

On a new class of positive definite RBFs by using Fourier cosine transform

Maryam Mohammadi^{a,b} · Mohammad Heidari^a · Stefano De Marchi^b

Dedicated to Professor Ioan Raşa on the occasion of his 70th birthday

Abstract

In this article, we use Fourier cosine transform in order to introduce a new family of infinitely smooth positive definite radial basis functions from completely monotone functions. These bases are represented in terms of positive Borel measures and their Fourier transforms are also given. The proposed theory is used for reconstructing the well-known Matérn RBF and presenting a new positive definite RBF. Numerical results show an accurate reconstruction of the Franke's function and also mitigating the Runge phenomenon as a key error mechanism.

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1 Introduction

Given a continuous function f mapping from the Euclidean space \mathbb{R}^d to the real numbers \mathbb{R} , a common objective is to approximate this function through interpolation at specified points or centers $X = \{x_j\}_{j=1}^n$. This involves using finite-dimensional linear spaces of simpler functions. An effective and widely adopted method involves interpolation using linear combinations of shifts of radial basis functions (RBFs) $\phi(r)$, where $r = \|x\|_2$, $x \in \mathbb{R}^d$ [4, 14, 39]. The interpolation expression takes the form

$$s(x) = \sum_{j=1}^n \lambda_j \phi(\|x - x_j\|),$$

where the unknown coefficients are achieved by enforcing interpolation conditions

$$s(x_j) = f_j, \quad j = 1, \dots, n.$$

This results in a square linear system of size $n \times n$ given by

$$A\lambda = f,$$

where $A = [\phi(\|x_i - x_j\|)]_{1 \leq i, j \leq n}$, is referred to as the system matrix.

Numerous researchers have extensively explored the theory of RBFs (refer to, for example, [5, 16, 19, 23, 38, 39, 40, 41]). As a result, over the past few decades, RBFs have found widespread applications in various fields such as multivariate function approximation, neural networks, and the solution of differential and integral equations (see, for instance, [8, 9, 21, 22, 29, 30, 34]).

The immediate concern arises regarding the uniqueness of the solution to this interpolation problem. Researchers such as Micchelli [19], Powell [27], and the collaboration of Buhmann and Micchelli [6], as well as others outlined in the book [4], have addressed this particular question.

A class of functions for which the interpolation problem is uniquely solvable for any distinct set of points, is the class of positive definite (PD) RBFs.

Definition 1.1. A radial basis function $\phi \in C([0, \infty))$ is called positive (semi)-definite on \mathbb{R}^d if and only if for any finite set of distinct points $\{x_j\}_{j=1}^n \subset \mathbb{R}^d$, the matrix $A = [\phi(\|x_i - x_j\|)]_{1 \leq i, j \leq n}$, is positive (semi)-definite.

Examples of PD RBFs are Gaussian $\phi(r) = \exp(-\frac{r^2}{2c^2})$, and Inverse multiquadrics $\phi(r) = (1 + \frac{r^2}{c^2})^{-\beta}$, $\beta > 0$, where c is a positive factor called shape parameter. Choosing the scale or shape parameter of RBFs is a well-documented but still an open problem in kernel-based methods. It is common to tune it according to the applications, and it plays a crucial role both for the accuracy and stability of the method [2]. Changing the shape parameter c from a large value to a small one reshapes the Gaussian function from a flat profile to a peaked one. For the longest time, people in the approximation theory and numerical

^aFaculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran

^bDepartment of Mathematics "Tullio Levi-Civita", University of Padova, Italy

analysis community went mostly with ad-hoc choices of the shape parameter or ignored its effect by treating it as a constant. Much more systematic approaches have been suggested in the statistics literature for a long time (see, e.g., [37] and many other references). In the radial basis community many researchers seek a scheme to find an optimal value for it, but research is on-going [7, 13, 17, 18, 24, 28, 36]. An extended discussion of a “good” shape parameter choice was also included in [9]. Recently, Heidari et al. devised a direct relation between the shape parameter of RBFs and their curvature at each point [17]. They used the fundamental theory of plane curves in order to recover univariate functions from scattered data, by enforcing the exact and approximate solutions have the same curvature at the point where they meet. This led to introducing curvature-based scaled RBFs with shape parameters depending on the function values and approximate curvature values of the function to be approximated.

PD RBFs have also become notably influential in numerous applications, including but not limited to numerical solution of partial differential equations (PDEs), computer experiments, machine learning, rapid prototyping, and computer graphics. The prevalence of PD RBFs in these diverse fields is emphasized by their effectiveness in addressing challenges and providing solutions in areas where accurate modeling, interpolation, and computational efficiency are crucial. For further details and a comprehensive overview of their applications, refer to [10] and the references cited there in. The fundamental work of Bochner [1], and subsequently Schoenberg [33], on the characterization of positive definite radial functions in terms of completely monotone functions, has played an important role in the development of both the theory and application of PD RBFs.

Recently, Buhmann et al. concentrated on examining the relationships between monotonicity properties and the positive definiteness of functions defined on the Euclidean space \mathbb{R}^d [3]. They introduced an innovative technique for constructing PD RBFs from multiply monotone functions.

In this paper, we utilize the Fourier cosine transform to propose a novel set of indefinitely smooth PD RBFs. These functions are derived from completely monotone functions, building upon the fundamental contribution by Schoenberg [33].

The outline of the paper is as follows. We start in Section 2 by summarizing the known results connecting monotonicity and positive definiteness. We then present the method of finding PD RBFs from the Fourier cosine transform of the squared of the completely monotone functions. An exact representation for the d -variate Fourier transform of the proposed class of RBFs is also given. In Section 3, the proposed theory is used for obtaining two PD RBFs. The first one is the common Matérn RBF and the second one is a new PD RBF. In Section 4, the numerical experiments are presented. The last section summarizes the conclusion and some further works.

2 Construction

We first introduce a class of functions that is closely related to PD RBFs and leads to a simple characterization of such functions.

Definition 2.1. A function g is called completely monotone (CM) on $(0, \infty)$ if it satisfies $g \in C^\infty(0, \infty)$ and $(-1)^l g^{(l)}(t) \geq 0$, for all $l \in \mathbb{N}_0$, $t > 0$. If in addition $g \in C[0, \infty)$ then g is called CM on $[0, \infty)$.

The following connection between PD RBFs and CM functions was first pointed out by Schoenberg in 1938 [33].

Theorem 2.1 (Schoenberg). *A non-constant function $g : [0, \infty) \rightarrow \mathbb{R}$ is CM on $[0, \infty)$ if and only if $\phi(r) = g(r^2)$ is PD on every \mathbb{R}^d .*

The following Theorem is our main result: we provide a technique that allows us to construct a new class of PD RBFs starting from a CM function g and considering the Fourier cosine transform of $f = g \circ x^2$.

Theorem 2.2. *Let g be a CM function on $[0, \infty)$ and $f \in L^1(0, \infty)$ with $f(x) = g(x^2)$. Then the Fourier cosine transform of f*

$$\phi(r) = F_{cs}(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(rx) f(x) dx, \quad r \geq 0, \tag{1}$$

is a bounded, infinitely smooth, and PD RBF on every \mathbb{R}^d .

Proof. It is clear that ϕ is a bounded and infinitely smooth RBF. Since g is CM on $[0, \infty)$, it is the Laplace transform of a nonnegative finite Borel measure ν , i.e. it is of the form

$$g(x) = \int_0^\infty e^{-sx} d\nu(s). \tag{2}$$

Then

$$\begin{aligned} \phi(r) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(rx) \int_0^\infty e^{-sx^2} d\nu(s) dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty e^{-sx^2} \cos(rx) dx d\nu(s). \end{aligned}$$

Now by using the Fourier cosine transform of the function e^{-sx^2} (see the formula (12.34.16) from [15]), we have

$$\phi(r) = \frac{1}{\sqrt{2}} \int_0^\infty \frac{e^{-\frac{r^2}{4s}}}{\sqrt{s}} d\nu(s). \tag{3}$$

So $\phi(r) = h(r^2)$, where

$$h(r) = \frac{1}{\sqrt{2}} \int_0^\infty \frac{e^{-\frac{r}{4s}}}{\sqrt{s}} d\nu(s).$$

Since

$$(-1)^l h^{(l)}(r) = \frac{1}{\sqrt{2}} \int_0^\infty \frac{e^{-\frac{r}{4s}}}{\sqrt{s} (4s)^l} d\nu(s) \geq 0, \quad \text{for all } l \in \mathbb{N}_0 \text{ and all } r > 0,$$

h is CM on $[0, \infty)$, and the positive definiteness is proved according to the Schoenberg theorem. □

Remark 1. The condition that the function g has to be CM is weaker than f being CM, because complete monotonicity of f implies complete monotonicity of $g(\cdot) = f(\sqrt{\cdot})$ (cf. [20]).

2.1 Fourier analysis of the new bases

It is a well-known fact that Fourier analysis plays a major role in the study of RBFs (see e.g. [9, 39]). It appears in the characterization of RBFs, rigorous convergence order estimates, and interpolation error analysis, specifying the native spaces corresponding to RBFs, stability analysis, and many other places. At first, we will take a look at the Fourier transform of a radial function, which is radial as well (see [39]).

Theorem 2.3. Suppose $\Phi \in L_1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ is radial, i.e. $\Phi(x) = \phi(\|x\|_2)$, $x \in \mathbb{R}^d$. Then, its Fourier transform $\widehat{\Phi}$ is also radial, i.e. $\widehat{\Phi}(w) = \mathcal{F}_d \phi(\|w\|_2)$, with

$$\mathcal{F}_d \phi(r) = r^{-\frac{(d-2)}{2}} \int_0^\infty \phi(t) t^{\frac{d}{2}} J_{\frac{(d-2)}{2}}(rt) dt,$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν .

In the following, we compute the Fourier transform of the suggested class of PD RBFs.

Theorem 2.4. The d -variate Fourier transform of $\phi(r)$ in (1) is given by

$$\mathcal{F}_d \phi(r) = 2^{\frac{(d-1)}{2}} \int_0^\infty s^{\frac{(d-1)}{2}} e^{-r^2 s} d\nu(s). \tag{4}$$

Proof. According to (3) and the Theorem 2.3, we have

$$\mathcal{F}_d \phi(r) = r^{-\frac{(d-2)}{2}} \int_0^\infty \left(\frac{1}{\sqrt{2}} \int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{\sqrt{s}} d\nu(s) \right) t^{\frac{d}{2}} J_{\frac{(d-2)}{2}}(rt) dt,$$

Fubini's theorem yields,

$$\mathcal{F}_d \phi(r) = \frac{1}{\sqrt{2}} r^{-\frac{(d-2)}{2}} \int_0^\infty \frac{1}{\sqrt{s}} \left(\int_0^\infty e^{-\frac{t^2}{4s}} t^{\frac{d}{2}} J_{\frac{(d-2)}{2}}(rt) dt \right) d\nu(s).$$

Now, we use the formula (see (6.631.4) from [15])

$$\int_0^\infty t^{\nu+1} e^{-\alpha t^2} J_\nu(bt) dt = \frac{b^\nu}{(2\alpha)^{\nu+1}} \exp\left(-\frac{b^2}{4\alpha}\right),$$

and letting $\nu = \frac{d-2}{2}$, $\alpha = \frac{1}{4s}$, $b = r$. Then

$$\mathcal{F}_d \phi(r) = \frac{1}{\sqrt{2}} r^{-\frac{(d-2)}{2}} \int_0^\infty \frac{1}{\sqrt{s}} \left((2s)^{\frac{d}{2}} r^{\frac{(d-2)}{2}} e^{-r^2 s} \right) d\nu(s) = 2^{\frac{(d-1)}{2}} \int_0^\infty s^{\frac{(d-1)}{2}} e^{-r^2 s} d\nu(s). \tag{4}$$

□

3 Examples of positive definite functions

In this section, we give two examples of basis functions constructed using the results of Section 2. It should be noted that some other common and new RBFs can be obtained from this theory.

Example 3.1. Consider the function $g(x) = (x + 1)^{-\nu-\frac{1}{2}}$, $\nu > 0$, which is CM. Then according to Theorem 2.2, the function

$$\phi(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(rx) (x^2 + 1)^{-\nu-\frac{1}{2}} dx,$$

is PD for all d . It has the representation (see 12.34.9 from [15])

$$\phi(r) = \frac{1}{2^{\nu-\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right)} r^\nu K_\nu(r),$$

where K_ν is the modified Bessel function of the second kind of order ν [32]. It is the Matérn kernel frequently used in statistics and probability theory, In order to compute its d -variate Fourier transform by (4), we first compute the inverse Laplace transform of the function g as (see 12.13.26 from [15])

$$\mathcal{L}^{-1}\left((x+1)^{-\nu-\frac{1}{2}}\right) = \frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)} s^{\nu-\frac{1}{2}} e^{-s},$$

which means that

$$d\nu(s) = \frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)} s^{\nu-\frac{1}{2}} e^{-s} ds. \tag{5}$$

By substituting (5) in (4), we get

$$\mathcal{F}_d\phi(r) = \frac{2^{\frac{(d-1)}{2}}}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\infty s^{\nu+\frac{d}{2}-1} e^{-(r^2+1)s} ds. \tag{6}$$

Now by considering the Mellin transform of the function e^{-ax^p} as (see 3.14 from [25])

$$\int_0^\infty x^{z-1} e^{-ax^p} dx = \frac{1}{p} a^{-\frac{z}{p}} \Gamma\left(\frac{z}{p}\right),$$

and letting $z = \nu + \frac{d}{2}$, $a = r^2 + 1$, $p = 1$, the equation (6) leads to

$$\mathcal{F}_d\phi(r) = \frac{2^{\frac{(d-1)}{2}} \Gamma\left(\nu+\frac{d}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)} (r^2+1)^{-\left(\nu+\frac{d}{2}\right)}.$$

Therefore the following common result yields

$$\mathcal{F}_d\left(r^{\nu-\frac{d}{2}} K_{\nu-\frac{d}{2}}(r)\right) = 2^{\nu-1} \Gamma(\nu) (r^2+1)^{-\nu}.$$

Example 3.2. Consider the function $g(x) = x^{-\frac{1}{4}} e^{-c\sqrt{x}}$, $c > 0$ which is the multiplication of two CM functions and hence it is CM. Then according to the Theorem 2.2, the function

$$\phi(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(rx) x^{-\frac{1}{2}} e^{-cx} dx,$$

is PD for all d . It has the representation (see 12.34.15 from [15])

$$\phi(r) = \sqrt{\frac{(c^2+r^2)^{\frac{1}{2}}+c}{c^2+r^2}}.$$

In order to compute its d -variate Fourier transform by (4), we first compute the inverse Laplace transform of the function g as (see 5.94 from [26])

$$\mathcal{L}^{-1}\left(x^{-\frac{1}{4}} e^{-c\sqrt{x}}\right) = \frac{2^{-\frac{1}{4}}}{\sqrt{\pi}} s^{-\frac{3}{4}} e^{-\frac{c^2}{8s}} D_{\frac{1}{2}}\left(\frac{c}{\sqrt{2s}}\right), \tag{7}$$

where $D_\alpha(x)$ is the parabolic cylinder function which is in the simple case $\alpha = \frac{1}{2}$ as

$$D_{\frac{1}{2}}(x) = \sqrt{\frac{x^3}{8\pi}} \left(K_{\frac{1}{4}}\left(\frac{x^2}{4}\right) + K_{\frac{3}{4}}\left(\frac{x^2}{4}\right) \right). \tag{8}$$

More details about this function can be found in [35]. According to (7)

$$d\nu(s) = \frac{2^{-\frac{1}{4}}}{\sqrt{\pi}} s^{-\frac{3}{4}} e^{-\frac{c^2}{8s}} D_{\frac{1}{2}}\left(\frac{c}{\sqrt{2s}}\right) ds. \tag{9}$$

By substituting (9) in (4), we get

$$\mathcal{F}_d\phi(r) = \frac{2^{\frac{(d-1)}{2}} 2^{-\frac{1}{4}}}{\sqrt{\pi}} \int_0^\infty s^{\left(\frac{d}{2}-\frac{3}{4}\right)} e^{-r^2s} e^{-\frac{c^2}{8s}} D_{\frac{1}{2}}\left(\frac{c}{\sqrt{2s}}\right) ds. \tag{10}$$

Since there is no explicit solution for (10), we consider the following common cases:

i) $d=1$.

According to (10), we have

$$\mathcal{F}_1\phi(r) = \frac{2^{-\frac{1}{4}}}{\sqrt{\pi}} \int_0^\infty s^{-\frac{3}{4}} e^{-r^2s} e^{-\frac{c^2}{8s}} D_{\frac{1}{2}}\left(\frac{c}{\sqrt{2s}}\right) ds.$$

Now by considering the following formula (see 7.728 from [15])

$$\int_0^\infty (2t)^{-\frac{\nu}{2}} e^{-pt} e^{-\frac{q^2}{8t}} D_{\nu-1}\left(\frac{q}{\sqrt{2t}}\right) dt = \sqrt{\frac{\pi}{2}} p^{\frac{1}{2}\nu-1} e^{-q\sqrt{p}},$$

and letting $\nu = \frac{3}{2}$, $p = r^2$, $q = c$, we get

$$\mathcal{F}_1\phi(r) = \frac{e^{-cr}}{\sqrt{r}}.$$

ii) $d = 2$ & $d = 3$

By substituting (8) in (10), we get

$$\mathcal{F}_d\phi(r) = \frac{2^{\frac{d}{2}-3}\sqrt{c^3}}{\pi} \int_0^\infty s^{(\frac{d}{2}-2)} e^{-\frac{c^2}{8s}} \left(K_{\frac{1}{4}}\left(\frac{c^2}{8s}\right) + K_{\frac{3}{4}}\left(\frac{c^2}{8s}\right) \right) e^{-r^2s} ds. \tag{11}$$

Then for $d = 2$, we have

$$\mathcal{F}_2\phi(r) = \frac{\sqrt{c^3}}{4\pi} \int_0^\infty s^{-1} e^{-\frac{c^2}{8s}} \left(K_{\frac{1}{4}}\left(\frac{c^2}{8s}\right) + K_{\frac{3}{4}}\left(\frac{c^2}{8s}\right) \right) e^{-r^2s} ds. \tag{12}$$

Now by considering the following formula (see 15.58 from [26])

$$\int_0^\infty t^{-1} e^{-\frac{b}{t}} K_\nu\left(\frac{b}{t}\right) e^{-pt} dt = 2 \left(K_\nu(\sqrt{2bp}) \right)^2,$$

and letting $b = \frac{c^2}{8}$, $p = r^2$, we have

$$\mathcal{F}_2\phi(r) = \frac{\sqrt{c^3}}{2\pi} \left(\left(K_{\frac{1}{4}}\left(\frac{cr}{2}\right) \right)^2 + \left(K_{\frac{3}{4}}\left(\frac{cr}{2}\right) \right)^2 \right).$$

Moreover, by substituting $d = 3$ in (11), we have

$$\mathcal{F}_3\phi(r) = \frac{2^{-\frac{3}{2}}\sqrt{c^3}}{\pi} \int_0^\infty s^{-\frac{1}{2}} e^{-\frac{c^2}{8s}} \left(K_{\frac{1}{4}}\left(\frac{c^2}{8s}\right) + K_{\frac{3}{4}}\left(\frac{c^2}{8s}\right) \right) e^{-r^2s} ds.$$

Now by considering the following formula (see 15.56 from [26])

$$\int_0^\infty t^{-\frac{1}{2}} e^{-\frac{b}{t}} K_\nu\left(\frac{b}{t}\right) e^{-pt} dt = 2\sqrt{\frac{\pi}{p}} K_{2\nu}(\sqrt{8bp}),$$

and letting $b = \frac{c^2}{8}$, $p = r^2$, we have

$$\mathcal{F}_3\phi(r) = \sqrt{\frac{c^3}{2\pi}} \frac{1}{r} \left(K_{\frac{1}{2}}(cr) + K_{\frac{3}{2}}(cr) \right). \tag{13}$$

4 Numerical Results

We now provide some examples in order to demonstrate the efficiency and superiority of our new PD RBF

$$\phi(r) = \sqrt{\frac{(c^2 + r^2)^{\frac{1}{2}} + c}{c^2 + r^2}}, \tag{14}$$

discussed in Example 2. We take different number of uniform center points

$$x_j = a + (j-1) \frac{b-a}{n-1}, \quad j = 1, \dots, n$$

and non-uniform Chebyshev-Gauss-Lobatto center points

$$x_j = \frac{(a+b)}{2} - \frac{(b-a)}{2} \cos\left(\frac{\pi(j-1)}{n-1}\right), \quad j = 1, \dots, n$$

in $[a, b]$. We use the maximum absolute error norm

$$L_\infty = \max_{1 \leq i \leq m} |f_i - \bar{f}_i|,$$

where f and \bar{f} represent the exact and approximate solutions, respectively. The optimal shape parameter c is found by trial and error. The numerical experiments have been carried out with Matlab on an Intel(R) Core(TM) i5-10210U CPU @ 1.60GHz, 2.11 GHz. In figure 1, we have plotted ϕ centered at $x_j = 0$ for different values of the shape parameter c .

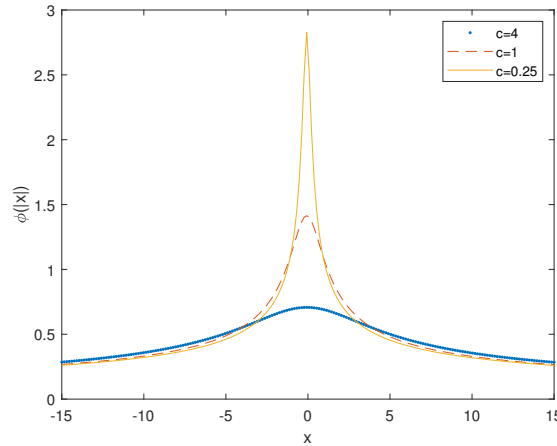


Figure 1: plots of $\phi(r)$ for different values of the shape RBF parameter c .

4.1 Test problem 1

In the first test problem, consider the function (cf. [11])

$$f_1(x) = \frac{\sinh(x)}{1 + \cosh(x)}, \quad x \in [-3, 3].$$

The exact and approximate solutions of f_1 and relative errors using the proposed PD RBF interpolation method with $c = 4.5$ for $n = 400$ uniform center points and $m = 200$ uniform evaluation points are given in Figures 2-(a) and 2-(b), respectively. The L_∞ error norms using the proposed RBF and classical Gaussian and Inverse Multiquadric ($\beta = \frac{1}{2}$) RBFs interpolation methods with different number of uniform and non-uniform center points, and $m = 350$ uniform evaluation points are reported in Tables 1 and 2, respectively. It can be noted from Tables 1-2, that the new PD RBF leads to more accurate results than the classical RBF interpolation methods. The results in Tables 1-2 are also in agreement with the ones given in [12] which evaluates the Gaussian RBF interpolants in a stable way for large values of the shape parameter c . But that approach works for certain values of n and c , and encounters limitations at large values of n due to the computational cost.

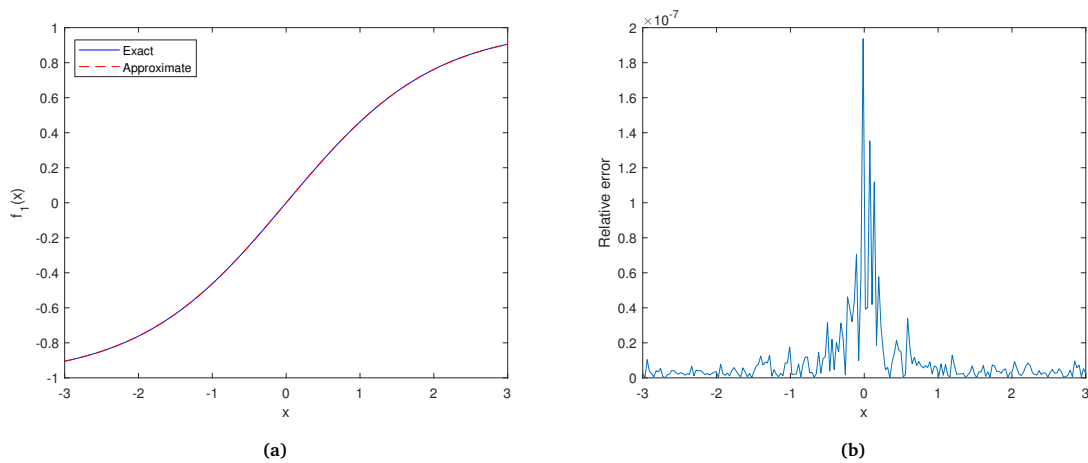


Figure 2: Exact and approximate solutions of f_1 (a), Relative errors (b) with the proposed PD RBF interpolation method for $n = 400$ uniform center points; Test problem 1.

Table 1: Comparison of approximation accuracy of the proposed PD RBF and classical Gaussian and Inverse Multiquadric RBFs interpolation methods for uniform center points; Test problem 1.

n	PD RBF (14)	Gaussian RBF	Inverse Multiquadric RBF
	($c = 4.5$)	($c = 1$)	($c = 0.8$)
	L_∞	L_∞	L_∞
80	6.6×10^{-9}	1.4×10^{-6}	8.8×10^{-7}
100	2.3×10^{-8}	5.2×10^{-7}	1.5×10^{-7}
200	1.8×10^{-9}	6.6×10^{-6}	2.7×10^{-8}
300	1.7×10^{-9}	2.3×10^{-5}	8.6×10^{-8}

Table 2: Comparison of approximation accuracy of the proposed PD RBF and classical Gaussian and Inverse Multiquadric RBFs interpolation methods for non-uniform center points; Test problem 1.

n	PD RBF (14)	Gaussian RBF	Inverse Multiquadric RBF
	($c = 4.5$)	($c = 0.5$)	($c = 0.6$)
	L_∞	L_∞	L_∞
80	2.6×10^{-9}	5.9×10^{-7}	5.8×10^{-8}
100	8.1×10^{-10}	5.1×10^{-6}	4.7×10^{-8}
200	2.7×10^{-9}	2.1×10^{-6}	1.4×10^{-7}
300	1.9×10^{-10}	1.3×10^{-6}	4.5×10^{-8}

4.2 Test problem 2

In this experiment, consider an oscillatory function (cf. [31])

$$f_2(x) = \sin(50\pi x)e^{(100(x-\frac{1}{2}))^2}, \quad x \in [0, 1].$$

The exact and approximate solutions of f_2 and relative errors using the proposed PD RBF interpolation method with $c = 1$ for $n = 200$ uniform center points and $m = 300$ uniform evaluation points are given in Figures 2-(a) and 2-(b), respectively. It should be noted that the numerical errors are really smaller than those reported in [31] for the Gaussian RBF interpolation method with constant, random, linearly, and exponentially varying shape parameters which are of at most 10^{-3} .

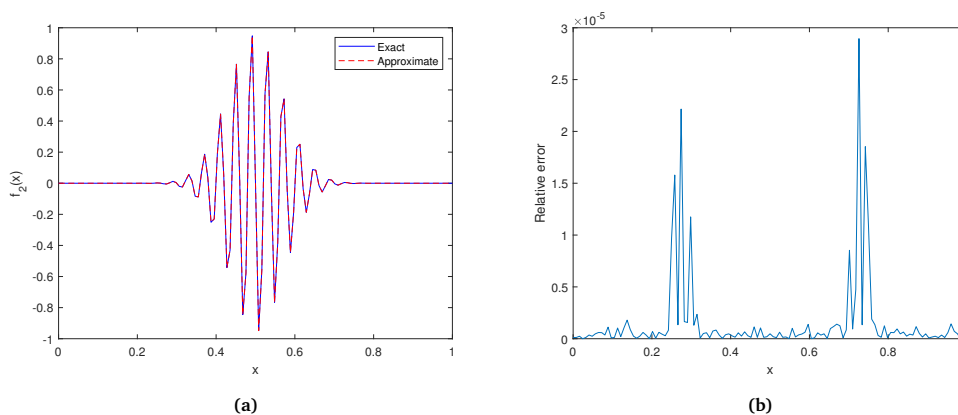


Figure 3: Exact and approximate solutions of f_2 (a), Relative errors (b) with the proposed PD RBF interpolation method for $n = 200$ uniform center points; Test problem 2.

4.3 Test problem 3 (Runge function)

Let us consider the Runge function on $[-1, 1]$, that is $f_3(x) = \frac{1}{1 + 25x^2}$. The exact and approximate solutions of f_3 and relative errors using the proposed PD RBF interpolation method for $c = 0.1$, $n = 400$ uniform center points, and $m = 200$ uniform evaluation points are given in Figures 4-(a) and 4-(b), respectively. We see that no Runge-type oscillations arise. In order to show the efficiency of our PD RBF, we multiply 25 by the large number 10^9 , and work with

$$f_4(x) = \frac{1}{1 + 25 \times 10^9 \times x^2}.$$

The exact and approximate solutions of f_4 using the proposed PD RBF interpolation method with $c = 0.1$ for $n = 101$ uniform center points is given in Figure 5. As we can see in the Figure 5, this change leads a sharp gradient at the points $x = 0$. Now, we plot the Relative errors using the proposed PD RBF ($c = 0.1$) and classical Inverse Multiquadric RBF ($c = 0.08$) interpolation

methods for $n = 200$ non-uniform center points and $m = 300$ uniform evaluation points in Figure 6. This figure reveals superiority of the proposed RBF.

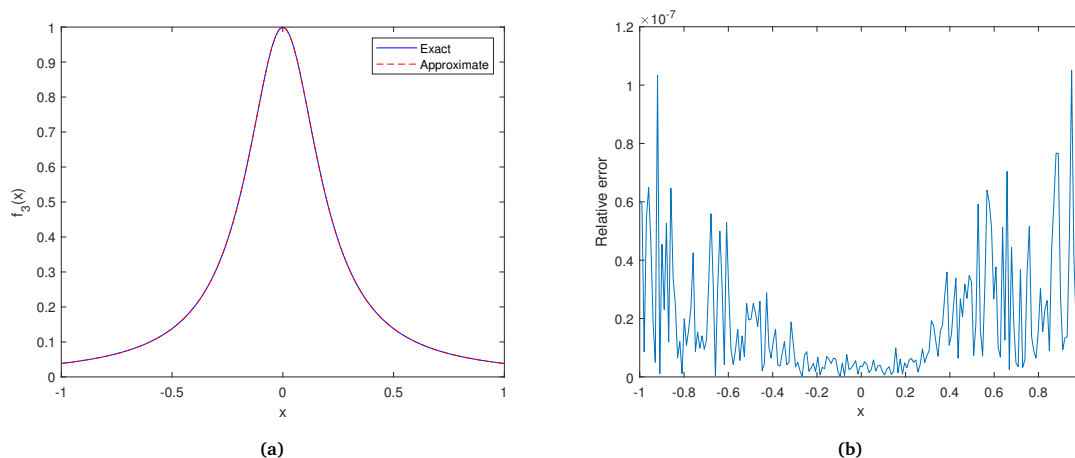


Figure 4: Exact and approximate solutions of f_3 (a), Relative errors (b) with the proposed PD RBF interpolation method for $c = 0.1$ and $n = 400$ uniform center points; Test problem 3.

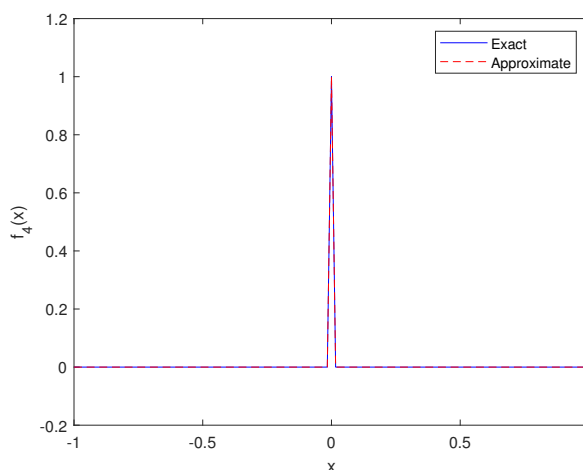


Figure 5: Exact and approximate solutions of f_4 with the proposed PD RBF interpolation method for $c = 0.1$ and $n = 101$ uniform center points; Test problem 3.

4.4 Test problem 4

Consider the interpolation of Franke’s function on $[0, 1]^2$. We plot the exact and approximate Franke’s function in Figures 7(a)-7(b) and point-wise error distributions in Figures 7(c)-7(d) for $c = 0.3$, $n = 441$ uniform center points and $m = 2601$ uniform evaluation points.

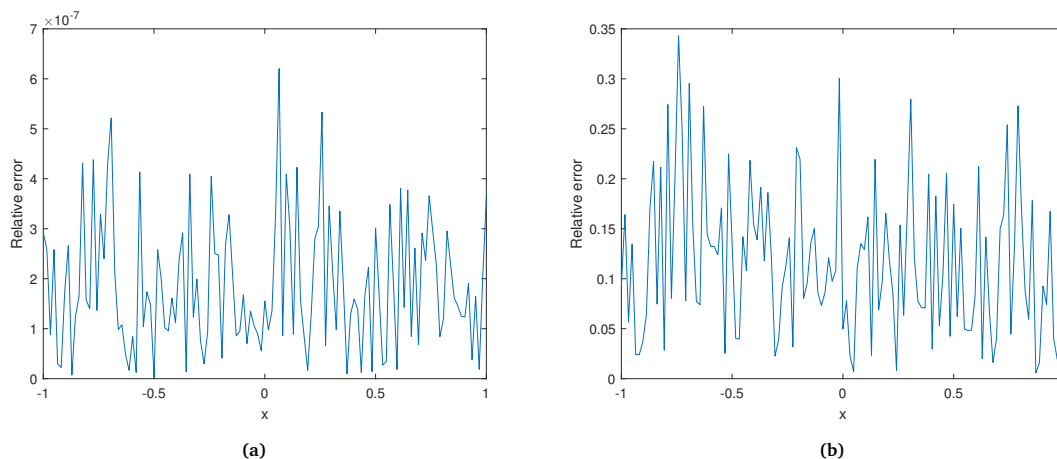


Figure 6: Relative errors: proposed PD RBF (a), Inverse Multiquadric RBF (b), for $n = 200$ non-uniform center points; Test problem 3.

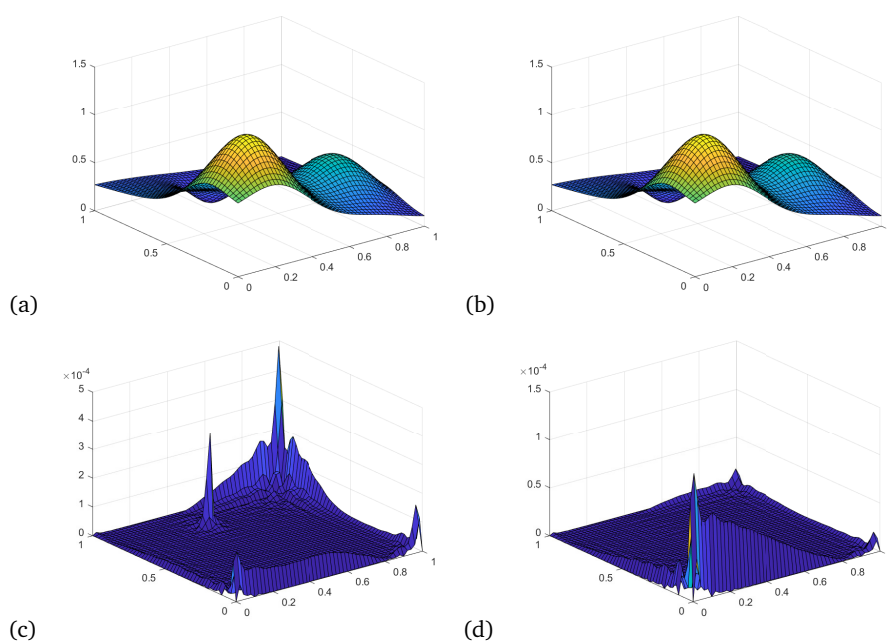


Figure 7: Exact (a) and approximate (b) Franke's function; Relative (c) and absolute (d) errors distributions, with $n = 441$, $c = 0.3$.

5 Conclusion

The Fourier cosine transform is used in order to obtain a new class of infinitely smooth positive definite RBFs from completely monotone functions, which are represented in terms of positive Borel measures. The proposed theory is used for reconstructing the well-known Matérn RBF and presenting a new positive definite RBF. Numerical results show an accurate reconstruction of functions in which the Runge phenomenon is substantially mitigated.

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