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# A Stancu type generalization of the Balázs operator

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*Dedicated to Professor Ioan Raşa on the occasion of his $70^{th}$ birthday*

**Abstract**

In this paper we investigate certain properties of Stancu type generalization of the Balázs operator.

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## 1 The Balázs-Stancu operators

For $f \in C[0,\infty)$, the Balázs operators [5] are defined by

$$\begin{aligned}(R_n f)(x) &= \frac{1}{(1+a_n x)^n}\sum_{j=0}^{n}\binom{n}{j}(a_n x)^j f\left(\frac{j}{b_n}\right) \qquad (1)\\ &= \sum_{j=0}^{n} p_{n,j}\left(\frac{a_n x}{1+a_n x}\right) f\left(\frac{j}{b_n}\right),\ x\ge 0,\ n\in\mathbb{N},\end{aligned}$$

where

$$p_{n,j}(z)=\binom{n}{j}z^j(1-z)^{n-j}, z\ge 0,$$

and $(a_n)_n$, $(b_n)_n$ are two sequences of positive real numbers suitably chosen.

These operators have been studied and generalized in many directions [6], [11], [7], [1], [2], [3], [9].

In this paper we consider a generalization of Balázs operators in the manner of the generalization of Bernstein operators introduced by D. D. Stancu in [10]

$$(S_{n,r,s}f)(x)=\sum_{j=0}^{n-rs}p_{n-rs,j}(x)\sum_{i=0}^{s}p_{s,i}(x)f\left(\frac{j+ir}{n}\right), \qquad (2)$$

$f \in C[0,1]$, $x\in[0,1]$, where $n\in\mathbb{N}$ and $r,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ are fixed such that $rs<n$. Bernstein's operators are obtained for $s=0$ or $s=1$, $r=0$ or $s=1$, $r=1$.

We consider the Balázs-Stancu operators, defined as follows:

$$(R_{n,r,s}f)(x)=\sum_{j=0}^{n-rs}p_{n-rs,j}\Big(\frac{a_n x}{1+a_n x}\Big)\sum_{i=0}^{s}p_{s,i}\Big(\frac{a_n x}{1+a_n x}\Big)f\Big(\frac{j+ir}{na_n}\Big), \qquad (3)$$

$f\in C[0,\infty)$, $x\ge 0$, where $n\in\mathbb{N}$, $r,s\in\mathbb{N}_0$ such that $rs<n$, $(a_n)_n$ being a sequence of positive real numbers.

If $a_n=1, (\forall)n\in\mathbb{N}$, we have $(R_{n,r,s}f)(x)=(S_{n,r,s}f)\left(\frac{x}{1+x}\right)$.

## 2 Convergence properties

**Lemma 2.1.** *The operator $S_{n,r,s}$ satisfies the following relations:*

*(i)* $(S_{n,r,s}e_0)(x)=1;$

*(ii)* $(S_{n,r,s}e_1)(x)=x;$

*(iii)* $(S_{n,r,s}e_2)(x)=x^2+\left(1+\frac{rs(r-1)}{n}\right)\cdot\frac{x(1-x)}{n}.$

*where $x\in[0,\infty)$ and $e_i(y)=y^i, i=0,1,2$.*

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*Proof.* For $p \in \mathbb{N}_0$, we have

$$\begin{aligned}(S_{n,r,s}e_{p+1})(x) &= \sum_{j=0}^{n-rs} p_{n-rs,j}(x) \sum_{i=0}^{s} p_{s,i}(x) \left(\frac{j+ir}{n}\right)^{p+1} \\ &= \sum_{j=0}^{n-rs} p_{n-rs,j}(x) \sum_{i=0}^{s} p_{s,i}(x) \sum_{k=0}^{p+1} \binom{p+1}{k} \left(\frac{ir}{n}\right)^k \left(\frac{j}{n}\right)^{p+1-k} \\ &= \sum_{k=0}^{p+1} \binom{p+1}{k} \left(\frac{rs}{n}\right)^k \left(\frac{n-rs}{n}\right)^{p+1-k} \cdot \\ &\quad \cdot \sum_{j=0}^{n-rs} p_{n-rs,j}(x) \left(\frac{j}{n-rs}\right)^{p+1-k} \sum_{i=0}^{s} p_{s,i}(x) \left(\frac{i}{s}\right)^k \\ &= \sum_{k=0}^{p+1} \binom{p+1}{k} \left(\frac{rs}{n}\right)^k \left(1-\frac{rs}{n}\right)^{p+1-k} (B_s e_k)(x) \left(B_{n-rs} e_{p+1-k}\right)(x),\end{aligned}$$

where $(B_n f)(x)$ are the Bernstein operators.

From the above relation, one has:

(i) $(S_{n,r,s}e_0)(x) = \sum_{j=0}^{n-rs} p_{n-rs,j}(x) \sum_{i=0}^{s} p_{s,i}(x) = 1.$

(ii)

$$\begin{aligned}(S_{n,r,s}e_1)(x) &= \left(1-\frac{rs}{n}\right)(B_s e_0)(x)(B_{n-rs}e_1)(x) + \frac{rs}{n}(B_s e_1)(x)(B_{n-rs}e_0)(x) \\ &= \left(1-\frac{rs}{n}\right)x + \frac{rs}{n}x = x.\end{aligned}$$

(iii)

$$\begin{aligned}(S_{n,r,s}e_2)(x) &= \left(1-\frac{rs}{n}\right)^2 (B_s e_0)(x)(B_{n-rs}e_2)(x) \\ &\quad +2\frac{rs}{n}\left(1-\frac{rs}{n}\right)(B_s e_1)(x)(B_{n-rs}e_1)(x) \\ &\quad +\left(\frac{rs}{n}\right)^2 (B_s e_2)(x)(B_{n-rs}e_0)(x) \\ &= \left(1-\frac{rs}{n}\right)^2 \left(x^2 + \frac{x(1-x)}{n-rs}\right) + 2\frac{rs}{n}\left(1-\frac{rs}{n}\right)x^2 \\ &\quad +\left(\frac{rs}{n}\right)^2 \left(x^2 + \frac{x(1-x)}{s}\right) \\ &= x^2 + \left(1+\frac{rs(r-1)}{n}\right) \cdot \frac{x(1-x)}{n}.\end{aligned}$$

□

**Lemma 2.2.** *The operator $R_{n,r,s}$ satisfies the following relations:*

*(i)* $R_{n,r,s}f \geq 0$, $(\forall) f \in C[0,\infty)$, $f \geq 0$;

*(ii)* $(R_{n,r,s}e_0)(x) = 1$;

*(iii)* $(R_{n,r,s}e_1)(x) = \dfrac{x}{1+a_n x}$;

*(iv)* $(R_{n,r,s}e_2)(x) = \dfrac{x^2}{(1+a_n x)^2} + \left(1+\dfrac{rs(r-1)}{n}\right) \cdot \dfrac{x}{na_n(1+a_n x)^2}$;

*where $x \in [0,\infty)$ and $e_p(y) = y^p, p = 0,1,2$.*

*Proof.* We specify that we have

$$\begin{aligned}(R_{n,r,s}e_p)(x) &= \sum_{j=0}^{n-rs} p_{n-rs,j}\Big(\frac{a_n x}{1+a_n x}\Big) \sum_{i=0}^{s} p_{s,i}\Big(\frac{a_n x}{1+a_n x}\Big)\Big(\frac{j+ir}{na_n}\Big)^p \\ &= \frac{1}{a_n^p}\left(S_{n,r,s}e_p\right)\left(\frac{a_n x}{1+a_n x}\right).\end{aligned}$$

(i) It is obvious by definition;

(ii) It is clear that

$$(R_{n,r,s}e_0)(x) = (S_{n,r,s}e_0)\left(\frac{a_nx}{1+a_nx}\right) = 1.$$

(iii) From $(S_{n,r,s}e_1)(x) = x$ it is obtained

$$(R_{n,r,s}e_1)(x) = \frac{1}{a_n}(S_{n,r,s}e_1)\left(\frac{a_nx}{1+a_nx}\right) = \frac{x}{1+a_nx}.$$

(iv) From $(S_{n,r,s}e_2)(x) = x^2 + \left(1+\frac{rs(r-1)}{n}\right)\cdot\frac{x(1-x)}{n}$ it is obtained

$$\begin{aligned}(R_{n,r,s}e_2)(x) &= \frac{1}{a_n^2}(S_{n,r,s}e_2)\left(\frac{a_nx}{1+a_nx}\right)\\ &= \frac{x^2}{(1+a_nx)^2} + \left(1+\frac{rs(r-1)}{n}\right)\cdot\frac{x}{na_n(1+a_nx)^2}.\end{aligned}$$

□

**Lemma 2.3.** *Let the m-th order moment for the operator be denoted as follows:*

$$\left(R_{n,r,s}(e_1-xe_0)^m\right)(x) = \sum_{j=0}^{n-rs} p_{n-rs,j}\Big(\frac{a_nx}{1+a_nx}\Big)\sum_{i=0}^{s} p_{s,i}\Big(\frac{a_nx}{1+a_nx}\Big)\cdot\Big(\frac{j+ir}{na_n}-x\Big)^m, m = 1,2,\ldots.$$

*Then we have*

*(i)*

$$\left(R_{n,r,s}(e_1-xe_0)\right)(x) = -\frac{a_nx^2}{1+a_nx};$$

*(ii)*

$$\left(R_{n,r,s}(e_1-xe_0)^2\right)(x) = r\frac{a_n^2x^4}{(1+a_nx)^2} + \Big(1+\frac{rs(r-1)}{n}\Big)\cdot\frac{x}{na_n(1+a_nx)^2}$$

*Proof.* (i)

$$\begin{aligned}\left(R_{n,r,s}(e_1-xe_0)\right)(x) &= \sum_{j=0}^{n-rs} p_{n-rs,j}\Big(\frac{a_nx}{1+a_nx}\Big)\sum_{i=0}^{s} p_{s,i}\Big(\frac{a_nx}{1+a_nx}\Big)\cdot\Big(\frac{j+ir}{na_n}-x\Big)\\ &= (R_{n,r,s}e_1)(x) - x = -\frac{a_nx^2}{1+a_nx};\end{aligned}$$

(ii)

$$\begin{aligned}\left(R_{n,r,s}(e_1-xe_0)^2\right)(x) &= \sum_{j=0}^{n-rs} p_{n-rs,j}\Big(\frac{a_nx}{1+a_nx}\Big)\sum_{i=0}^{s} p_{s,i}\Big(\frac{a_nx}{1+a_nx}\Big)\cdot\Big(\frac{j+ir}{na_n}-x\Big)^2\\ &= (R_{n,r,s}e_2)(x) - 2x(R_{n,r,s}e_1)(x) + x^2\\ &= \frac{a_n^2x^4}{(1+a_nx)^2} + \Big(1+\frac{rs(r-1)}{n}\Big)\cdot\frac{x}{na_n(1+a_nx)^2}.\end{aligned}$$

□

**Theorem 2.4.** *If* $\lim_{n\to\infty} a_n = 0$ *and* $\lim_{n\to\infty} na_n = \infty$, *then for a bounded function* $f \in C[0,\infty)$ *it follows*

$$\lim_{n\to\infty} R_{n,r,s}f = f \text{ uniformly on any compact interval } K \subset [0,\infty).$$

*Proof.* Let $K \subset [0,\infty)$ be a compact interval, $K = [m,M]$, $0 \le m < M < \infty$.
It is obvious that

$$\lim_{n\to\infty} \|R_{n,r,s}e_0 - e_0\|_{[m,M]} = 0.$$

Since

$$|\left(R_{n,r,s}e_1\right)(x) - e_1(x)| = \frac{a_nx^2}{1+a_nx} \le a_nM^2, (\forall)x \in [m,M]$$

and $a_n M^2 \underset{n\to\infty}{\longrightarrow} 0$, result

$$\lim_{n\to\infty} \|R_{n,r,s}e_1 - e_1\|_{[m,M]} = 0.$$

Since

$$\begin{aligned}
|\left(R_{n,r,s}e_2\right)(x) - e_2(x)| &= |-\frac{a_n x^3 (2+a_n x)}{(1+a_n x)^2} + \left(1+\frac{rs(r-1)}{n}\right)\frac{x}{na_n(1+a_n x)^2}| \\
&\le a_n M^3(2+a_n M) + \left(1+\frac{rs(r-1)}{n}\right)\frac{M}{na_n}, (\forall)x\in[m,M]
\end{aligned}$$

and $a_n M^3(2+a_n M) + \left(1+\frac{rs(r-1)}{n}\right)\frac{M}{na_n} \underset{n\to\infty}{\longrightarrow} 0$, result

$$\lim_{n\to\infty} \|R_{n,r,s}e_2 - e_2\|_{[m,M]} = 0.$$

Finally, Theorem 2.4 results by applying [4]-Theorem 4.1. □

The modulus of continuity of a continuous function $f$ on $[0,\infty)$, is defined by

$$\omega(f,t) = \sup\{|f(y)-f(x)| : x,y\in[0,\infty), |y-x|\le t\}, t>0.$$

**Theorem 2.5.** *For any function $f\in C[0,\infty)$ such that $\omega(f,t)<\infty$, $(\forall)t>0$, the following inequality holds*

$$|\left(R_{n,r,s}f\right)(x) - f(x)| \le 2\omega\left(f,\theta_{n,r,s,x}\right), \tag{4}$$

*where*

$$\theta_{n,r,s,x} = \sqrt{\frac{a_n^2 x^4}{(1+a_n x)^2} + \left(1+\frac{rs(r-1)}{n}\right)\cdot\frac{x}{na_n(1+a_n x)^2}}.$$

*Proof.* Since

$$|f(y)-f(x)| \le \omega(f,|y-x|) \le \left(1+\frac{(y-x)^2}{\theta^2}\right)\omega(f,\theta)$$

turns out that

$$\begin{aligned}
|\left(R_{n,r,s}f\right)(x) - f(x)| &\le \left(R_{n,r,s}|f - f(x)e_0|\right)(x) \\
&\le \left(1+\frac{\left(R_{n,r,s}(e_1 - xe_0)^2\right)(x)}{\theta^2}\right)\omega(f,\theta).
\end{aligned}$$

The result is obtained by choosing

$$\begin{aligned}
\theta &= \theta_{n,r,s,x} = \sqrt{\left(R_{n,r,s}(e_1-xe_0)^2\right)(x)} \\
&= \sqrt{\frac{a_n^2 x^4}{(1+a_n x)^2} + \left(1+\frac{rs(r-1)}{n}\right)\cdot\frac{x}{na_n(1+a_n x)^2}}.
\end{aligned}$$

□

*Remark* 1. For $f\in C[0,\infty)$ and $M>0$ we have

$$\|R_{n,r,s}f - f\|_{[0,M]} \le 2\omega\left(f, \sqrt{a_n^2 M^4 + \left(1+\frac{rs(r-1)}{n}\right)\cdot\frac{M}{na_n}}\right). \tag{5}$$

**Corollary 2.6.** *If $f$ is a function which is uniformly continuous on $[0,\infty)$, then $f$ can be uniformly approximated on any compact interval $K\subset[0,\infty)$.*

## 3 Some preservation properties

**Lemma 3.1.** *For $f\in C[0,\infty)$, $0\le x<y$, $\lambda\in[0,1]$ we have*

$$\begin{aligned}
&(R_{n,r,s}f)((1-\lambda)x+\lambda y) \\
&= \sum_{k_1+l_1=0}^{s}\sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}\left(\frac{a_n x}{1+a_n x}, \frac{a_n(y-x)}{(1+a_n x)(1+a_n y)}\right)\cdot \\
&\cdot p_{n-rs,k_2,l_2}\left(\frac{a_n x}{1+a_n x}, \frac{a_n(y-x)}{(1+a_n x)(1+a_n y)}\right)\cdot \\
&\cdot \sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}\left(\frac{\lambda(1+a_n y)}{\lambda(1+a_n y)+(1-\lambda)(1+a_n x)}\right)\cdot \\
&\cdot p_{l_2,m_2}\left(\frac{\lambda(1+a_n y)}{\lambda(1+a_n y)+(1-\lambda)(1+a_n x)}\right) f\left(\frac{k_2+m_2+r(k_1+m_1)}{na_n}\right),
\end{aligned} \tag{6}$$

*where* $p_{m,k,l}(u,v)=\dfrac{m!}{k!l!(m-k-l)!}u^k v^l(1-u-v)^{m-k-l}$ *is the two-variable Bernstein basis.*

*Proof.* Let $f\in C[0,\infty)$, $0\le x<y$, $\lambda\in[0,1]$.

We denote by $\alpha_n(x)=\dfrac{a_n x}{1+a_n x}$, $n\in\mathbb{N}$ and note that $\alpha_n(x)<\alpha_n(y)$.

If $F_{n,j}:C[0,\infty)\longrightarrow\mathbb{R}$, $i=0,\dots,n$, are linear positive functionals, proceeding similarly as in [8], we obtain:

$$\begin{aligned}
&\sum_{j=0}^{n} p_{n,j}\left(\alpha_n((1-\lambda)x+\lambda y)\right)F_{n,j}(f)\\
&=\sum_{j=0}^{n}\binom{n}{j}\left[\alpha_n(x)+\alpha_n((1-\lambda)x+\lambda y)-\alpha_n(x)\right]^j\cdot\\
&\cdot\left[1-\alpha_n(y)+\alpha_n(y)-\alpha_n((1-\lambda)x+\lambda y)\right]^{n-j}F_{n,j}(f)\\
&=\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}\binom{j}{k}\alpha_n(x)^k\left[\alpha_n((1-\lambda)x+\lambda y)-\alpha_n(x)\right]^{j-k}\cdot\\
&\cdot\sum_{p=0}^{n-j}\binom{n-j}{p}\left[\alpha_n(y)-\alpha_n((1-\lambda)x+\lambda y)\right]^p\left[1-\alpha_n(y)\right]^{n-j-p}F_{n,j}(f)\\
&=\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{p=0}^{n-j}\frac{n!}{k!(j-k)!p!(n-j-p)!}\alpha_n(x)^k\left[\alpha_n((1-\lambda)x+\lambda y)-\alpha_n(x)\right]^{j-k}\cdot\\
&\cdot\left[\alpha_n(y)-\alpha_n((1-\lambda)x+\lambda y)\right]^p\left[1-\alpha_n(y)\right]^{n-j-p}F_{n,j}(f)\\
&=\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{p=0}^{n-j}\frac{n!}{k!(j-k+p)!(n-j-p)!}\alpha_n(x)^k\left[\alpha_n(y)-\alpha_n(x)\right]^{j-k+p}\left[1-\alpha_n(y)\right]^{n-j-p}\cdot\\
&\cdot\frac{(j-k+p)!}{(j-k)!p!}\left[\frac{\alpha_n((1-\lambda)x+\lambda y)-\alpha_n(x)}{\alpha_n(y)-\alpha_n(x)}\right]^{j-k}\left[\frac{\alpha_n(y)-\alpha_n((1-\lambda)x+\lambda y)}{\alpha_n(y)-\alpha_n(x)}\right]^p F_{n,j}(f)\\
&=\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{p=0}^{n-j}p_{n,k,j-k+p}\left(\alpha_n(x),\alpha_n(y)-\alpha_n(x)\right)\cdot\\
&\cdot p_{j-k+p,j-k}\left(\frac{\alpha_n((1-\lambda)x+\lambda y)-\alpha_n(x)}{\alpha_n(y)-\alpha_n(x)}\right)F_{n,j}(f)
\end{aligned}$$

We reverse the summation order and change the index $j-k+p=l$:

$$\begin{aligned}
&\sum_{j=0}^{n}p_{n,j}\left(\alpha_n((1-\lambda)x+\lambda y)\right)F_{n,j}(f)\\
&=\sum_{k=0}^{n}\sum_{j=k}^{n}\sum_{l=j-k}^{n-k}p_{n,k,l}\left(\alpha_n(x),\alpha_n(y)-\alpha_n(x)\right)p_{l,j-k}\left(\frac{\alpha_n((1-\lambda)x+\lambda y)-\alpha_n(x)}{\alpha_n(y)-\alpha_n(x)}\right)F_{n,j}(f)
\end{aligned}$$

We reverse the summation order and change the index $j-k=m$ and we obtain the following representation:

$$\begin{aligned}
&\sum_{j=0}^{n}p_{n,j}\left(\alpha_n((1-\lambda)x+\lambda y)\right)F_{n,j}(f)\\
&=\sum_{k+l=0}^{n}p_{n,k,l}\left(\alpha_n(x),\alpha_n(y)-\alpha_n(x)\right)\cdot\\
&\cdot\sum_{m=0}^{l}p_{l,m}\left(\frac{\alpha_n((1-\lambda)x+\lambda y)-\alpha_n(x)}{\alpha_n(y)-\alpha_n(x)}\right)F_{n,k+m}(f).
\end{aligned}\tag{7}$$

Repeating the application of an adapted version of relation (7) yields

$$\begin{aligned}
&(R_{n,r,s}f)((1-\lambda)x+\lambda y)\\
&=\sum_{k_1+l_1=0}^{s}\sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\cdot\\
&\cdot p_{n-rs,k_2,l_2}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\cdot\\
&\cdot\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}\left(\frac{\alpha_n((1-\lambda)x+\lambda y)-\alpha_n(x)}{\alpha_n(y)-\alpha_n(x)}\right)\cdot\\
&\cdot p_{l_2,m_2}\left(\frac{\alpha_n((1-\lambda)x+\lambda y)-\alpha_n(x)}{\alpha_n(y)-\alpha_n(x)}\right) f\left(\frac{k_2+m_2+r(k_1+m_1)}{na_n}\right).
\end{aligned}$$

□

**Theorem 3.2.** *Let $f\in C[0,\infty)$. If $f$ is a non-increasing function, then $R_{n,r,s}f$ is a non-increasing function.*

*Proof.* Let $0\le x<y<\infty$. We have

$$\begin{aligned}
&(R_{n,r,s}f)(y)-(R_{n,r,s}f)(x)\\
&=\sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\sum_{k_2+l_2=0}^{n-rs} p_{n-rs,k_2,l_2}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\cdot\\
&\quad\cdot\left[f\left(\frac{k_2+l_2+(k_1+l_1)r}{na_n}\right)-f\left(\frac{k_2+k_1r}{na_n}\right)\right]\le 0.
\end{aligned}$$

□

**Theorem 3.3.** *Let $f\in C[0,\infty)$ a non-increasing function. If $f$ is a convex function, then $R_{n,r,s}f$ is a convex function.*

*Proof.* Let $0\le x<y$ and $\lambda\in[0,1]$. From 3.1, we have

$$\begin{aligned}
&(R_{n,r,s}f)((1-\lambda)x+\lambda y)\\
&=\sum_{k_1+l_1=0}^{s}\sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}\left(\frac{a_nx}{1+a_nx},\frac{a_n(y-x)}{(1+a_nx)(1+a_ny)}\right)\cdot\\
&\cdot p_{n-rs,k_2,l_2}\left(\frac{a_nx}{1+a_nx},\frac{a_n(y-x)}{(1+a_nx)(1+a_ny)}\right)\cdot\\
&\cdot\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}\left(\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}\right)\cdot\\
&\cdot p_{l_2,m_2}\left(\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}\right) f\left(\frac{k_2+m_2+r(k_1+m_1)}{na_n}\right).
\end{aligned}$$

For $l_2+rl_1\neq 0$ we have

$$\begin{aligned}
&\frac{k_2+m_2+r(k_1+m_1)}{na_n}\\
&=\left(1-\frac{m_2+rm_1}{l_2+rl_1}\right)\frac{k_2+rk_1}{na_n}+\frac{m_2+rm_1}{l_2+rl_1}\cdot\frac{k_2+l_2+r(k_1+l_1)}{na_n}.
\end{aligned}$$

Since $f$ is a convex function it results

$$\begin{aligned}
&f\left(\frac{k_2+m_2+r(k_1+m_1)}{na_n}\right)\\
&\le\left(1-\frac{m_2+rm_1}{l_2+rl_1}\right)f\left(\frac{k_2+rk_1}{na_n}\right)+\frac{m_2+rm_1}{l_2+rl_1}f\left(\frac{k_2+l_2+r(k_1+l_1)}{na_n}\right),
\end{aligned}$$

from where

$$
\begin{aligned}
& (R_{n,r,s}f)((1-\lambda)x+\lambda y) \\
\leq\; & \sum_{k_1=0}^{s}\sum_{k_2=0}^{n-rs} p_{s,k_1}\left(\frac{a_nx}{1+a_nx}\right)p_{n-rs,k_2}\left(\frac{a_nx}{1+a_nx}\right)f\left(\frac{k_2+rk_1}{na_n}\right) \\
& +\sum_{\substack{k_1+l_1=0 \\ l_2+rl_1\neq 0}}^{s}\sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}\left(\frac{a_nx}{1+a_nx},\frac{a_n(y-x)}{(1+a_nx)(1+a_ny)}\right)\times \\
& \cdot p_{n-rs,k_2,l_2}\left(\frac{a_nx}{1+a_nx},\frac{a_n(y-x)}{(1+a_nx)(1+a_ny)}\right)\times \\
& \times\Big[f\left(\frac{k_2+rk_1}{na_n}\right)\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}\left(\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}\right)\times \qquad (8) \\
& \times p_{l_2,m_2}\left(\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}\right)\left(1-\frac{m_2+rm_1}{l_2+rl_1}\right) \\
& +f\left(\frac{k_2+l_2+r(k_1+l_1)}{na_n}\right)\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}\left(\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}\right)\times \\
& \times p_{l_2,m_2}\left(\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}\right)\frac{m_2+rm_1}{l_2+rl_1}\Big] \\
=\; & \sum_{k_1=0}^{s}\sum_{k_2=0}^{n-rs} p_{s,k_1}\left(\frac{a_nx}{1+a_nx}\right)p_{n-rs,k_2}\left(\frac{a_nx}{1+a_nx}\right)f\left(\frac{k_2+rk_1}{na_n}\right) \\
& +\sum_{\substack{k_1+l_1=0 \\ l_2+rl_1\neq 0}}^{s}\sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}\left(\frac{a_nx}{1+a_nx},\frac{a_n(y-x)}{(1+a_nx)(1+a_ny)}\right)\times \qquad (9) \\
& \times p_{n-rs,k_2,l_2}\left(\frac{a_nx}{1+a_nx},\frac{a_n(y-x)}{(1+a_nx)(1+a_ny)}\right)\times \\
& \times\Big[\left(1-\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}\right)f\left(\frac{k_2+rk_1}{na_n}\right) \\
& +\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}f\left(\frac{k_2+l_2+r(k_1+l_1)}{na_n}\right)\Big] \\
=\; & \sum_{k_1+l_1=0}^{s}\sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}\left(\frac{a_nx}{1+a_nx},\frac{a_n(y-x)}{(1+a_nx)(1+a_ny)}\right)\times \\
& \cdot p_{n-rs,k_2,l_2}\left(\frac{a_nx}{1+a_nx},\frac{a_n(y-x)}{(1+a_nx)(1+a_ny)}\right)\times \\
& \times\Big[\left(1-\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}\right)f\left(\frac{k_2+rk_1}{na_n}\right) \\
& +\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}f\left(\frac{k_2+l_2+r(k_1+l_1)}{na_n}\right)\Big] \\
=\; & \left(1-\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}\right)(R_{n,r,s}f)(x) \\
& +\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}(R_{n,r,s}f)(y). \qquad (10)
\end{aligned}
$$

Since $x<y$, we obtain

$$\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}\geq\lambda. \qquad (11)$$

Because $f$ is non-increasing it follows from Theorem 3.2 that $(R_{n,r,s}f)(x)\geq(R_{n,r,s}f)(y)$ and hence from (11) we have

$$\left(1-\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}\right)(R_{n,r,s}f)(x)+\frac{\lambda(1+a_ny)}{\lambda(1+a_ny)+(1-\lambda)(1+a_nx)}(R_{n,r,s}f)(y)\leq(1-\lambda)R_{n,r,s}f)(x)+\lambda R_{n,r,s}f)(y).$$

Then using (10) we get

$$R_{n,r,s}f)((1-\lambda)x+\lambda y)\leq(1-\lambda)R_{n,r,s}f)(x)+\lambda R_{n,r,s}f)(y).$$

□

We denote by $Lip_M\alpha$ the class of Lipschitz continuous functions on $[0,\infty)$ with exponent $\alpha\in(0,1]$ and the Lipschitz constant $M>0$ i.e. the set of all real valued continuous functions $f$ defined on $[0,\infty)$ that verify the condition

$$|f(x)-f(y)|\le M\cdot|x-y|^{\alpha},\ (\forall)x,y\in[0,\infty).$$

**Theorem 3.4.** *Let $f\in C[0,\infty)$, $M>0$ and $\alpha\in(0,1]$. If $f\in Lip_M\alpha$, then $R_{n,r,s}f\in Lip_M\alpha$.*

*Proof.* Let $0\le x<y<\infty$. We have

$$\begin{aligned}
&|(R_{n,r,s}f)(y)-(R_{n,r,s}f)(x)|\\
&\le \sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\sum_{k_2+l_2=0}^{n-rs} p_{n-rs,k_2,l_2}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\cdot\\
&\quad\cdot\left|f\left(\frac{k_2+l_2+(k_1+l_1)r}{na_n}\right)-f\left(\frac{k_2+k_1r}{na_n}\right)\right|\\
&\le \sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\sum_{k_2+l_2=0}^{n-rs} p_{n-rs,k_2,l_2}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\cdot\\
&\quad\cdot M\left(\frac{l_2+l_1r}{na_n}\right)^{\alpha}\\
&\le M\sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\cdot\\
&\quad\cdot\left[\sum_{k_2+l_2=0}^{n-rs} p_{n-rs,k_2,l_2}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\frac{l_2+l_1r}{na_n}\right]^{\alpha}\\
&= M\sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\cdot\left[\frac{n-rs}{na_n}(\alpha_n(y)-\alpha_n(x))+\frac{l_1r}{na_n}\right]^{\alpha}\\
&\le M\left[\sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(\alpha_n(x),\alpha_n(y)-\alpha_n(x))\cdot\left[\frac{n-rs}{na_n}(\alpha_n(y)-\alpha_n(x))+\frac{l_1r}{na_n}\right]\right]^{\alpha}\\
&= M\left[\frac{n-rs}{na_n}(\alpha_n(y)-\alpha_n(x))+\frac{rs}{na_n}(\alpha_n(y)-\alpha_n(x))\right]^{\alpha}\\
&= M\left[\frac{1}{(1+a_nx)(1+a_ny)}\right]^{\alpha}(y-x)^{\alpha}\\
&\le M(y-x)^{\alpha}.
\end{aligned}$$

□

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