
\sum_{j=0}^{n} p_{n,j}(\alpha_n ((1-\lambda)x + \lambda y)) F_{n,j}(f) = \sum_{j=0}^{n} \binom{n}{j} [\alpha_n(x) + \alpha_n ((1-\lambda)x + \lambda y) - \alpha_n(x)]^j \cdot [1 - \alpha_n(y) - \alpha_n(x)]^{n-j} F_{n,j}(f)
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{j} \frac{j!}{(j-k)! k!} [\alpha_n(x)]^k [\alpha_n ((1-\lambda)x + \lambda y) - \alpha_n(x)]^{j-k} \cdot [1 - \alpha_n(y)]^{n-j-k} F_{n,j}(f)
\]

\[
= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{n!}{k!(j-k)!} \alpha_n(x)^k [\alpha_n ((1-\lambda)x + \lambda y) - \alpha_n(x)]^{j-k} \cdot [1 - \alpha_n(y)]^{n-j-k} F_{n,j}(f)
\]

We reverse the summation order and change the index \( j-k+p = l \):

\[
\sum_{j=0}^{n} p_{n,j}(\alpha_n ((1-\lambda)x + \lambda y)) F_{n,j}(f) = \sum_{k=0}^{n} \sum_{j=k}^{n-k} n! \alpha_n(x)^k [\alpha_n ((1-\lambda)x + \lambda y) - \alpha_n(x)]^{j-k} \cdot [1 - \alpha_n(y)]^{n-j-k} F_{n,j}(f)
\]

We reverse the summation order and change the index \( j-k = m \) and we obtain the following representation:

\[
\sum_{j=0}^{n} p_{n,j}(\alpha_n ((1-\lambda)x + \lambda y)) F_{n,j}(f)
\]

\[
= \sum_{k=0}^{n} \sum_{j=k}^{n-k} p_{n,k,l}(\alpha_n(x),\alpha_n(y) - \alpha_n(x)) p_{l-j-k}(\alpha_n((1-\lambda)x + \lambda y) - \alpha_n(x)) \frac{\alpha_n((1-\lambda)x + \lambda y) - \alpha_n(x)}{\alpha_n(y) - \alpha_n(x)} F_{n,j}(f)
\]

Repeating the application of an adapted version of relation (7) yields
Theorem 3.2. Let $f \in C[0, \infty)$. If $f$ is a non-increasing function, then $R_{n,r}f$ is a non-increasing function.

Proof. Let $0 \leq x < y < \infty$. We have

$$(R_{n,r}f)(y) - (R_{n,r}f)(x)$$

$$= \sum_{k_l+1+1=0}^{n-r_1} \sum_{l_2+1=0}^{n-r_2} p_{k_l+1,l_1} \left( \frac{a_n(x)}{(1+a_n)(1+a_n)} \right) \cdot p_{n-r_1,k_l+1} \left( \frac{a_n(y)}{(1+a_n)(1+a_n)} \right) \cdot$$

$$\cdot \left[ f \left( \frac{k_2 + m_2 + r(k_1 + m_1)}{na_n} \right) \right] \leq 0.$$

$\square$

Theorem 3.3. Let $f \in C[0, \infty)$ a non-increasing function. If $f$ is a convex function, then $R_{n,r}f$ is a convex function.

Proof. Let $0 \leq x < y$ and $\lambda \in [0,1]$. From 3.1, we have

$$(R_{n,r}f)((1-\lambda)x + \lambda y)$$

$$= \sum_{k_l+1=0}^{n-r_1} \sum_{l_2=0}^{n-r_2} p_{k_l+1,l_1} \left( \frac{a_n(x)}{1+a_n} \right) \cdot p_{n-r_1,k_l+1} \left( \frac{a_n(y)}{1+a_n} \right) \cdot$$

$$\cdot \left[ f \left( \frac{k_2 + m_2 + r(k_1 + m_1)}{na_n} \right) \right] \leq 0.$$

For $l_2 + r l_1 \neq 0$ we have

$$\frac{k_2 + m_2 + r(k_1 + m_1)}{na_n}$$

$$= \left( 1 - \frac{m_2 + r m_1}{l_2 + r l_1} \right) f \left( \frac{k_2 + r k_1}{na_n} \right) + \frac{m_2 + m_1}{l_2 + r l_1} \cdot \frac{k_2 + l_2 + r(k_1 + l_1)}{na_n}.$$

Since $f$ is a convex function it results

$$f \left( \frac{k_2 + m_2 + r(k_1 + m_1)}{na_n} \right)$$

$$\leq \left( 1 - \frac{m_2 + r m_1}{l_2 + r l_1} \right) f \left( \frac{k_2 + r k_1}{na_n} \right) + \frac{m_2 + m_1}{l_2 + r l_1} \cdot f \left( \frac{k_2 + l_2 + r(k_1 + l_1)}{na_n} \right).$$

$\square$
from where

\[
\begin{align*}
&\leq \sum_{k=0}^{n-1} \sum_{l=0}^{\infty} p_{k,l} \left( \frac{a_n x}{1 + a_n x} \right) p_{n-r,k_1} \left( \frac{a_n x}{1 + a_n x} \right) f \left( \frac{k_2 + r k_1}{na_n} \right) \\
&\quad + \sum_{k_1 + l_1 = 0}^{\infty} \sum_{k_2 + l_2 = 0}^{\infty} p_{k_1,l_1} \left( \frac{a_n x}{1 + a_n x} \right) f \left( \frac{a_n(y - x)}{(1 + a_n x)(1 + a_n y)} \right) \times (1 - \frac{\lambda(1 + a_n y)}{\lambda(1 + a_n y) + (1 - \lambda)(1 + a_n x)}) \times \lambda(1 + a_n y) \\
&\quad \times \left[ 1 - \frac{\lambda(1 + a_n y)}{\lambda(1 + a_n y) + (1 - \lambda)(1 + a_n x)} \right] \times \left[ 1 - \frac{\lambda(1 + a_n y)}{\lambda(1 + a_n y) + (1 - \lambda)(1 + a_n x)} \right] \times \left[ 1 - \frac{\lambda(1 + a_n y)}{\lambda(1 + a_n y) + (1 - \lambda)(1 + a_n x)} \right] \times (R_{n,r,f})(x) \\
&\quad + \frac{\lambda(1 + a_n y)}{\lambda(1 + a_n y) + (1 - \lambda)(1 + a_n x)} (R_{n,r,f})(y)
\end{align*}
\]

Since \( x < y \), we obtain

\[
\frac{\lambda(1 + a_n y)}{\lambda(1 + a_n y) + (1 - \lambda)(1 + a_n x)} \geq \lambda.
\]

Because \( f \) is non-increasing it follows from Theorem 3.2 that \((R_{n,r,f})(x) \geq (R_{n,r,f})(y)\) and hence from (11) we have

\[
\left( 1 - \frac{\lambda(1 + a_n y)}{\lambda(1 + a_n y) + (1 - \lambda)(1 + a_n x)} \right) (R_{n,r,f})(x) + \frac{\lambda(1 + a_n y)}{\lambda(1 + a_n y) + (1 - \lambda)(1 + a_n x)} (R_{n,r,f})(y) \leq (1 - \lambda) (R_{n,r,f})(x) + \lambda (R_{n,r,f})(y).
\]

Then using (10) we get

\[
R_{n,r,f} ((1 - \lambda)x + \lambda y) \leq (1 - \lambda) R_{n,r,f}(x) + \lambda R_{n,r,f}(y).
\]

\[
\square
\]
We denote by \( \text{Lip}_M \alpha \) the class of Lipschitz continuous functions on \([0, \infty)\) with exponent \( \alpha \in (0, 1] \) and the Lipschitz constant \( M > 0 \) i.e. the set of all real valued continuous functions \( f \) defined on \([0, \infty)\) that verify the condition

\[
|f(x) - f(y)| \leq M \cdot |x - y|^{\alpha}, \quad (x, y) \in [0, \infty).
\]

**Theorem 3.4.** Let \( f \in \mathcal{C}[0, \infty) \), \( M > 0 \) and \( \alpha \in (0, 1] \). If \( f \in \text{Lip}_M \alpha \), then \( R_{n, r, f} \in \text{Lip}_M \alpha \).

**Proof.** Let \( 0 \leq x < y < \infty \). We have

\[
|\langle R_{n, r, f} \rangle(y) - \langle R_{n, r, f} \rangle(x)\rangle \leq \sum_{k_1 + l_1 = 0}^{n-r} p_{k_1, l_1} (a_n(x), a_n(y) - a_n(x)) \sum_{k_2 + l_2 = 0}^{n-r} p_{k_2, l_2} (a_n(x), a_n(y) - a_n(x)) \cdot |f\left(\frac{k_2 + l_2 + (k_1 + l_1)r}{n a_n}\right) - f\left(\frac{k_2 + k_1 r}{n a_n}\right)| \leq \sum_{k_1 + l_1 = 0}^{n-r} p_{k_1, l_1} (a_n(x), a_n(y) - a_n(x)) \sum_{k_2 + l_2 = 0}^{n-r} p_{k_2, l_2} (a_n(x), a_n(y) - a_n(x)) \cdot \cdot \cdot M \left(\frac{l_2 + l_1 r}{n a_n}\right)^\alpha \leq M \sum_{k_1 + l_1 = 0}^{n-r} p_{k_1, l_1} (a_n(x), a_n(y) - a_n(x)) \cdot \left[\sum_{k_2 + l_2 = 0}^{n-r} p_{k_2, l_2} (a_n(x), a_n(y) - a_n(x)) \cdot \left(\frac{l_2 + l_1 r}{n a_n}\right)\right]^\alpha \leq M \sum_{k_1 + l_1 = 0}^{n-r} p_{k_1, l_1} (a_n(x), a_n(y) - a_n(x)) \cdot \left[\sum_{k_2 + l_2 = 0}^{n-r} p_{k_2, l_2} (a_n(x), a_n(y) - a_n(x)) \cdot \left(\frac{l_2 + l_1 r}{n a_n}\right)\right]^\alpha \leq M \left[\sum_{k_1 + l_1 = 0}^{n-r} p_{k_1, l_1} (a_n(x), a_n(y) - a_n(x)) \cdot \left(\frac{n-r s}{n a_n} (a_n(y) - a_n(x)) + \frac{l_1 r l_2 l_1}{n a_n}\right)\right]^\alpha \leq M \left[\frac{n-r s}{n a_n} (a_n(y) - a_n(x)) + \frac{r s}{n a_n} (a_n(y) - a_n(x))\right]^\alpha \leq M \left(1 + \alpha_n x\right)^\alpha (y-x)^\alpha \leq M (y-x)^\alpha.
\]

\[\square\]

**References**


