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Bernstein numerical method for solving nonlinear fractional and weakly singular Volterra integral equations of the second kind

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Dedicated to Professor Ioan Raşa on the occasion of his 70th birthday

Abstract

An iterative numerical method is proposed for solving nonlinear fractional Volterra integral equations. This method is based on Picard iterations and at each iterative step we will apply a piecewise Bernstein polynomial approximation technique. The convergence of the method is proved by providing the error estimate in the discrete and continuous approximation. The theoretical results are tested on some numerical examples, illustrating the performances of the proposed method.

Keywords and phrases: nonlinear Volterra fractional integral equations, iterative numerical method, Bernstein splines.

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1 Introduction

In this work we present the effectiveness of the piecewise Bernstein polynomial functions when are applied for approximating the solution of nonlinear fractional Volterra integral equations. The study of fractional integral and differential equations is motivated by their applicability in mapping real world phenomena and processes. The exact solution computation sometimes is very difficult, especially for automatic calculations made by modern computers, so an approximation method can be useful in this situations.

Fractional calculus is widely used in many applications in science such as physics, engineering, modeling, biomedical applications [28], pulses of sound reflections [8], neural networks [21], [13], some optimal control problems which are typically nonlinear, are ruled by Volterra integral or Volterra integral derivative systems [25].

There are different definitions for the fractional integral like Riemann-Liouville, Hadamard, Atangana–Baleanu (see [2] and [6]) and the most intensive studied fractional order integral equation is the Abel's equation (see [2], [4], [12], [14], [16], [24], [29]). Several numerical methods were developed for solving fractional Volterra integral equations, such as Galerkin method, collocation, Taylor series (see [1], [4], [22], [26], [27]), product integration (see [1], [3], [7], [11], [18], [19]), multistep Adams-Bashforth techniques (see [5], [9], and [10]), fast Fourier transform techniques (see [13]), Runge-Kutta procedures (see [16]), Bernstein polynomial approximation with Voronovskaia's type error estimate (see [29]), Tau method using Jacobi functions (see [24]), Haar, Legendre and Riesz wavelet (see [23] and [31]), piecewise linear functions (see [20]), Lagrange polynomials collocation (see [7]), Legendre spectral collocation (see [32]), Nyström methods (see [3]), Adomian decomposition (see [17]), homotopy perturbation (see [14]), variational iteration (see [30]).

In that follows, we construct an iterative method based on Picard iterations and piecewise Bernstein polynomials applied at each iterative step, for approximating the solution of the following fractional type Riemann-Liouville nonlinear Volterra integral equation:

$$x(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t H(t,s)(t-s)^{\alpha-1} f(s,x(s))\,ds, \quad t \in [0,T],\ \alpha \in (0,1). \tag{1}$$

The first section of this work is devoted to some boundedness and Lipshitz properties of the sequence of Picard iterations, while in the second section we present the construction of the Bernstein splines iterative method. Then, the convergence of this method is proved by providing the error estimates in the discrete and continuous approximation. In the last two sections we present some numerical experiments and concluding remarks.

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2 The sequence of successive approximations

In order to obtain the existence and uniqueness of the solution of equation (1), we consider the integral operator $A : C[0,T] \to C[0,T]$ given by the following expression:

$$A(x)(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t H(t,s)(t-s)^{\alpha-1} f(s,x(s))\,ds, \quad \alpha \in (0,1) \tag{2}$$

Theorem 2.1. *If $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, $g : [0,T] \to \mathbb{R}$ and $H : [0,T] \times [0,T] \to \mathbb{R}_+$ are continuous, $\alpha \in (0,1)$ and $M_H \geq 0$ is such that $\max\limits_{t,s\in[0,T]} H(t,s) = M_H$ and if f is Lipschitz on its second argument, with Lipschitz constant L, then under the following condition*

$$w \overset{not.}{=} \frac{LT^\alpha M_H}{\Gamma(\alpha+1)} < 1 \tag{3}$$

the integral equation (1) has a unique solution $x^* \in C(0,T)$.

Proof. After elementary computation, the integral operator (2) is a contraction, having

$$\begin{aligned} |A(x)(t) - A(y)(t)| &= \tfrac{1}{\Gamma(\alpha)} \left| \int_0^t H(t,s)(t-s)^{\alpha-1} [f(s,x(s)) - f(s,y(s))]\,ds \right| \\ &\leq \tfrac{LT^\alpha M_H}{\Gamma(\alpha+1)} \|x-y\|_\infty, \quad \forall x,y \in C([0,T]), \forall t \in [0,T] \end{aligned}$$

which means

$$\|A(x) - A(y)\|_\infty \leq \frac{LT^\alpha M_H}{\Gamma(\alpha+1)} \|x-y\|_\infty, \quad \forall x,y \in C([0,T])$$

and based on the condition (3), by using the Banach Fixed Point Principle, we obtain the desired result. □

Defining the sequence of succesive approximation as

$$\begin{aligned} x_0(t) &= g(t),\ t \in [0,T] \\ x_{m+1}(t) &= A(x_m)(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t H(t,s)(t-s)^{\alpha-1} f(s,x_m(s))\,ds \end{aligned}$$

we get $\lim\limits_{m\to\infty} x_m(t) = x^*(t)$, $\forall t \in [0,T]$ and the following error estimates:

$$\begin{aligned} \|x^* - x_m\|_\infty &\leq \frac{w^m}{1-w} \|x_1 - x_0\|_\infty, \quad \forall m \in \mathbb{N}^* \\ \|x^* - x_m\|_\infty &\leq \frac{w}{1-w} \|x_m - x_{m-1}\|_\infty, \quad \forall m \in \mathbb{N}^*. \end{aligned}$$

Definition 2.1. For a given $\alpha, \beta > 0$, a function $f : [a,b] \to \mathbb{R}$ is $(\alpha,\beta)-$Lipschitz if there exist $L_1, L_2 \geq 0$ such that

$$|f(x) - f(y)| \leq L_1 |x-y|^\alpha + L_2 |x-y|^\beta, \quad \forall x,y \in [a,b].$$

The classical Lipchitz property is equivalent with the case $(1,1)-$Lipschitz. Concerning the properties of Picard iterations we can obtain the following result.

Theorem 2.2. *Under the conditions of Theorem 2.1, the sequence of the successive approximations is uniformly bounded and under supplementary conditions:*

$$\exists L_{H_t} > 0 \text{ such that } |H(t_1,s) - H(t_2,s)| \leq L_{H_t} |t_1 - t_2|,\ \forall t_1,t_2 \in [0,T]$$

and

$$\exists L_g > 0 \text{ such that } |g(t_1) - g(t_2)| \leq L_g |t_1 - t_2|,\ \forall t_1,t_2 \in [0,T]$$

it is uniform $(1,\alpha)-$Lipschitz.

Proof. We have the following relation:

$$|x_m(t)| \leq |x_m(t) - x_{m-1}(t)| + \ldots + |x_1(t) - x_0(t)| + |x_0(t)|$$

Let us consider $M_0 = \max\limits_{s\in[0,T]} |f(s,g(s))|$. Now we compute

$$|x_1(t) - x_0(t)| \leq \frac{1}{\Gamma(\alpha)} \left| \int_0^t H(t,s)(t-s)^{\alpha-1} f(s,x_0(s))\,ds \right| \leq \frac{T^\alpha M_H M_0}{\Gamma(\alpha+1)}$$

obtaining $\|x_1 - x_0\|_\infty \le \frac{T^\alpha M_H M_0}{\Gamma(\alpha+1)}$ and also

$$\begin{aligned}|x_m(t) - x_{m-1}(t)| &\le \frac{M_H L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x_{m-1}(s) - x_{m-2}(s)| \, ds \\ &\le \frac{L T^\alpha M_H}{\Gamma(\alpha+1)} \|x_{m-1} - x_{m-2}\|_\infty .\end{aligned}$$

By induction we get

$$\|x_m - x_{m-1}\|_\infty \le w^{m-1} \|x_1 - x_0\|_\infty , \quad \forall m \in \mathbb{N}^*$$

giving us

$$\begin{gathered}|x_m(t)| \le \left(w^{m-1} + w^{m-2} + \ldots + w + 1\right) \|x_1 - x_0\|_\infty + \|x_0\|_\infty \\ \le \tfrac{1}{1-w} \cdot \tfrac{T^\alpha M_H M_0}{\Gamma(\alpha+1)} + M_g \overset{not.}{=} M, \quad \forall t \in [0,T], \ m \in \mathbb{N}^*\end{gathered}$$

that is the uniform boundedness of the sequence $(x_m)_{m\in\mathbb{N}}$, where $M_g = \max\limits_{t\in[0,T]} |g(t)| = \max\limits_{t\in[0,T]} |x_0(t)|$. Moreover, we have

$$\begin{aligned}|f(s, x_m(s))| &\le |f(s, x_m(s)) - f(s, x_0(s))| + |f(s, x_0(s))| \\ &\le L |x_m(s) - x_0(s)| + M_0 \\ &\le \frac{L}{1-w} \cdot \frac{T^\alpha M_H M_0}{\Gamma(\alpha+1)} + M_0 \overset{not.}{=} M_f, \forall s \in [0,T], m \in \mathbb{N}\end{aligned} \tag{4}$$

that is the uniform boundedness of the sequence of functions $(F_m)_{m\in\mathbb{N}}$, $F_m(t) \overset{not.}{=} f(t, x_m(t))$. Now, let us consider arbitrary $0 \le t_1 \le t_2 \le T$, and we see that

$$\begin{gathered}|x_{m+1}(t_1) - x_{m+1}(t_2)| \le |g(t_1) - g(t_2)| + \frac{1}{\Gamma(\alpha)} \cdot \\ \cdot \left| \int_0^{t_1} H(t_1, s)(t_1 - s)^{\alpha-1} f(s, x_m(s)) \, ds - \int_0^{t_2} H(t_2, s)(t_2 - s)^{\alpha-1} f(s, x_m(s)) \, ds \right| \\ \le \tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| H(t_1, s)(t_1 - s)^{\alpha-1} - H(t_2, s)(t_2 - s)^{\alpha-1} \right| |f(s, x_m(s))| \, ds + \\ + L_g |t_1 - t_2| + \tfrac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} H(t_2, s)(t_2 - s)^{\alpha-1} f(s, x_m(s)) \, ds \right|, \quad \forall m \in \mathbb{N}.\end{gathered}$$

In the case $t_1 \le t_2$ (the case $t_2 \le t_1$ being approached similarly) we have $(t_2 - s)^{\alpha-1} \le (t_1 - s)^{\alpha-1}$ and $t_1^\alpha - t_2^\alpha \le 0$, for $s \in (0,t)$, obtaining

$$|x_{m+1}(t_1) - x_{m+1}(t_2)| \le L_g |t_1 - t_2| + \frac{M_f L_{H_t} T^\alpha}{\Gamma(\alpha+1)} |t_1 - t_2| + \frac{M_f M_H}{\Gamma(\alpha)} \int_0^{t_1} \left| (t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right| ds + \frac{M_H M_f}{\Gamma(\alpha+1)} |t_1 - t_2|^\alpha$$

The expression inside the modulus is positive, because of our supposition, so we obtain for the integral:

$$\int_0^{t_1} \left| (t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right| ds = \int_0^{t_1} (t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \, ds = \frac{t_1^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha}{\alpha} \le \frac{(t_2 - t_1)^\alpha}{\alpha} = \frac{|t_1 - t_2|^\alpha}{\alpha}$$

resulting

$$|x_{m+1}(t_1) - x_{m+1}(t_2)| \le \left[L_g + \frac{M_f L_{H_t} T^\alpha}{\Gamma(\alpha+1)} \right] |t_1 - t_2| + \frac{2 M_H M_f}{\Gamma(\alpha+1)} |t_1 - t_2|^\alpha, \forall m \in \mathbb{N}.$$

and denoting $L_{x_1} = L_g + \frac{T^\alpha M_f L_{H_t}}{\Gamma(\alpha+1)}$ and $L_{x_2} = \frac{2 M_H M_f}{\Gamma(\alpha+1)}$ we can write

$$|x_{m+1}(t_1) - x_{m+1}(t_2)| \le L_{x_1} |t_1 - t_2| + L_{x_2} |t_1 - t_2|^\alpha, \forall t_1, t_2 \in [0,T], \quad m \in \mathbb{N}$$

which is the uniform $(1,\alpha)$-Lipschitz property of the Picard iterations $(x_m)_{m\in\mathbb{N}}$. □

Remark 1. For an arbitrary $t \in [0,T]$, if f is γ−Lipschitz on its first argument and H is Lipschitz on its second argument, the product HF_m is uniform $(1,\alpha)$-Lipschitz too. Indeed, we get

$$\begin{gathered}|H(t, s_1) F_m(s_1) - H(t, s_2) F_m(s_2)| \le \\ \le |H(t, s_1)(F_m(s_1) - F_m(s_2))| + |(H(t, s_1) - H(t, s_2)) F_m(s_2)| \le \\ \le M_H \left(\gamma |s_1 - s_2| + L \left(L_{x_1} |s_1 - s_2| + L_{x_2} |s_1 - s_2|^\alpha \right) \right) + L_{H_s} M |s_1 - s_2| \le \\ \le \left(\gamma M_H + L_{H_s} M + L M_H L_{x_1} \right) |s_1 - s_2| + L M_H L_{x_2} |s_1 - s_2|^\alpha\end{gathered}$$

for all $m \in \mathbb{N}^*$ and for any $s_1, s_2 \in [0,T]$. Let us denote $L_0 = \gamma M_H + L_{H_s} M + L M_H L_{x_1}$ and $L' = L M_H L_{x_2}$, and we see that

$$\left| H(t,s) F_m(s) - H\left(t, s'\right) F_m\left(s'\right) \right| \le L_0 \left| s - s' \right| + L' \left| s - s' \right|^\alpha, \forall s, s' \in [0,T] \tag{5}$$

for all $t \in [0,T]$, $m \in \mathbb{N}$.

3 Bernstein splines approximation

On each iterative step, instead of calculating the value of x_m, which implies the computation of a fractional integral, we will approximate a part of the expression inside the integral with Bernstein type splines.

Let us consider a uniform partition of $[0,T]$, with the knots $t_i = ih$, $i=\overline{1,n}$, $n\in\mathbb{N}$, where $h=\frac{T}{n}$ is the stepsize. On each subinterval $[t_i,t_{i+1}]$, $i=\overline{0,n-1}$ we will consider the following Bernstein polynomial of degree q. The Bernstein polynomial of degree q approximating a given function $f\in C[a,b]$ has the expression,

$$B_q f(s)=\frac{1}{(b-a)^q}\sum_{j=0}^{q} C_q^j (s-a)^j (b-s)^{q-j} f\left(a+\frac{(b-a)j}{q}\right),\quad \forall s\in[a,b].$$

In the approximation formula,

$$f(s)=B_q f(s)+R_q f(s)$$

the reminder $R_q f(s)$ is estimated by using the inequality of Lorentz (see [15]) :

$$\left|R_q f(s)\right|\le \frac{5}{4}\omega\left(f,\frac{b-a}{\sqrt{q}}\right),\ \forall s\in[a,b]$$

where ω refers to the modulus of continuity.

Integrating the approximation formula, we get the following quadrature:

$$\int_a^b f(s)\,ds=\int_a^b B_q f(s)\,ds+\int_a^b R_q f(s)\,ds.$$

Let us introduce now the sequence of functions

$$F_{m,k}(s)\overset{def.}{=}H(t_k,s)\cdot f(s,x_m(s)),\quad \forall s\in[0,T],\quad m\in\mathbb{N},\quad k=\overline{0,n}$$

which is uniformly bounded, according to (4):

$$\left|F_{m,k}(s)\right|\le M_H M_f,\qquad \forall s\in[0,T],\quad m\in\mathbb{N},\quad k=\overline{0,n}.$$

Now, the sequence of successive approximations becomes

$$\begin{aligned} x_{m+1}(t_k) &= g(t_k)+\frac{1}{\Gamma(\alpha)}\int_0^{t_k} F_{m,k}(s)(t_k-s)^{\alpha-1}\,ds \\ &= g(t_k)+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[B_{q,i}\left(F_{m,k}\right)(s)+R_{m,i}(s)\right](t_k-s)^{\alpha-1}\,ds \end{aligned}$$

where $B_{q,i}\left(F_{m,k}\right)$ is the Bernstein polynomial approximating the function $F_{m,k}$ on each subinterval $[t_{i-1},t_i]$ and at each iterative step m :

$$B_{q,i}\left(F_{m,k}\right)(s)=\frac{1}{h^q}\sum_{j=0}^{q} C_q^j (s-t_{i-1})^j (t_i-s)^{q-j} F_{m,k}\left(t_{i-1}+\frac{jh}{q}\right),\ s\in[t_{i-1},t_i].$$

We define

$$\begin{gathered} \overline{x_{m+1}}(t_k)=g(t_k)+\tfrac{1}{h^q\Gamma(\alpha)}\cdot \\ \cdot\textstyle\sum_{i=1}^{k}\displaystyle\int_{t_{i-1}}^{t_i}\left[\sum_{j=0}^{q} C_q^j (s-t_{i-1})^j (t_i-s)^{q-j}\,\overline{F_{m,k}}\left(t_{i-1}+\tfrac{jh}{q}\right)\right]\cdot \\ \cdot(t_k-s)^{\alpha-1}\,ds=g(t_k)+\tfrac{1}{h^q\Gamma(\alpha)}\cdot \\ \cdot\sum_{i=1}^{k}\sum_{j=0}^{q} C_q^j\int_{t_{i-1}}^{t_i}(s-t_{i-1})^j (t_i-s)^{q-j}(t_k-s)^{\alpha-1}\,ds\cdot\overline{F_{m,k}}\left(t_{i-1}+\tfrac{jh}{q}\right) \end{gathered}$$

with $\overline{F_{m,k}}\left(t_{i-1}+\frac{jh}{q}\right)=H\left(t_k,t_{i-1}+\frac{jh}{q}\right)f\left(t_{i-1}+\frac{jh}{q},\overline{x_m}\left(t_{i-1}+\frac{jh}{q}\right)\right)$, where $\overline{x_m}$ is the approximated value of x_m calculated at the previous step on the knots, resulting the formula

$$x_{m+1}(t_k)=\overline{x_{m+1}}(t_k)+\overline{R_{m+1}}(t_k),\quad \forall m\in\mathbb{N}$$

In the integrals from this last formula we will make the change of variable $s=t_{i-1}+uh$, $ds=hdu$ obtaining:

$$\begin{gathered}\int_{t_{i-1}}^{t_i}(s-t_{i-1})^j(t_i-s)^{q-j}(t_k-s)^{\alpha-1}\,ds= \\ =h^{q+\alpha}\int_0^1 (u)^j(1-u)^{q-j}(k-i-u+1)^{\alpha-1}\,du\end{gathered}$$

and denoting $\psi_{j,k}(i) \overset{not.}{=} \int_0^1 (u)^j (1-u)^{q-j} (k-i-u+1)^{\alpha-1} du$ we get the expression of $\overline{x_{m+1}}$:

$$\overline{x_{m+1}}(t_k) = g(t_k) + \frac{h^\alpha}{\Gamma(\alpha)} \sum_{i=1}^{k} \sum_{j=0}^{q} C_q^j \cdot \psi_{j,k}(i) \cdot \overline{F_{m,k}}\left(t_{i-1} + \frac{jh}{q}\right)$$

Now, for $m \in \mathbb{N}$ and $l = \overline{0,q}$ we obtain the following iterative algorithm:

$$\begin{aligned}
\overline{x_0}\left(t_k + \frac{lh}{q}\right) &= g\left(t_k + \frac{lh}{q}\right) \\
\overline{x_{m+1}}\left(t_k + \frac{lh}{q}\right) &= g\left(t_k + \frac{lh}{q}\right) + \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha)} \sum_{i=1}^{k} \sum_{j=0}^{q} C_q^j \cdot \psi_{j,k+\frac{l}{q}}(i) \cdot \overline{F_{m,k}}\left(t_{i-1} + \frac{jh}{q}\right) + \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha)} \sum_{j=0}^{q} C_q^j \cdot \psi_{j,k+\frac{l}{q}}(k+1) \cdot \overline{F_{m,k}}\left(t_k + \frac{jh}{q}\right), \quad k = \overline{0, n-1} \\
\overline{x_{m+1}}(t_n) &= g(t_n) + \frac{h^\alpha}{\Gamma(\alpha)} \sum_{i=1}^{n} \sum_{j=0}^{q} C_q^j \cdot \psi_{j,n}(i) \cdot \overline{F_{m,n}}\left(t_{i-1} + \frac{jh}{q}\right)
\end{aligned} \tag{6}$$

having

$$\psi_{j,k+\frac{l}{q}}(i) = \begin{cases} \int_0^1 (u)^j (1-u)^{q-j} \left(k + \frac{l}{q} - (i-1) - u\right)^{\alpha-1} du & i = \overline{1,k} \\ \int_0^{\frac{l}{q}} (u)^j (1-u)^{q-j} \left(\frac{l}{q} - u\right)^{\alpha-1} du & i = k+1. \end{cases}$$

The relation (6) can be written as $x_{m+1}\left(t_k + \frac{lh}{q}\right) = \overline{x_{m+1}}\left(t_k + \frac{lh}{q}\right) + \overline{R_{m,k+\frac{l}{q}}}$. The algorithm stops when the difference between two consecutive iterations are under a given tolerance $\varepsilon > 0$ for all $t_k, k = \overline{0,n}$, so the at the first $m \in \mathbb{N}^*$ for which $|\overline{x_m}(t_k) - \overline{x_{m-1}}(t_k)| < \varepsilon, \forall k = \overline{0,n}$.

After obtaining the computed values at the last iterative step on the knots, we can provide the continuous approximation of the solution by using a Bernstein spline approximation:

$$\overline{B_{q,m}}(t) = \frac{1}{h^q} \sum_{j=0}^{q} C_q^j (t - t_{i-1})^j (t_i - t)^{q-j} \overline{x_m}\left(t_{i-1} + \frac{jh}{q}\right), \ t \in [t_{i-1}, t_i], \ i = \overline{1,n}.$$

4 Convergence analysis

Concerning the convergence of this proposed iterative method we obtain the main result of this work, as follows.

Theorem 4.1. *Suppose that the following conditions are fulfilled:*

1. $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, $g : [0,T] \to \mathbb{R}$ *and* $H : [0,T] \times [0,T] \to \mathbb{R}_+$ *are continuous functions*
2. f *is* L*—Lipschitz on its second argument*
3. $w \overset{not.}{=} \frac{LT^\alpha M_H}{\Gamma(\alpha+1)} < 1$.

   Then the sequence $(\overline{x_m}(t_k))_{m \in \mathbb{N}^*}$, $k = \overline{0,n}$ *approximates the solution* x^* *of the Volterra integral equation (1), having the error estimate on the mesh knots:*

$$|x^*(t_k) - \overline{x_m}(t_k)| \le \frac{T^\alpha}{(1-w)\Gamma(\alpha+1)} \cdot \left[w^m M_0 M_H + \frac{5}{4}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^\alpha}{(\sqrt{q})^\alpha}\right)\right] \tag{7}$$

The error estimate in the continuous Bernstein-spline approximation is:

$$\begin{aligned}
\left|x^*(t) - \overline{B_{q,m}}(t)\right| &\le \frac{T^\alpha M_0 M_H w^m}{(1-w)\Gamma(\alpha+1)} + \\
&+ \frac{5}{4}\left(\frac{L_{x_1} h}{\sqrt{q}} + \frac{L_{x_2} h^\alpha}{(\sqrt{q})^\alpha}\right) + \frac{5T^\alpha\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^\alpha}{(\sqrt{q})^\alpha}\right)}{4(1-w)\Gamma(\alpha+1)}, \ \forall t \in [0,T], \ m \in \mathbb{N}^*.
\end{aligned} \tag{8}$$

Proof. Let us calculate some of the first Picard iterations $(x_m)_{m\in\mathbb{N}}$:

$$\begin{aligned} x_1(t_k) &= g(t_k)+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} F_{0,k}(s)(t_k-s)^{\alpha-1}\,ds \\ &= g(t_k)+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \left[B_{q,i}F_{0,k}(s)+R_{0,i}(s)\right](t_k-s)^{\alpha-1}\,ds \\ &= \overline{x_1}(t_k)+\overline{R_1}(t_k) \end{aligned}$$

where

$$\begin{aligned} \overline{x_1}(t_k) &= g(t_k)+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} B_{q,i}F_{0,k}(s)(t_k-s)^{\alpha-1}\,ds \\ &= g(t_k)+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\frac{1}{h^q}\sum_{j=0}^{q}C_q^j h^{q+\alpha}\psi_{j,k}(i)F_{0,k}\left(t_{i-1}+\frac{jh}{q}\right) \end{aligned}$$

and

$$\overline{R_1}(t_k)=\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} R_{0,i}(s)(t_k-s)^{\alpha-1}\,ds$$

with $\left|R_{0,i}(s)\right|\le\frac{5}{4}\omega\left(F_{0,k},\frac{h}{\sqrt{q}}\right),\forall s\in[t_{i-1},t_i],i=\overline{1,k}$. Therefore we get

$$\left|\overline{R_1}(t_k)\right|\le\frac{1}{\Gamma(\alpha)}\frac{5}{4}\omega\left(F_{0,k},\frac{h}{\sqrt{q}}\right)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_k-s)^{\alpha-1}\,ds\le\frac{5\omega\left(F_{0,k},\frac{h}{\sqrt{q}}\right)T^\alpha}{4\Gamma(\alpha+1)}.$$

Let us denote $\overline{R_{F_1}}(t_j)=\left|F_{1,k}(t_j)-\overline{F_{1,k}}(t_j)\right|$. We have the following inequality

$$\begin{aligned} \left|F_{1,k}(t_j)-\overline{F_{1,k}}(t_j)\right| &= \left|H(t_k,t_j)\cdot f(s,x_m(t_j))-H(t_k,t_j)\cdot f(s,\overline{x_m}(t_j))\right| \\ &\le M_H L\left|x_1(t_j)-\overline{x_1}(t_j)\right|\le M_H L\left|\overline{R_1}(t_j)\right| \end{aligned}$$

and it obtains

$$\begin{aligned} x_2(t_k) &= g(t_k)+\tfrac{1}{\Gamma(\alpha)}\textstyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} H(t_k,s)(t_k-s)^{\alpha-1}f(s,x_1(s))\,ds \\ &= g(t_k)+\tfrac{1}{\Gamma(\alpha)}\textstyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[B_{q,1,i}(s)+R_{1,i}(s)\right](t_k-s)^{\alpha-1}\,ds=g(t_k)+ \\ &+\tfrac{1}{\Gamma(\alpha)}\cdot\textstyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[\tfrac{1}{h^q}\sum_{j=0}^{q}C_q^j(s-t_{i-1})^j(t_i-s)^{q-j}F_{1,k}\left(t_{i-1}+\tfrac{jh}{q}\right)\right]\cdot \\ &\cdot(t_k-s)^{\alpha-1}\,ds+\tfrac{1}{\Gamma(\alpha)}\textstyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}R_{1,i}(s)(t_k-s)^{\alpha-1}\,ds=g(t_k)+\tfrac{1}{\Gamma(\alpha)}\cdot \\ &\cdot\textstyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[\tfrac{1}{h^q}\sum_{j=0}^{q}C_q^j(s-t_{i-1})^j(t_i-s)^{q-j}\overline{F_{1,k}}\left(t_{i-1}+\tfrac{jh}{q}\right)\right]\cdot(t_k-s)^{\alpha-1}\,ds \\ &+\tfrac{1}{\Gamma(\alpha)}\cdot\textstyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[\tfrac{1}{h^q}\sum_{j=0}^{q}C_q^j(s-t_{i-1})^j(t_i-s)^{q-j}\overline{R_{F_1}}\left(t_{i-1}+\tfrac{jh}{q}\right)\right]\cdot \\ &\cdot(t_k-s)^{\alpha-1}\,ds+\tfrac{1}{\Gamma(\alpha)}\textstyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}R_{1,i}(s)(t_k-s)^{\alpha-1}\,ds=\overline{x_2}(t_k)+\overline{R_2}(t_k) \end{aligned}$$

where

$$\begin{aligned} \overline{x_2}(t_k) &= g(t_k)+\frac{1}{\Gamma(\alpha)}\cdot \\ &\cdot\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left[\frac{1}{h^q}\sum_{j=0}^{q}C_q^j(s-t_{i-1})^j(t_i-s)^{q-j}\overline{F_{1,k}}\left(t_{i-1}+\frac{jh}{q}\right)\right]\cdot \\ &\cdot(t_k-s)^{\alpha-1}\,ds \\ &= g(t_k)+\frac{1}{\Gamma(\alpha)}h^\alpha\sum_{i=1}^{k}\sum_{j=0}^{q}C_q^j\psi_{j,k}(i)\overline{F_{1,k}}\left(t_{i-1}+\frac{jh}{q}\right) \end{aligned}$$

and

$$\begin{aligned}\left|\overline{R_2}(t_k)\right| &\le \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i}\left[\frac{1}{h^q}\sum_{j=0}^{q}C_q^j(s-t_{i-1})^j(t_i-s)^{q-j}M_H L\left|\overline{R_1}\left(t_{i-1}+\frac{jh}{q}\right)\right|\right]\cdot\\ &\quad\cdot(t_k-s)^{\alpha-1}\,ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i}\left|R_{1,i}(s)\right|(t_k-s)^{\alpha-1}\,ds\\ &\le \frac{5M_H L\omega\left(F_{0,k},\frac{h}{\sqrt{q}}\right)T^\alpha}{4\Gamma(\alpha)\Gamma(\alpha+1)}\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i}\frac{1}{h^q}\left[\sum_{j=0}^{q}C_q^j(s-t_{i-1})^j(t_i-s)^{q-j}\right]\\ &\quad\cdot(t_k-s)^{\alpha-1}\,ds+\frac{5\omega\left(F_{1,k},\frac{h}{\sqrt{q}}\right)}{4\Gamma(\alpha)}\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i}(t_k-s)^{\alpha-1}\,ds.\end{aligned}$$

Since

$$\sum_{j=0}^{q}C_q^j(s-t_{i-1})^j(t_i-s)^{q-j}=h^q$$

and

$$\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i}(t_k-s)^{\alpha-1}\,ds=\frac{t_k^\alpha}{\alpha}\le\frac{T^\alpha}{\alpha}$$

we obtain

$$\left|\overline{R_2}(t_k)\right|\le\frac{5M_H L\omega\left(F_{0,k},\frac{h}{\sqrt{q}}\right)T^\alpha}{4\Gamma(\alpha+1)\Gamma(\alpha+1)}T^\alpha+\frac{5\omega\left(F_{1,k},\frac{h}{\sqrt{q}}\right)}{4\Gamma(\alpha+1)}T^\alpha$$

and denoting $\overline{\omega}\left(F_m,\frac{h}{\sqrt{q}}\right)=\max\limits_{\substack{i=\overline{0,m}\\k=\overline{1,n}}}\omega\left(F_{i,k},\frac{h}{\sqrt{q}}\right)$ we get

$$\left|\overline{R_2}(t_k)\right|\le\frac{5\overline{\omega}\left(F_1,\frac{h}{\sqrt{q}}\right)T^\alpha}{4\Gamma(\alpha+1)}\left[\frac{M_H L T^\alpha}{\Gamma(\alpha+1)}+1\right],\quad\forall k=\overline{1,n}.$$

Since $\frac{LT^\alpha M_H}{\Gamma(\alpha+1)}=w$, by induction we obtain:

$$\begin{aligned}\left|\overline{R_m}(t_k)\right| &\le \frac{5\overline{\omega}\left(F_{m-1},\frac{h}{\sqrt{q}}\right)T^\alpha}{4\Gamma(\alpha+1)}\left[w^{m-1}+w^{m-2}+\cdots+w^2+w+1\right]\\ &\le \frac{5\overline{\omega}\left(F_{m-1},\frac{h}{\sqrt{q}}\right)T^\alpha}{4\Gamma(\alpha+1)}\frac{1-w^m}{1-w}\le\frac{5\overline{\omega}\left(F_{m-1},\frac{h}{\sqrt{q}}\right)T^\alpha}{4(1-w)\Gamma(\alpha+1)}\end{aligned}$$

We obtain the same relation for $\left|\overline{R_m}\left(t_k+\frac{lh}{q}\right)\right|$ but in this case we will denote

$$\overline{\omega}\left(F_m,\frac{h}{\sqrt{q}}\right)=\max_{\substack{i=\overline{0,m}\\k=\overline{1,n}\\l=\overline{0,q}}}\omega\left(F_{i,k+\frac{l}{q}},\frac{h}{\sqrt{q}}\right),\quad\text{where}\quad F_{i,k+\frac{l}{q}}(s)\overset{def.}{=}H\left(t_k+\frac{lh}{q},s\right)\cdot f(s,x_i(s)),\quad\forall s\in[0,T]$$

The estimate $|x^*(t_k)-x_m(t_k)|\le\frac{w^m}{1-w}\frac{T^\alpha M_H M_0}{\Gamma(\alpha+1)}$ can be deduced from relations

$$\|x_m-x_{m-1}\|_\infty\le w^{m-1}\|x_1-x_0\|_\infty\quad\text{and}\quad\|x_1-x_0\|_\infty\le\frac{T^\alpha M_H M_0}{\Gamma(\alpha+1)}$$

proved in Section 2, and thus

$$\begin{aligned}|x^*(t_k)-\overline{x_m}(t_k)| &\le |x^*(t_k)-x_m(t_k)|+|x_m(t_k)-\overline{x_m}(t_k)|\\ &\le \frac{w^m}{1-w}\frac{T^\alpha M_H M_0}{\Gamma(\alpha+1)}+\frac{5\overline{\omega}\left(F_{m-1},\frac{h}{\sqrt{q}}\right)T^\alpha}{4(1-w)\Gamma(\alpha+1)}\\ &= \frac{T^\alpha}{(1-w)\Gamma(\alpha+1)}\left[w^m M_0 M_H+\frac{5}{4}\overline{\omega}\left(F_{m-1},\frac{h}{\sqrt{q}}\right)\right].\end{aligned}$$

According to the Lipschitz property (5), we get the error estimate (7):

$$|x^*(t_k) - \overline{x_m}(t_k)| \le \frac{T^\alpha M_0 M_H w^m}{(1-w)\Gamma(\alpha+1)} + \frac{5T^\alpha\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^\alpha}{(\sqrt{q})^\alpha}\right)}{4(1-w)\Gamma(\alpha+1)}$$

for all $k = \overline{0,n}$, $m \in \mathbb{N}^*$. For the last part of demonstration we consider the Bernstein spline approximating the Picard iterations $(x_m)_{m\in\mathbb{N}^*}$, given by

$$B_{m,q}(t) = \frac{1}{h^q}\sum_{k=0}^{q} C_q^k (t-t_{i-1})^k (t_i - t)^{q-k} \cdot x_m\left(t_{i-1} + \frac{kh}{q}\right),\ t \in [t_{i-1}, t_i],\ i = \overline{1,n}$$

and since

$$\left|x^*(t) - \overline{B_{m,q}}(t)\right| \le |x^*(t) - x_m(t)| + \left|x_m(t) - B_{m,q}(t)\right| + \left|B_{m,q}(t) - \overline{B_{m,q}}(t)\right|$$

we get

$$\left|x^*(t) - \overline{B_{m,q}}(t)\right| \le \frac{T^\alpha M_0 M_H w^m}{(1-w)\Gamma(\alpha+1)} + \frac{5}{4}\left(\frac{L_{x_1} h}{\sqrt{q}} + \frac{L_{x_2} h^\alpha}{\left(\sqrt{q}\right)^\alpha}\right) +$$

$$+\frac{1}{h^q}\sum_{k=0}^{q} C_q^k (t-t_{i-1})^k (t_i-t)^{q-k}\left|x_m\left(t_{i-1} + \frac{kh}{q}\right) - \overline{x_m}\left(t_{i-1} + \frac{kh}{q}\right)\right| \le$$

$$\le \frac{T^\alpha M_0 M_H w^m}{(1-w)\Gamma(\alpha+1)} + \frac{5}{4}\left(\frac{L_{x_1} h}{\sqrt{q}} + \frac{L_{x_2} h^\alpha}{\left(\sqrt{q}\right)^\alpha}\right) + \frac{5T^\alpha\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^\alpha}{(\sqrt{q})^\alpha}\right)}{4(1-w)\Gamma(\alpha+1)}$$

obtaining the error estimate (8). From (8) we see that the order of convergence is $\left\|x_m - \overline{B_{m,q}}\right\| = O(h^\alpha)$. □

Remark 2. In the same manner, the method of Bernstein splines can be applied for nonlinear weakly singular Volterra integral equations too. Since the case of using Bernstein splines with degree $q = 1$ corresponds to the trapezoidal product integration, as a particular case of the Bernstein splines method, the accuracy of this method for degree $q > 1$ will be better due to the uniform approximation properties of the Bernstein polynomials. This aspect will be tested in the next section on some numerical examples.

5 Numerical experiments

In order to test the theoretical convergence stated in Theorem 4.1 and to illustrate the accuracy of the proposed method we present below some numerical examples.

Example 5.1. Consider the following fractional integral equation:

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{-\frac{1}{2}} x^2(s)\,ds + t^{\frac{1}{2}}\left(1 - \frac{4t}{3\sqrt{\pi}}\right), \quad t \in [0,1] \tag{9}$$

where $\alpha = \frac{1}{2}, H(t,s) = 1,\ f(s,x(s)) = [x(s)]^2,\ g(t) = t^{\frac{1}{2}}\left(1 - \frac{4t}{3\sqrt{\pi}}\right)$ and $T = 1$. The exact solution is $x^*(t) = \sqrt{t}$. We will consider $n = 10$ and 100, and the number of iterations is $m = 30$. The pointwise errors are $e_{n,i} = |\overline{x_m}(t_i) - x^*(t_i)|$, $i = \overline{0,n}$, observing that $\max\limits_{i=\overline{0,n}} |\overline{x_m}(t_i) - x^*(t_i)| = |\overline{x_m}(t_n) - x^*(t_n)|$. We put $q = 1$ and $q = 5$ and the numerical results are presented in Tables 1 and 2.

$t_i \backslash e_{n,i}$	$m=30,\ n=10$	$m=30,\ n=100$
0,0	0	0
0,2	$3,24E-017$	$3,17E-017$
0,4	$3,32E-016$	$2,68E-016$
0,6	$2,05E-012$	$5,63E-016$
0,8	$1,47E-09$	$2,76E-012$
1,0	$3,11E-07$	$2,73E-09$

Table 1. Numerical results for (9) with $q = 1$

$t_i \backslash e_{n,i}$	$m=30,\ n=10$	$m=30,\ n=100$
0,0	0	0
0,2	$3,47E-017$	$6,09E-020$
0,4	$2,64E-018$	$2,58E-016$
0,6	$1,92E-014$	$5,28E-016$
0,8	$3,60E-011$	$2,43E-012$
1,0	$1,59E-08$	$2,51E-09$

Table 2. Numerical results for (9) with $q = 5$

Example 5.2. Now, we consider the weakly singular integral equation (Example 1 in [18])

$$x(t) = \frac{1}{12}\int_0^t x^2(s)(t-s)^{-\frac{1}{2}}\,ds + t^{\frac{1}{2}}\left(1-\frac{1}{9}t\right), \quad t\in[0,1] \tag{10}$$

having the exact solution $x^*(t)=\sqrt{t}$, and $\alpha=\frac{1}{2}$, $f(s,x(s))=x^2(s)$, $g(t)=t^{\frac{1}{2}}\left(1-\frac{1}{9}t\right)$, $H(t,s)=\frac{1}{12}$, $T=1$. The numerical results obtained with $q=5$, $m=5$ and $m=10$ iterations, and taking $n=10$ and $n=20$ are presented in Table 3 in terms of the pointwise errors $e_{n,i}=|\overline{x_m}(t_i)-x^*(t_i)|$, $i=\overline{0,n}$. By considering $e_m=\max\limits_{i=\overline{0,n}}|\overline{x_m}(t_i)-x^*(t_i)|$, we present in Table 4 a comparison with the results from [18], Table 1, page 12.

	$m=5$		$m=10$	
$t_i\backslash e_{n,i}$	$n=10$	$n=20$	$n=10$	$n=20$
0,0	0	0	0	0
0,1	$4,86E-012$	$2,56E-012$	$2,24E-018$	$1,07E-018$
0,2	$2,31E-010$	$1,80E-010$	$6,17E-018$	$3,20E-018$
0,3	$2,68E-09$	$2,36E-09$	$1,11E-016$	$1,81E-017$
0,4	$1,61E-08$	$1,50E-08$	$2,04E-015$	$1,39E-015$
0,5	$6,62E-08$	$6,30E-08$	$2,23E-014$	$1,68E-014$
0,6	$2,12E-07$	$2,04E-07$	$1,61E-013$	$1,32E-013$
0,7	$5,67E-07$	$5,52E-07$	$8,84E-013$	$7,60E-013$
0,8	$1,33E-06$	$1,31E-06$	$3,91E-012$	$3,48E-012$
0,9	$2,84E-06$	$2,79E-06$	$1,46E-011$	$1,33E-011$
1,0	$5,58E-06$	$5,51E-06$	$4,78E-011$	$4,43E-011$

Table 3. Numerical results for (10) with $q=5$

n/e_m	$m=5$	$m=10$
$n=12$, in Table 1, [18]	$2.799443E-04$	$6.91396E-07$
$n=10$, in Table 3	$5,58E-06$	$4,78E-011$
$n=24$, in Table 1, [18]	$5.567188E-06$	$4.690204E-09$
$n=20$, in Table 3	$5,51E-06$	$4,43E-011$

Table 4. Comparison between the results in Table 3 and Table 1 from [18]

Example 5.3. Let us consider the following weakly singular integral equation (Example 2 in [18])

$$x(t) = \frac{1}{18}\int_0^t \left(\sin^2(s)+x^2(s)\right)(t-s)^{-\frac{2}{3}}\,ds + \cos(t) - \frac{1}{6}t^{\frac{1}{3}}, \quad t\in\left[0,\frac{\pi}{4}\right]. \tag{11}$$

The exact solution is $x^*(t)=\cos(t)$ and we have $\alpha=\frac{1}{3}$, $H(t,s)=1$, $T=\frac{\pi}{4}$, $f(s,x(s))=\frac{1}{18}\left(\sin^2(s)+x^2(s)\right)$, $g(t)=\cos(t)-\frac{1}{6}t^{\frac{1}{3}}$. The iterative algorithm of Bernstein-splines was applied with $q=4$, $m=5$ and $m=10$ iterations, and taking $n=10$ and $n=20$. The numerical results $e_m=\max\limits_{i=\overline{0,n}}|\overline{x_m}(t_i)-x^*(t_i)|, i=\overline{0,n}$, are presented in Table 5, including a comparison with the results from [18], Table 2, page 13.

n/e_m	$m=5$	$m=10$
$n=12$, in Table 2, [18]	$2.315358E-04$	$9.363611E-07$
$n=10$, in Table 5	$4,40E-05$	$5,50E-09$
$n=24$, in Table 2, [18]	$4.412851E-05$	$5.525447E-09$
$n=20$, in Table 5	$4,45E-05$	$5,60E-09$

Table 5. Comparison of our results for (11) with $q=4$
and the results from Table 2 in [18]

Example 5.4. Finally, we test the performances of the proposed method on the following fractional Volterra integral equation:

$$x(t) = \frac{1}{4\Gamma(\alpha)}\int_0^t (t-s)^{-\frac{1}{2}}x^2(s)\,ds + \sqrt{t(1-t)} - \frac{t\sqrt{t}(5-4t)}{15\sqrt{\pi}} \tag{12}$$

where we have $\alpha=\frac{1}{2}, H(t,s)=\frac{1}{4}$, $f(s,x(s))=[x(s)]^2$, $g(t)=\sqrt{t(1-t)}-\frac{t\sqrt{t}(5-4t)}{15\sqrt{\pi}}$ and $T=1$. The exact solution is: $x^*(t)=\sqrt{t(1-t)}$. For the test of convergence we consider $n=30$, 60 and 120, and choose the number of iterations $m=30$. In

Tables 6 and 7 we give the pointwise errors $e_{n,i} = |\overline{x_m}(t_i) - x^*(t_i)|$, $i = \overline{0,n}$, by taking $q = 1$ and $q = 5$:

t_i	$n = 30$	$n = 60$	$n = 120$
0,0	0	0	0
0,2	$2,30E-05$	$5,89E-06$	$1,50E-06$
0,4	$3,53E-05$	$8,99E-06$	$2,28E-06$
0,6	$4,56E-05$	$1,16E-05$	$2,93E-06$
0,8	$5,39E-05$	$1,37E-05$	$3,45E-06$
1,0	$5,83E-05$	$1,48E-05$	$3,73E-06$

Table6. Numerical results for (12) with $q = 1$

t_i	$n = 30$	$n = 60$	$n = 120$
0,0	0	0	0
0,2	$4,62E-06$	$1,18E-06$	$3,01E-07$
0,4	$7,11E-06$	$1,81E-06$	$4,57E-07$
0,6	$9,19E-06$	$2,33E-06$	$5,87E-07$
0,8	$1,09E-05$	$2,75E-06$	$6,92E-07$
1,0	$1,18E-05$	$2,97E-06$	$7,48E-07$

Table7. Numerical results for (12) with $q = 5$

6 Conclusions

In our main result, Theorem 4.1, we proved the convergence of the Bernstein splines iterative method applied to nonlinear fractional Volterra integral equations, providing the error estimate in the discrete and continuous approximation. The order of convergence is $O(h^\alpha)$ for $\alpha \in (0,1)$, where h is the stepsize.

After analysing our numerical results we concluded two things: first, under the contraction condition, the Picard-Banach iterations combined with Bernstein type splines generate an effective approximation method. The second observation is that increasing the degree of Bernstein polynomials the accuracy is improved, but doesn't have a huge impact on the obtained numerical results. This is because when we have more intermediate nodes on a given iterative step, m, we obtain a better appproximation for x_m, not for the exact solution x^*. But increasing the number of iterations will give us better results. Although, the use of Berstein splines with degree $q = 4$ or $q = 5$, will provide better results than those provided with the degree $q = 1$ (see Example 5.4). This is expected because of the nice uniform approximation and shape preserving properties of the Bernstein polynomials.

Comparing the obtained results at Example 5.1 we observe better results when the number of points are increased from $n = 10$ to $n = 100$, which confirm the convergece of the method. Moreover, the results in Table 2 are improved when the Bernstein polynomial degree increases by $q = 1$ to $q = 5$. Since the case $q = 1$ corresponds to the trapezoidal product integration technique, by comparing Tables 1 and 2, we infer that the iterative Bernstein splines method provides better results, as was expected from theoretical point of view. Since the trapezoidal product integration is the particular case $q = 1$ in our method, the results for $q = 5$ are better, as can be viewed at Example 5.2, too (see Table 4). Comparing the results obtained at Example 5.3, in Table 5, with the results from [18], Table 2, we observe similar results in the case of $n = 20$ and $m = 24$ points, respectively. At the last numerical example we see that the convergence of the Bernstein splines method is confirmed again (by comparing the results for $n = 30$, $n = 60$, and $n = 120$ points) better accuracy being observed when the Bernstein polynomial degree increases from $q = 1$ to $q = 5$.

As a final remark, we observe the following advantage of the Bernstein splines method: when the polynomial degree is changed to high values, the accuracy is improved without a significant enhancement of the computational complexity. This is not the case when the product integration techniques are based on Newton-Cotes quadrature formulas. In this context, the performances and limitations of the techniques using high degree Lagrange polynomials in the product integration rule are mentioned in [10] and [11].

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