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Convergence Analysis with Self-Comparative Rate Assessment of a Novel Iterative Method Based on Kirk's Iteration

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Abstract

This research presents a comprehensive theoretical and computational analysis of the Kirk order-2 iteration method for approximating fixed points of operators that satisfy a weak contractive condition within the framework of a real Banach space. The primary objectives are to establish the strong convergence, stability, and computational efficiency of this iterative scheme. A key contribution of this work is a detailed self-comparative analysis of the convergence rate among six distinct permutations of the iterative scheme's coefficients. We derive a precise analytical condition that determines which permutation yields a faster convergence rate, providing a theoretical framework for optimizing the algorithm's performance. Numerical results are presented to validate the theoretical findings and demonstrate the algorithm's efficiency, confirming that specific permutations of the coefficients can significantly accelerate convergence.

Keywords: Fixed point iteration procedures; Strong convergence; Stability; Banach spaces.
Mathematics Subject Classification: 47H09; 47H10; 54H25.

1 Introduction

Fixed point theory plays a crucial role in addressing a wide spectrum of mathematical problems across diverse domains. Systems of linear and nonlinear equations, differential and integral equations, optimization problems, and variational analysis can be expressed as fixed point problems. The literature is replete with recent contributions to fixed point theory and its applications in related optimization areas, such as minimization problems, equilibrium problems, and variational inequalities, as highlighted in [4, 5, 6, 27].

Considerable attention has been devoted to studying fixed points of various classes and their generalization to nonlinear mappings in several spaces such as metric spaces, Banach spaces, Hilbert spaces, orthogonal metric-like space, ultrametric spaces, perturbed metric spaces, soft metric spaces, b-rectangular metric spaces etc. Numerous authors have made significant contributions in this area, including [1, 2, 3, 11, 12, 13, 14, 15, 20, 22, 23, 24, 25, 26]. The ability to approximate fixed points and characterize the conditions under which solutions exist is a fundamental pursuit in fixed point theory.

While closed-form solutions for finding fixed points of arbitrary nonlinear mappings are generally nonexistence, fixed point iterative methods provide a powerful computational framework. These iterative schemes leverage the intrinsic structure of the fixed point problem to construct sequences that converge to fixed points, often with remarkable efficiency and accuracy.

To establish the framework for our discussion, we adopt the following definitions unless otherwise specified in this paper. Let $(X, \|\cdot\|)$ be a real Banach space, C is a nonempty closed convex subset of X, and T is a self-mapping on C. Additionally, let $x_0 \in C$ be an initial point, and denote the set of fixed points of T in C by $F(T) := \{p \in C : Tp = p\}$. This setting provides the foundation for exploring various fixed point iterative schemes.

The pioneering work of Picard in 1890 introduced the iteration scheme bearing his name, given by:

$$x_{n+1} = Tx_n \tag{1}$$

for all $n \in \mathbb{N} \cup \{0\}$. This scheme, also known as the functional iteration scheme, generates a sequence $\{x_n\}_{n=0}^{\infty}$ called the Picard sequence with an initial point x_0. Regrettably, this scheme does not converge for a nonexpansive mapping T, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, prompting researchers to seek alternative iterative methods tailored for such mapping. In 1953, Mann [18] proposed the Mann iteration $\{x_n\}_{n=0}^{\infty}$, which can be formulated as:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n \tag{2}$$

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for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of real numbers in $[0,1]$. When $\alpha_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$, the Mann iteration reduces to the Picard iteration (1). One of generalizations of the Mann iteration is an iteration process called Kirk order-k iteration due in [16], which is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + \ldots + \alpha_k T^k x_n \tag{3}$$

for all $n \in \mathbb{N} \cup \{0\}$, where k is a natural number, $\alpha_1 > 0$ and $\alpha_i \geq 0$ for all $i = 0, 2, 3, \ldots, k$ such that $\sum_{i=0}^{k} \alpha_i = 1$. This scheme reduces to Picard iteration, for $k = 0$, and to Krasnoselskij iteration, for $k = 1$.

Three years later, Ishikawa [11] generalized the Mann iteration (2) to the iteration process $\{x_n\}_{n=0}^{\infty}$, which is defined by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \end{cases} \tag{4}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0,1]$. This iteration is also called a two-step Mann iteration with two parameter sequences, reducing to the Mann iteration (2) when $\beta_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$. In 2000, Noor [19] extended the idea of the Ishikawa iteration (4) to the new iteration named the Noor iteration, which is defined by

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0,1]$. Clearly, Mann and Ishikawa iterations are special cases of Noor iteration.

Over the years, numerous authors have developed various iteration schemes to approximate fixed points of nonlinear operators and solutions of operator equations in appropriate normed spaces. For instance, in 2017, Karakaya et al. [14] introduced a three-step iteration process $\{x_n\}_{n=0}^{\infty}$, which is defined by

$$\begin{cases} z_n = T x_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} = T y_n \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}$ is a sequence of real numbers in $[0,1]$. They also established convergence results for this process to approximate a fixed point of a mapping T satisfying two conditions, including the weak contractive condition due to Berinde [8]:

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|y - Tx\| \tag{5}$$

for all $x, y \in C$, where $\delta \in [0,1)$ and $L \geq 0$, and the following condition:

$$\|Tx - Ty\| \leq \delta_1\|x - y\| + L_1\|x - Tx\| \tag{6}$$

for all $x, y \in C$, where $\delta_1 \in [0,1)$ and $L_1 \geq 0$. One year later, Ullah and Arshad [26] introduced the M-iteration $\{x_n\}_{n=0}^{\infty}$, which is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T y_n \end{cases} \tag{7}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of real numbers in $[0,1]$. They also investigated convergence results for this process to approximate a fixed point of a mapping T satisfying the following condition for all $x, y \in C$:

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|.$$

At the same year, Abbas et al.[1] introduced another novel three-step iteration process as follows:

$$\begin{cases} z_n = T x_n, \\ y_n = T z_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of real numbers in $[0,1]$. They also established convergence results for this process to approximate a fixed point of a mapping T satisfying (5) and (6). Moreover, they proved that this iterative process converges faster than most existing iterative schemes in the literature.

Motivated by the limited availability of efficient three-step fixed point schemes, Kanwar et al. [13] developed a new scheme based on a geometrically constructed iteration proposed by:

$$x_{n+1} = \frac{mx_n + Tx_n}{m+1}$$

for all $n \in \mathbb{N} \cup \{0\}$, where m is a positive real number. In addition, Sharma et al. [23] recently proposed a new three-step scheme $\{x_n\}_{n=0}^{\infty}$, which is defined by

$$\begin{cases} z_n = \dfrac{mx_n + Tx_n}{m+1}, \\ y_n = Tz_n, \\ x_{n+1} = Ty_n \end{cases} \tag{8}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $m > 0$. They also derived convergence results for this process, demonstrating its effectiveness in approximating a fixed point of the mapping T under the following condition:

$$\|Tx - Ty\| \le \varphi(\|x - Tx\|) + \delta\|x - y\| \tag{9}$$

for all $x, y \in C$, where $\varphi : [0, \infty) \to [0, \infty)$ is a monotone increasing continuous function with $\varphi(0) = 0$ and $\delta \in [0, 1)$. Their numerical experiments are presented showing that this scheme outperforms all the well-known existing three-step schemes available in the literature.

Motivated by the active research in expanding and refining fixed point iterative algorithms, and building on Kirk's classical scheme (3) and its recent multi-step variants, in this paper, we propose the new scheme covering several existing scheme in the literature as follows:

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n Tx_n + \beta_n T^2 x_n, \tag{10}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $x_0 \in C$, $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0, 1]$ and T satisfies the weak contractive condition (9). It is worth noting that the proposed iterative scheme (10) serves as a generalization of several fundamental iterative methods. The most significant structural reduction occurs when the parameter $\beta_n = 0$, which reduces our algorithm to the well-known Mann iteration (2). If this is further specialized by constraining the sequence of coefficients $\{\alpha_n\}$ to be a constant, $\alpha_n = \alpha \in [0, 1]$, it becomes the Krasnoselskij iteration [17]. In the most specific case where $\beta_n = 0$ and $\alpha_n = 1$ for all n, the iteration (10) simplifies to the classical Picard iteration (1). Alternatively, if all sequential parameters are set to constants, namely $\alpha_n = \alpha > 0$ and $\beta_n = 0 \ge 0$, the proposed scheme reduces to Kirk's iteration (3), which retains the original three-term structure. To visually synthesize the relationships between the proposed method and the existing literature, Figure 1 presents a genealogy of the iterative schemes discussed. This diagram maps the structural reductions that connect the proposed scheme (10) to fundamental algorithms. As illustrated, the central column traces the historical evolution from the classical Picard iteration through the Mann, Ishikawa, and Noor iterations. In this lineage, each subsequent scheme generalizes the former by introducing additional steps and parameters. This visual map emphasizes that the proposed scheme (10) is not merely an isolated algorithm, but a unifying framework that encompasses these foundational fixed point iterations.

Furthermore, we aim to establish strong convergence results and to investigate both the theoretical stability of the method. We will also benchmark the proposed scheme's performance through itself-comparing and demonstrate its practical value through numerical experiments.

The remainder of this article is organized as follows. In Section 2, we recall essential definitions and preliminaries; In Section 3, we proposed convergence and stability results of the proposed iteration (10) under the weak contractive condition (9). In addition, we showed self-comparison on rate of convergence and also numerical experiments on selective examples in Section 4 and Section 5, respectively.

2 Preliminaries

In this section, we introduce the fundamental concepts necessary for the study in the subsequent sections. This includes the definitions of key functions relevant to this research, the formal definitions of stability for iteration processes converging to a fixed point, as well as essential auxiliary results required for the convergence analysis of the iteration process in this study.

Lemma 2.1 ([7]). *If δ is a real number such that $0 \le \delta < 1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequences of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \le \delta u_n + \varepsilon_n \text{ for all } n = 0, 1, 2, \dots,$$

we have $\lim_{n\to\infty} u_n = 0$.

Definition 2.1 ([10]). Let T be a self-mapping on a nonempty subset D of a normed space $(X, \|\cdot\|)$ and $\{t_n\}_{n=0}^{\infty}$ be any arbitrary sequence in D . Then, an iteration procedure $x_{n+1} = f(T, x_n)$ in D, converging to a fixed point p, is said to be T-stable, if for $\varepsilon_n := \|t_{n+1} - f(T, t_n)\|$ $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \varepsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n = p$.

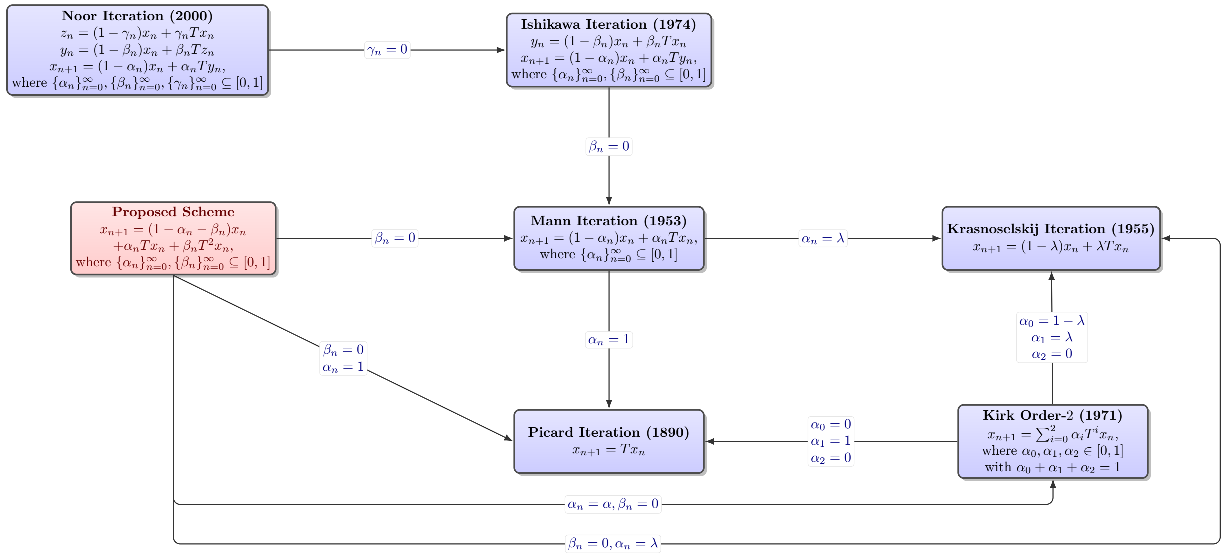

Figure 1: Genealogy of fixed point iteration schemes. The diagram illustrates the parameter constraints required to reduce the proposed scheme (10) and other multi-step iterations to fundamental methods like Mann, Krasnoselskij, and Picard.

Definition 2.2 ([9]). Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two iteration procedures in a nonempty subset D of a normed space $(X, \|\cdot\|)$ such that converge to the same point p and $\|u_n - p\| \leq a_n$ and $\|v_n - p\| \leq b_n$ for all $n \in \mathbb{N} \cup \{0\}$. If the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ converge to a and b, respectively, and $\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0$, then we say that $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to p.

Phuengrattana and Suantai [21] observed that the comparison in the above definition depends on the choice of the error-bound sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$. Because this could make the comparison unclear, they modified the definition as follows:

Definition 2.3 ([21]). Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be two iteration procedures in a nonempty subset D of a normed space $(X, \|\cdot\|)$ such that converge to the same point z. We say that $\{x_n\}_{n=0}^{\infty}$ converges faster than $\{y_n\}_{n=0}^{\infty}$ to z if $\lim_{n\to\infty} \frac{\|x_n - z\|}{\|y_n - z\|} = 0$.

3 Main Results

From our theoretical and computational investigations, we will show that the proposed iteration (10) under the weak contractive condition (9) not only converges reliably but also exhibits desirable stability and competitive efficiency. We begin with the following convergence theorem:

Theorem 3.1. *Suppose that D is a nonempty closed convex subset of a real Banach space $(E, \|\cdot\|)$, and let $T : D \to D$ be a mapping satisfying (9). Assume that $p \in F(T) \neq \emptyset$ is a fixed point of T. Define the sequence $\{x_n\}_{n=0}^{\infty}$ through the iterative process:*

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n T x_n + \beta_n T^2 x_n,$$

starting from $x_0 \in D$, and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequence in $[0,1]$ which satisfy at least one of the following conditions:

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \tag{11}$$

or

$$\sum_{n=1}^{\infty} \beta_n = \infty. \tag{12}$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point p of T.

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Proof. To show that $\lim_{n\to\infty} \|x_n - p\| = 0$, we proceed as follows: for each $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n-\beta_n)x_n + \alpha_n T x_n + \beta_n T^2 x_n - p\| \\
&\le (1-\alpha_n-\beta_n)\|x_n - p\| + \alpha_n\|Tx_n - p\| + \beta_n\|T^2x_n - p\| \\
&= (1-\alpha_n-\beta_n)\|x_n - p\| + \alpha_n\|Tp - Tx_n\| + \beta_n\|T^2p - T^2x_n\| \\
&\le (1-\alpha_n-\beta_n)\|x_n - p\| + \alpha_n\left[\varphi\left(\|p - Tp\|\right) + \delta\|p - x_n\|\right] \\
&\quad + \beta_n\left[\varphi\left(\|Tp - T^2p\|\right) + \delta\|Tp - Tx_n\|\right] \\
&= (1-\alpha_n-\beta_n)\|x_n - p\| + \alpha_n\delta\|p - x_n\| + \beta_n\delta\|Tp - Tx_n\| \\
&\le (1-\alpha_n-\beta_n)\|x_n - p\| + \alpha_n\delta\|x_n - p\| + \beta_n\delta\left[\varphi\left(\|p - Tp\|\right) + \delta\|p - x_n\|\right] \\
&= (1-\alpha_n-\beta_n)\|x_n - p\| + \alpha_n\delta\|x_n - p\| + \beta_n\delta^2\|x_n - p\| \\
&= (1-\alpha_n-\beta_n+\alpha_n\delta+\beta_n\delta^2)\|x_n - p\|.
\end{aligned}$$

From this inequality, we derive that

$$\begin{aligned}
\|x_{n+1} - p\| &\le \prod_{i=0}^{n}\left(1-\alpha_i-\beta_i+\alpha_i\delta+\beta_i\delta^2\right)\|x_0 - p\| \\
&= \prod_{i=0}^{n}\left(1-(1-\delta)\alpha_i-(1-\delta^2)\beta_i\right)\|x_0 - p\| \\
&\le \prod_{i=0}^{n}\left(1-(1-\delta)\alpha_i-(1-\delta)\beta_i\right)\|x_0 - p\| \\
&\le \prod_{i=0}^{n}\left(1-(1-\delta)(\alpha_i+\beta_i)\right)\|x_0 - p\| \qquad (13)
\end{aligned}$$

for each $n \in \mathbb{N} \cup \{0\}$. Since $\delta \in [0,1)$, $\alpha_i \in [0,1]$ and $\sum_{i=0}^{\infty} \alpha_i = \infty$ or $\sum_{i=0}^{\infty} \beta_i = \infty$ implied that $\sum_{i=0}^{\infty}(\alpha_i + \beta_i) = \infty$, we obtain

$$\lim_{n\to\infty} \prod_{i=0}^{n}(1-(1-\delta)(\alpha_i+\beta_i)) = 0,$$

which by (13) implies

$$\lim_{n\to\infty} \|x_{n+1} - p\| = 0,$$

therefore, $\{x_n\}$ converges strongly to p.

Next, we verify that p is the unique fixed point of T. Suppose that p and p^* are fixed points of T with $p \ne p^*$. We have

$$\begin{aligned}
\|p - p^*\| &= \|Tp - Tp^*\| \\
&\le \varphi(\|p - Tp\|) + \delta\|p - p^*\| \\
&= \delta\|p - p^*\| \\
&< \|p - p^*\|,
\end{aligned}$$

which is a contradiction. Therefore, $p = p^*$. □

To aid in visualizing the structural relationships between the iteration (10) and classical schemes, we present the process flow in Figure 2. Crucially, the flowchart incorporates a practical termination condition, where the algorithm stops when the error $\|x_{n+1} - x_n\|$ falls below a prescribed tolerance ϵ. The diagram illustrates how x_{n+1} is constructed as a weighted average of three components: the current state (identity), the operator image (Tx_n), and the second iterate (T^2x_n). This visual representation clarifies the generalizations established in the following corollaries. By setting the weight $\beta_n = 0$ for all n, the branch utilizing T^2x_n in Figure 2 is effectively pruned. The scheme reduces to a convex combination of only the Identity and Tx_n, recovering the standard Mann iteration. Additionally, by further setting $\alpha_n = 1$ (implying the identity weight $1-\alpha_n-\beta_n = 0$), the process simplifies to a single path where $x_{n+1} = Tx_n$, recovering the classical Picard iteration.

Corollary 3.2. *Under the same conditions on E, D, T in Theorem 3.1, the Mann iterative process defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad (14)$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ with $\sum_{i=0}^{\infty} \alpha_i = \infty$, converges strongly to the unique fixed point p of T.

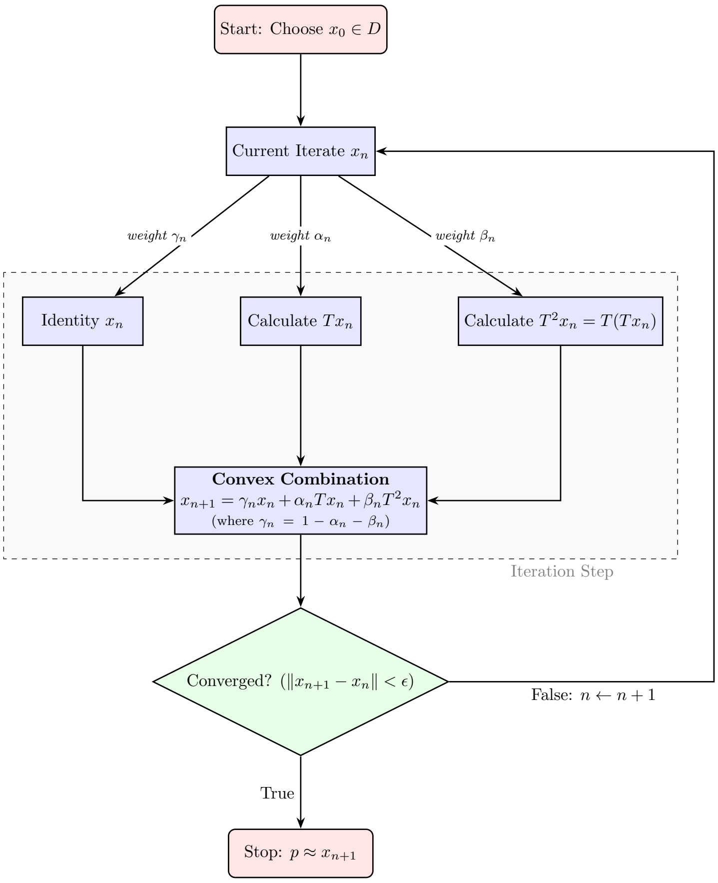

Figure 2: Schematic representation of the proposed iterative scheme (10). The process illustrates the convex combination of the current state x_n, its image Tx_n, and the second iterate T^2x_n.

Proof. Setting $\beta_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$ in Theorem 3.1. □

Corollary 3.3. *Under the same conditions on E, D, T in Theorem 3.1, the Picard iterative process defined by*

$$x_{n+1} = Tx_n, \tag{15}$$

for all $n \in \mathbb{N} \cup \{0\}$, converges strongly to the unique fixed point p of T.

Proof. Setting $\beta_n = 0$ and $\alpha_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$ in Theorem 3.1. □

The stability of an iterative algorithm is crucial for practical applications, as small computational errors introduced at each step can accumulate and lead to incorrect results. This robustness is formalized by the concept of T-stability, which requires that a sequence converges to the correct solution if and only if the errors introduced during the iteration eventually vanish. This property guarantees reliability in settings where perfect precision is unattainable. The following theorem demonstrates that our proposed iteration (10) satisfies this essential conditions.

Theorem 3.4. *Suppose that D is a nonempty closed convex subset of a real Banach space $(E, \|\cdot\|)$, and $T : D \to D$ is a mapping satisfying (9) with a unique fixed point p. If a sequence in D produced by (10), denoted by $x_{n+1} = f(T, x_n)$, converging to p, then the iteration (10) is T-stable.*

Proof. Assume that $\{t_n\}_{n=0}^{\infty}$ is an arbitrary sequence in D. Define ε_n for each $n \in \mathbb{N}$ by $\varepsilon_n := \|t_{n+1} - f(T, t_n)\|$. Let us begin by assuming that $\lim_{n\to\infty} \varepsilon_n = 0$. We aim to show that $\{t_n\}$ converges to p. Firstly, we consider the following expression for each

$n \in \mathbb{N} \cup \{0\}$:

$$\begin{aligned}
\|t_{n+1}-p\| &= \|t_{n+1}-f(T,t_n)+f(T,t_n)-p\| \\
&\le \|t_{n+1}-f(T,t_n)\|+\|f(T,t_n)-p\| \\
&= \varepsilon_n+\|f(T,t_n)-p\| \\
&= \varepsilon_n+\left\|(1-\alpha_n-\beta_n)t_n+\alpha_n Tt_n+\beta_n T^2t_n-p\right\| \\
&\le \varepsilon_n+(1-\alpha_n-\beta_n)\|t_n-p\|+\alpha_n\|Tt_n-p\|+\beta_n\|T^2t_n-p\| \\
&= \varepsilon_n+(1-\alpha_n-\beta_n)\|t_n-p\|+\alpha_n\|Tp-Tt_n\|+\beta_n\|T^2p-T^2t_n\| \\
&\le \varepsilon_n+(1-\alpha_n-\beta_n)\|t_n-p\|+\alpha_n[\varphi(\|p-Tp\|)+\delta\|p-t_n\|] \\
&\quad +\beta_n\left[\varphi\left(\|Tp-T^2p\|\right)+\delta\|Tp-Tt_n\|\right] \\
&= \varepsilon_n+(1-\alpha_n-\beta_n)\|t_n-p\|+\alpha_n\delta\|p-t_n\|+\beta_n\delta\|Tp-Tt_n\| \\
&\le \varepsilon_n+(1-\alpha_n-\beta_n)\|t_n-p\|+\alpha_n\delta\|t_n-p\|+\beta_n\delta[\varphi(\|p-Tp\|)+\delta\|p-t_n\|] \\
&= \varepsilon_n+(1-\alpha_n-\beta_n)\|t_n-p\|+\alpha_n\delta\|t_n-p\|+\beta_n\delta^2\|t_n-p\| \\
&= \varepsilon_n+(1-\alpha_n-\beta_n+\alpha_n\delta+\beta_n\delta^2)\|t_n-p\|. \\
&= \varepsilon_n+(1-\alpha_n(1-\delta)-\beta_n(1-\delta^2))\|t_n-p\|.
\end{aligned} \tag{16}$$

Since $0 \le 1-\alpha_n(1-\delta)-\beta_n(1-\delta^2) \le 1$ and $\lim_{n\to\infty} \varepsilon_n = 0$, by using Lemma 2.1 with (16), we conclude that

$$\lim_{n\to\infty} t_n = p.$$

Conversely, suppose that $\lim_{n\to\infty} t_n = p$. Using a similar approach, we get

$$\begin{aligned}
\varepsilon_n &= \|t_{n+1}-f(T,t_n)\| \\
&= \|t_{n+1}-p+p-f(T,t_n)\| \\
&\le \|t_{n+1}-p\|+\|f(T,t_n)-p\| \\
&\le \|t_{n+1}-p\|+\|t_n-p\|
\end{aligned} \tag{17}$$

for all $n \in \mathbb{N} \cup \{0\}$. Taking $n \to \infty$ in (17), we see that $\lim_{n\to\infty} \varepsilon_n = 0$. □

4 Self-comparing of the iteration (10)

Through analytical arguments and illustrative examples, we aim to compare the convergence rate of the iteration (10) with that of selected existing iterations developed under similar contractive hypotheses, highlighting scenarios in which our method achieves superior or comparable speed.

Currently, we considered six cases for writing the iteration (10). In the next result, we show that the coefficient sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ have effective roles to play in the rate of convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (10). To conduct this self-comparison analysis, we must first define the six distinct iterative forms. These forms are not fundamentally different algorithms but are rather "permutations" of the same underlying scheme (10). The core concept is to test how the convergence rate is impacted when we assign the three coefficients α_n, β_n, and $(1-\alpha_n-\beta_n)$ to the different terms in the iteration x_n, Tx_n, and T^2x_n. By analyzing all six possible permutations, we can effectively isolate the impact of each coefficient's position on the algorithm's convergence speed. To show this, we want to compare the iteration (10) with itself in the following six possible cases, using the same initial point:

$$x_{n+1}^{(1)} = (1-\alpha_n-\beta_n)x_n^{(1)}+\alpha_n Tx_n^{(1)}+\beta_n T^2x_n^{(1)}, \tag{18}$$

$$x_{n+1}^{(2)} = (1-\alpha_n-\beta_n)x_n^{(2)}+\beta_n Tx_n^{(2)}+\alpha_n T^2x_n^{(2)}, \tag{19}$$

$$x_{n+1}^{(3)} = \alpha_n x_n^{(3)}+(1-\alpha_n-\beta_n)Tx_n^{(3)}+\beta_n T^2x_n^{(3)}, \tag{20}$$

$$x_{n+1}^{(4)} = \alpha_n x_n^{(4)}+\beta_n Tx_n^{(4)}+(1-\alpha_n-\beta_n)T^2x_n^{(4)}, \tag{21}$$

$$x_{n+1}^{(5)} = \beta_n x_n^{(5)}+(1-\alpha_n-\beta_n)Tx_n^{(5)}+\alpha_n T^2x_n^{(5)}, \tag{22}$$

and

$$x_{n+1}^{(6)} = \beta_n x_n^{(6)}+\alpha_n Tx_n^{(6)}+(1-\alpha_n-\beta_n)T^2x_n^{(6)}, \tag{23}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $x_0^{(t)} = x_0 \in D$ for all $t \in \{1,2,3,4,5,6\}$ and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \subseteq [0,1]$ with $\alpha_n + \beta_n \in (0,1]$ for all $n \in \mathbb{N} \cup \{0\}$.

Observing from (18) to (23), it is evident that they share a fundamental structure. In each case of $t \in \{1,2,3,4,5,6\}$, $x_{n+1}^{(t)}$ is computed based on the terms $x_n^{(t)}, Tx_n^{(t)}$, and $T^2 x_n^{(t)}$. The variation among these equations arises solely from the coefficients multiplying these three terms; specifically, these coefficients correspond to the different permutations of the values α_n, β_n, and $(1-\alpha_n-\beta_n)$. All six iterative schemes can be represented by the following single generalized form:

$$x_{n+1}^{(t)} = A_n^{(t)} x_n^{(t)} + B_n^{(t)} T x_n^{(t)} + C_n^{(t)} T^2 x_n^{(t)} \tag{24}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $t \in \{1,2,3,4,5,6\}$.

In this expression:

- The index t ranges from 1 to 6, corresponding to (18) through (23).
- Crucially, for any given t, the coefficients $(A_n^{(t)}, B_n^{(t)}, C_n^{(t)})$ form one of the six possible permutations of the set $\{\alpha_n, \beta_n, 1-\alpha_n-\beta_n\}$.

For instance:

- In (18) ($t=1$), the coefficients are $(A_n^{(1)}, B_n^{(1)}, C_n^{(1)}) = (1-\alpha_n-\beta_n, \alpha_n, \beta_n)$.
- In (20) ($t=3$), the coefficients are $(A_n^{(3)}, B_n^{(3)}, C_n^{(3)}) = (\alpha_n, 1-\alpha_n-\beta_n, \beta_n)$.

This pattern encompasses all six distinct permutations of the three coefficient values. It is also noteworthy that the sum of these coefficients consistently equals one:

$$A_n^{(t)} + B_n^{(t)} + C_n^{(t)} = \alpha_n + \beta_n + (1-\alpha_n-\beta_n) = 1$$

for all $n \in \mathbb{N} \cup \{0\}$. This unified representation underscores the shared mathematical structure across the different iterative forms presented in (18)-(23).

Theorem 4.1. *Let $k,l \in \{1,2,3,4,5,6\}$ with $k \neq l$. Suppose that D is a nonempty closed convex subset of a real Banach space $(E, \|\cdot\|)$, and let $T : D \to D$ be a mapping satisfying (9). Assume that $p \in F(T) \neq \emptyset$ and let $\{x_n^{(k)}\}_{n=0}^{\infty}$ and $\{x_n^{(l)}\}_{n=0}^{\infty}$ be two sequences generated by the iterative scheme (24), while the following conditions hold:*

(S1) The sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are in $[0,1]$ which satisfy (11) or (12).

(S2) There exist $a^{(k)}, b^{(k)}, c^{(k)}, \alpha^{(k)}, \beta^{(k)}$ and $\gamma^{(k)}$ such that $0 < \alpha^{(k)} \leq A_n^{(k)} \leq a^{(k)} < 1$, $0 < \beta^{(k)} \leq B_n^{(k)} \leq b^{(k)} < 1$ and $0 < \gamma^{(k)} \leq C_n^{(k)} \leq c^{(k)} < 1$ for all $n \in \mathbb{N} \cup \{0\}$.

(S3) There exist $a^{(l)}, b^{(l)}, c^{(l)}, \alpha^{(l)}, \beta^{(l)}$ and $\gamma^{(l)}$ such that $0 < \alpha^{(l)} \leq A_n^{(l)} \leq a^{(l)} < 1$, $0 < \beta^{(l)} \leq B_n^{(l)} \leq b^{(l)} < 1$ and $0 < \gamma^{(l)} \leq C_n^{(l)} \leq c^{(l)} < 1$ for all $n \in \mathbb{N} \cup \{0\}$.

If $a^{(k)} + b^{(k)}\delta + c^{(k)}\delta^2 < \alpha^{(l)} - b^{(l)}\delta - c^{(l)}\delta^2$, then $\{x_n^{(k)}\}_{n=0}^{\infty}$ converges to p faster than $\{x_n^{(l)}\}_{n=0}^{\infty}$.

Proof. From condition (S1), we get $\lim_{n\to\infty} x_n^{(k)} = p$ and $\lim_{n\to\infty} x_n^{(l)} = p$. Applying (9) to the iteration (24), we immediately deduce

$$\|x_{n+1}^{(k)} - p\| \leq \prod_{i=0}^{n} \left(A_i^{(k)} + B_i^{(k)}\delta + C_i^{(k)}\delta^2\right) \|x_0^{(k)} - p\| \tag{25}$$

and

$$\|x_{n+1}^{(l)} - p\| \geq \prod_{i=0}^{n} \left(A_i^{(l)} - B_i^{(l)}\delta - C_i^{(l)}\delta^2\right) \|x_0^{(l)} - p\| \tag{26}$$

for all $n \in \mathbb{N} \cup \{0\}$. Next, we want to show that $\{x_n^{(k)}\}_{n=0}^{\infty}$ converges to p faster than $\{x_n^{(l)}\}_{n=0}^{\infty}$ by using Definition 2.3. Leveraging conditions (S2) and (S3) together with (25) and (26), the quotient of the norms can be bounded as follows:

$$\begin{aligned} \frac{\|x_{n+1}^{(k)} - p\|}{\|x_{n+1}^{(l)} - p\|} &\leq \frac{\prod_{i=0}^{n} \left(A_i^{(k)} + B_i^{(k)}\delta + C_i^{(k)}\delta^2\right) \|x_0^{(k)} - p\|}{\prod_{i=0}^{n} \left(A_i^{(l)} - B_i^{(l)}\delta - C_i^{(l)}\delta^2\right) \|x_0^{(l)} - p\|} \\ &= \frac{\prod_{i=0}^{n} \left(A_i^{(k)} + B_i^{(k)}\delta + C_i^{(k)}\delta^2\right)}{\prod_{i=0}^{n} \left(A_i^{(l)} - B_i^{(l)}\delta - C_i^{(l)}\delta^2\right)} \\ &\leq \frac{\left(a^{(k)} + b^{(k)}\delta + c^{(k)}\delta^2\right)^n}{\left(\alpha^{(l)} - b^{(l)}\delta - c^{(l)}\delta^2\right)^n} \end{aligned} \tag{27}$$

for all $n \in \mathbb{N} \cup \{0\}$. Define $\theta_n = \frac{\left(a^{(k)}+b^{(k)}\delta+c^{(k)}\delta^2\right)^n}{\left(\alpha^{(l)}-b^{(l)}\delta-c^{(l)}\delta^2\right)^n}$ for all $n \in \mathbb{N}$. We now turn to the classical ratio test. Observe that

$$\begin{aligned}\lim_{n\to\infty}\frac{\theta_{n+1}}{\theta_n} &= \lim_{n\to\infty}\frac{\left(a^{(k)}+b^{(k)}\delta+c^{(k)}\delta^2\right)^{n+1}}{\left(\alpha^{(l)}-b^{(l)}\delta-c^{(l)}\delta^2\right)^{n+1}}\cdot\frac{\left(\alpha^{(l)}-b^{(l)}\delta-c^{(l)}\delta^2\right)^n}{\left(a^{(k)}+b^{(k)}\delta+c^{(k)}\delta^2\right)^n}\\ &= \lim_{n\to\infty}\frac{a^{(k)}+b^{(k)}\delta+c^{(k)}\delta^2}{\alpha^{(l)}-b^{(l)}\delta-c^{(l)}\delta^2}\\ &= \frac{a^{(k)}+b^{(k)}\delta+c^{(k)}\delta^2}{\alpha^{(l)}-b^{(l)}\delta-c^{(l)}\delta^2}\\ &< 1.\end{aligned}$$

Therefore, the series $\sum_{n=0}^{\infty}\theta_n$ converges, which forces $\lim_{n\to\infty}\theta_n = 0$. As a conclusion from the previous step with (27), the sequence $\{x_n^{(k)}\}_{n=0}^{\infty}$ converges faster to p than the sequence $\{x_n^{(l)}\}_{n=0}^{\infty}$.

□

We now investigate the behavior of the sequences $\{x_n^{(k)}\}_{n=0}^{\infty}$ and $\{x_n^{(l)}\}_{n=0}^{\infty}$ in Theorem 4.1 under the condition that there exist $a^{'}, b^{'}, \alpha^{'}$ and $\beta^{'}$ satisfying

$$\text{(S4)}\quad \begin{cases} 0 < \alpha^{'} \le \alpha_n \le a^{'} < 1,\\ 0 < \beta^{'} \le \beta_n \le b^{'} < 1. \end{cases}$$

This condition follows that

$$\text{(S5)}\quad 0 < 1 - a^{'} - b^{'} \le 1 - \alpha_n - \beta_n \le 1 - \alpha^{'} - \beta^{'} < 1$$

Furthermore, considering the iterations (18)–(23), we obtain Table 1 lists the explicit forms of the coefficients $A_n^{(t)}, B_n^{(t)}$ and $C_n^{(t)}$ corresponding to each iterative scheme $x_{n+1}^{(t)}$ for all $t = 1, 2, 3, \ldots, 6$.

Table 1: Coefficients $A_n^{(t)}, B_n^{(t)}, C_n^{(t)}$ corresponding to $x_{n+1}^{(t)}$ for $t = 1, 2, 3, \ldots, 6$

t	Iteration $x_{n+1}^{(t)}$	$A_n^{(t)}$	$B_n^{(t)}$	$C_n^{(t)}$
1	$x_{n+1}^{(1)}$	$1-\alpha_n-\beta_n$	α_n	β_n
2	$x_{n+1}^{(2)}$	$1-\alpha_n-\beta_n$	β_n	α_n
3	$x_{n+1}^{(3)}$	α_n	$1-\alpha_n-\beta_n$	β_n
4	$x_{n+1}^{(4)}$	α_n	β_n	$1-\alpha_n-\beta_n$
5	$x_{n+1}^{(5)}$	β_n	$1-\alpha_n-\beta_n$	α_n
6	$x_{n+1}^{(6)}$	β_n	α_n	$1-\alpha_n-\beta_n$

Using Table 1 together with the bounds in (S4) and (S5) the admissible ranges of $\alpha^{(t)}$, $a^{(t)}$, $\beta^{(t)}$, $b^{(t)}$, $\gamma^{(t)}$, $c^{(t)}$ for all $t = 1, 2, 3, \ldots, 6$ are obtained as summarized in Table 2.

Table 2: Bounds for $(\alpha^{(t)}, a^{(t)})$, $(\beta^{(t)}, b^{(t)})$, $(\gamma^{(t)}, c^{(t)})$ from Conditions (S2)–(S5)

t	$A_n^{(t)}$-bounds $(\alpha^{(t)}, a^{(t)})$	$B_n^{(t)}$-bounds $(\beta^{(t)}, b^{(t)})$	$C_n^{(t)}$-bounds $(\gamma^{(t)}, c^{(t)})$
1	$(1-a'-b', 1-\alpha'-\beta')$	(α', a')	(β', b')
2	$(1-a'-b', 1-\alpha'-\beta')$	(β', b')	(α', a')
3	(α', a')	$(1-a'-b', 1-\alpha'-\beta')$	(β', b')
4	(α', a')	(β', b')	$(1-a'-b', 1-\alpha'-\beta')$
5	(β', b')	$(1-a'-b', 1-\alpha'-\beta')$	(α', a')
6	(β', b')	(α', a')	$(1-a'-b', 1-\alpha'-\beta')$

Let $k, l \in \{1, 2, 3, 4, 5, 6\}$ and $k \neq l$. From Theorem 4.1, $x_{n+1}^{(k)}$ faster than $x_{n+1}^{(l)}$, when $a^{(k)} + b^{(k)}\delta + c^{(k)}\delta^2 < \alpha^{(l)} - b^{(l)}\delta - c^{(l)}\delta^2$.

Table 3 provides a concise visual summary of the convergence rate conditions derived from Theorem 4.1. The table reveals a key principle: an iteration $\{x_n^{(k)}\}_{n=0}^{\infty}$, corresponding to a specific row, converges faster than all other iterations if its diagonal entry is less than all of its off-diagonal entries in that same row. For example, the first row demonstrates that the iteration $\{x_n^{(1)}\}_{n=0}^{\infty}$ is the fastest when its diagonal value (in the first column) is smaller than the values in every other column of that row.

Table 3: Summary of conditions for the Kirk order-2 iteration scheme, $\{x_n^{(k)}\}_{n=0}^{\infty}$ converges faster than $\{x_n^{(l)}\}_{n=0}^{\infty}$.

	$x_{n+1}^{(1)}$	$x_{n+1}^{(2)}$	$x_{n+1}^{(3)}$	$x_{n+1}^{(4)}$	$x_{n+1}^{(5)}$	$x_{n+1}^{(6)}$
$x_{n+1}^{(1)}$	$(1-\alpha'-\beta')+a'\delta+b'\delta^2$	$(1-a'-b')-b'\delta-a'\delta^2$	$\alpha'-(1-\alpha'-\beta')\delta-b'\delta^2$	$\alpha'-b'\delta-(1-\alpha'-\beta')\delta^2$	$\beta'-(1-\alpha'-\beta')\delta-a'\delta^2$	$\beta'-a'\delta-(1-\alpha'-\beta')\delta^2$
$x_{n+1}^{(2)}$	$(1-a'-b')-a'\delta-b'\delta^2$	$(1-\alpha'-\beta')+b'\delta+a'\delta^2$	$\alpha'-(1-\alpha'-\beta')\delta-b'\delta^2$	$\alpha'-b'\delta-(1-\alpha'-\beta')\delta^2$	$\beta'-(1-\alpha'-\beta')\delta-a'\delta^2$	$\beta'-a'\delta-(1-\alpha'-\beta')\delta^2$
$x_{n+1}^{(3)}$	$(1-a'-b')-a'\delta-b'\delta^2$	$(1-a'-b')-b'\delta-a'\delta^2$	$a'+(1-\alpha'-\beta')\delta+b'\delta^2$	$\alpha'-b'\delta-(1-\alpha'-\beta')\delta^2$	$\beta'-(1-\alpha'-\beta')\delta-a'\delta^2$	$\beta'-(1-\alpha'-\beta')\delta-a'\delta^2$
$x_{n+1}^{(4)}$	$(1-a'-b')-a'\delta-b'\delta^2$	$(1-a'-b')-b'\delta-a'\delta^2$	$\alpha'-(1-\alpha'-\beta')\delta-b'\delta^2$	$a'+b'\delta+(1-\alpha'-\beta')\delta^2$	$\beta'-(1-\alpha'-\beta')\delta-a'\delta^2$	$\alpha'-b'\delta-(1-\alpha'-\beta')\delta^2$
$x_{n+1}^{(5)}$	$(1-a'-b')-a'\delta-b'\delta^2$	$(1-a'-b')-b'\delta-a'\delta^2$	$\alpha'-(1-\alpha'-\beta')\delta-b'\delta^2$	$\alpha'-b'\delta-(1-\alpha'-\beta')\delta^2$	$b'+(1-\alpha'-\beta')\delta+a'\delta^2$	$\alpha'-b'\delta-(1-\alpha'-\beta')\delta^2$
$x_{n+1}^{(6)}$	$(1-a'-b')-a'\delta-b'\delta^2$	$(1-a'-b')-b'\delta-a'\delta^2$	$\alpha'-(1-\alpha'-\beta')\delta-b'\delta^2$	$\alpha'-b'\delta-(1-\alpha'-\beta')\delta^2$	$\beta'-(1-\alpha'-\beta')\delta-a'\delta^2$	$b'+a'\delta+(1-\alpha'-\beta')\delta^2$

5 Numerical experiments

We implemented the iteration (10) on representative numerical examples. These experiments validate the theoretical convergence and stability results, and will provide empirical evidence of the method's robustness and practical performance.

Example 5.1. To demonstrate the convergence result stated in Theorem 3.1, we consider a mapping

$$T : [0,1] \longrightarrow [0,1] \quad \text{defined by} \quad Tx = \frac{x}{2(x+1)}$$

for all $x \in [0,1]$. For any $x, y \in [0,1]$, we have

$$\begin{aligned} |Tx - Ty| &= \frac{1}{2}\left|\frac{x}{x+1} - \frac{y}{y+1}\right| \\ &\le \frac{1}{2}\frac{|x-y|}{(x+1)(y+1)} \\ &\le \frac{1}{2}\frac{|x-y|}{|x-y|+1} \\ &\le \frac{1}{2}|x-y| \\ &= \varphi(|x-Tx|) + \delta|x-y|, \end{aligned}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is defined by $\varphi(t) = 0$ for all $t \in [0,\infty)$ and $\delta = \frac{1}{2}$ for all $t \in [0,\infty)$. Then T, φ and δ satisfy (9) with respect to a usual normed space $(\mathbb{R}, |\cdot|)$. Furthermore, a unique fixed point of T is $p = 0$.

To empirically validate our theoretical results, we now present a series of numerical experiments designed to test the behavior of the iterative scheme (10). In these experiments, we set $\alpha_n = \alpha$ and $\beta_n = \beta$, where α, β are constants. Table 4 demonstrates that varying parameters α and β (under the constraint $\alpha + \beta = 0.9$) for a fixed initial value $x_0 = 0.9$ impacts the rate of convergence, with iterations ranging from 34 to 53; specifically, a smaller α relative to β accelerated convergence for the given operator in this context. Further emphasizing parameter sensitivity, Table 5 shows that permuting the roles of coefficients in the iteration (24) significantly alters efficiency, with iteration counts varying widely from 39 to 130 depending on the specific permutation. Importantly, all configurations in both Tables 4 and 5 achieved convergence to the fixed point with high precision, with errors consistently below 10^{-15}. In Tables 1 and 2, the computational time is not reported since the results are based on self-comparison, and therefore, it is sufficient to consider only the number of iterations.

In contrast to the significant influence of parameter choice and assignment, Table 6 highlights the scheme's robustness concerning the initial value x_0. When using fixed parameters $\alpha = 0.3$ and $\beta = 0.6$, varying x_0 across the interval $[0.1, 1.0]$ resulted in only minor variations in the number of iterations required for convergence (ranging from 37 to 39), with final errors consistently below 10^{-15} (see Figure 3). Collectively, these findings confirm that the iterative scheme (10) reliably converges to the unique fixed point $p = 0$ of T. While the convergence rate is evidently influenced by the specific selection and structural assignment of parameters α and β, the method exhibits stability with respect to the initial guess, aligning with the convergence results of Theorem 3.1.

Table 4: Convergence performance of the iteration (10) for $T(x) = \frac{x}{2(x+1)}$ with an initial seed $x_0 = 0.9$ and varying parameters α and β.

α	β	Iterations	$\|x_n - p\|$	$\|x_n - x_{n-1}\|$
0.10	0.80	34	3.832322009833200E-16	7.117169446833096E-16
0.20	0.70	36	5.172997667501350E-16	8.621662779168935E-16
0.30	0.60	39	3.116648568403815E-16	4.674972852605731E-16
0.40	0.50	41	5.533879174848491E-16	7.487013001265625E-16
0.50	0.40	44	4.863414371290738E-16	5.944173120466471E-16
0.60	0.30	47	5.215018566353107E-16	5.763967889127132E-16
0.70	0.20	50	6.738737128770300E-16	6.738737128770320E-16
0.80	0.10	53	1.037775113333616E-15	9.389393882542283E-16

Table 5: Effect of coefficient permutations on the convergence efficiency of the iteration schemes (10) for $T(x) = \frac{x}{2(x+1)}$ with $\alpha = 0.6, \beta = 0.3$ and $x_0 = 0.9$.

$x_n^{(k)}$	$1-\alpha-\beta$	α	β	Iterations	$\|x_n - p\|$
1	0.10	0.60	0.30	47	5.215018566353194E-16
2	0.10	0.30	0.60	39	3.116648568403848E-16
3	0.60	0.10	0.30	104	2.464091396349304E-16
4	0.60	0.30	0.10	130	2.734653386609106E-16
5	0.30	0.10	0.60	50	9.400485618894463E-16
6	0.30	0.60	0.10	72	1.406013442494477E-16

Table 6: Robustness of the iteration (10) to initial seeds x_0 for $T(x) = \frac{x}{2(x+1)}$ with $\alpha = 0.3$ and $\beta = 0.6$.

x_0	Iterations	$\|x_n - p\|$	$\|x_n - x_{n-1}\|$
0.10	37	4.097722210572873E-16	6.146583315859320E-16
0.20	38	2.910489509835331E-16	4.365734264753003E-16
0.30	38	3.941632577710987E-16	5.912448866566489E-16
0.40	38	4.806126772950470E-16	7.209190159425720E-16
0.50	38	5.549668685133309E-16	8.324503027699982E-16
0.60	38	6.202231753400581E-16	9.303347630100895E-16
0.70	39	2.713740043591565E-16	4.070610065387353E-16
0.80	39	2.924238698401458E-16	4.386358047602193E-16
0.90	39	3.116648568403832E-16	4.674972852605754E-16
1.00	39	3.294160669178576E-16	4.941241003767871E-16

6 Conclusion

Building upon classical scheme Kirk's order-k iteration (3), this paper introduces a Kirk order-2 iteration scheme (10) under the weak contraction (9), this method aims to enhance the efficiency and reliability of finding fixed points.

The theoretical investigations yielded several significant outcomes. The proposed Kirk order-2 iteration (10) was rigorously proven to converge strongly to a unique fixed point p. This holds when the mapping T satisfies a weak contractive condition (9). Notably, classic methods like Mann and Picard iterations can be derived as special cases of this scheme under certain parameter settings. Moreover, the method also demonstrates T-stability, a crucial property for iterative procedures. This means that if an arbitrary sequence generated by the iteration converges to the fixed point, it does so if and only if a corresponding error sequence converges to zero, affirming the method's reliability in the presence of small perturbations. Additionally, we demonstrate self-comparison of convergence rates among six possible permutations of the coefficients in the generalized iterative scheme (10). Theorem 4.1 provides precise conditions a clear criterion for the self-comparison of the Kirk order-2 iteration converges speed, allowing us to determine which of two different parameter sets will yield a faster solution.

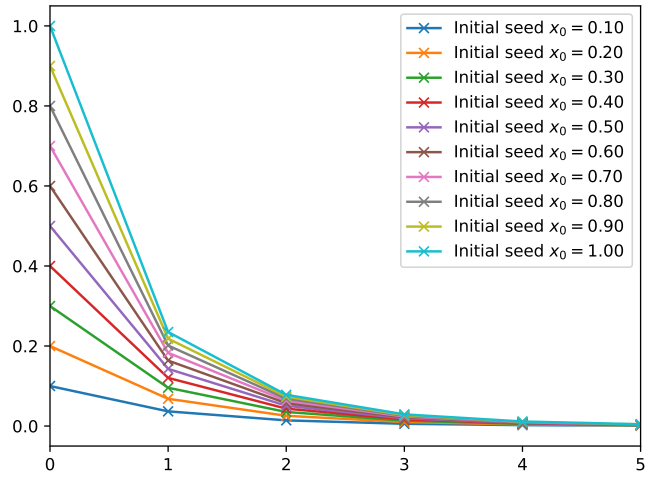

Figure 3: Effect of initial seeds on the convergence efficiency of the iteration schemes (10) for $T(x) = \dfrac{x}{2(x+1)}$ with $\alpha = 0.6, \beta = 0.3$ and $x_0 = 0.9$.

Furthermore, numerical experiments with two distinct examples confirmed the theoretical results and provided deeper insights into the algorithm's practical performance. The first example, using constant parameters α and β, a smaller α relative to β generally accelerated convergence for the tested operator. In contrast, the scheme (10) exhibited remarkable robustness concerning the initial value x_0, with minimal changes in iteration counts despite varying initial guesses. The second example extended this analysis by employing non-constant parameter sequences, directly testing the theoretical conditions of Theorem 3.1. The results compellingly showed that the choice of sequences is crucial, with the fastest convergence observed when both parameter sequences α_n and β_n were chosen from series that diverge, underscoring the practical implications of the theorem's convergence criteria. Across all configurations in both examples, the iterations (10) consistently achieved high precision since errors are below 10^{-15}.

The possible future works may include an investigation on the performance of the Kirk order-2 iteration with other classes of nonlinear mappings and in different mathematical spaces beyond the real Banach space studied here.

In conclusion, the proposed Kirk order-2 iteration (10) is a robust and efficient method for approximating fixed points. It reliably converges and exhibits desirable stability. While its convergence speed is notably influenced by the specific selection and structural assignment of its parameters, its robustness to the initial guess perfectly aligns with the theoretical convergence results. This research significantly contributes to the body of knowledge on fixed point iterative methods.

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