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On Semi Uniform Alpha Convergence and a Korovkin-Type Theorem

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Abstract

In this paper, we introduce the notion of semi uniform alpha convergence for sequences of functions between metric spaces. We also provide some results concerning the relationship between uniform exhaustiveness, semi uniform exhaustiveness and semi uniform alpha convergence. Finally, we give a Korovkin-type theorem related to uniform exhaustiveness.

1 Introduction and A Brief History

Over the years, various forms of convergence have been introduced and developed to address specific mathematical challenges , much like the rise of uniform convergence. One such concept is alpha convergence (also referred to as "continuous convergence") which was initially introduced by Courant [12] in 1914. Alpha convergence distinguishes itself from other types of convergence due to its property of ensuring the continuity of the limit function, even when the terms of the function sequence are not continuous themselves. Numerous studies, including several influential ones, have been conducted in this area [8, 22, 23].

In recent decades, the alpha convergence has received renewed attention. In 2003, Das and Papanastassiou [13] introduced several new convergence concepts for sequences of real-valued functions namely, alpha uniform equal convergence, alpha strong uniform equal convergence and alpha equal convergence. They also provided a characterization of these notions within the framework of compact metric spaces.

Gregoriades and Papanastassiou [18] introduced the concept of exhaustiveness, which is closely related with equicontinuity, and established a connection between alpha convergence and pointwise convergence through this new notion. In 2011, Caserta et al. [10] introduced the statistical analogues of classical convergence concepts such as Dini, Arzelà, and Alexandroff convergence. They also introduced statistical versions of strong uniform convergence and strong exhaustiveness. In [11] the statistical versions of notions of exhaustiveness alpha convergence were introduced by Caserta and Kočinac.

In 2020, Papanastassiou [21] introduced the "semi" notion for sequences of functions and introduced notions such as semi alpha convergence, semi uniform convergence and notion of semi exhaustiveness. In 2022, Das and Ghosh [14] introduced statistical analogues of certain convergence concepts for sequences of functions between metric spaces namely statistical versions of semi alpha convergence, semi exhaustiveness, and semi uniform convergence.

In 2024, Erdem and Tunç [15] analyzed several types of convergences and the relationships among them, depending on the elements and the domains of the function sequences. Also in 2025, Erdem and Tunç [16] introduced the notions of Cauchy alpha convergence and uniform alpha convergence for function sequences between metric spaces. They investigated their properties based on the structure of the functions and their domains.

In recent years, there has been a growing interest in the construction and generalization of Korovkin-type approximation theorems and positive linear operators. Various modifications and extensions of classical Korovkin theorem have been proposed to handle broader classes of functions and convergence modes. Several recent studies have introduced new types of operators and investigated their approximation properties under different settings, highlighting the flexibility and efficiency of such operators in modern approximation theory. These approaches allow the study of a wider class of operators and offer new insights into Korovkin-type theorems, see [3, 4, 5, 6, 7, 16, 17, 20].

In this study, we introduce the notion of semi uniform alpha convergence for sequences of functions between metric spaces. We also provide some results concerning the relationship between uniform exhaustiveness and semi uniform exhaustiveness. Moreover, we establish several connections involving semi uniform alpha convergence. Finally, we present a Korovkin-type theorem related to uniform exhaustiveness.

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2 Definitions and Auxiliary Results

To enhance readability, in this section, we recall some basic definitions and notations. Throughout the paper, we write $X = (X, d)$ and $Y = (Y, \rho)$ to denote metric spaces unless or otherwise mentioned. By $B(x_0, \delta) := B_d(x_0, \delta)$ denotes the open ball with center x_0 and radius $\delta > 0$ with respect to metric d. The set of all Y-valued functions defined on X is denoted by Y^X:

$$Y^X = \{f \mid f : X \to Y \text{ is a function.}\}$$

The family of continuous and uniformly continuous functions defined on X to Y are denoted by $C(Y^X)$ and $UC(Y^X)$, respectively. It is well known that $UC(Y^X) \subset C(Y^X)$. For simplicity, if the domain of the function is known, we write $C(Y^X)$ and $UC(Y^X)$ simply as C and UC, respectively. This convention is applied consistently to all related notations throughout the paper.

The family of sequences of functions and the family of sequence of continuous functions defined over X to Y is denoted by $\mathfrak{sf}(Y^X)$ and $\mathfrak{scf}(Y^X)$, respectively. For simplicity, if the domain of functions is known, $\mathfrak{sf}(Y^X)$ and $\mathfrak{sf}(Y^X)$ are abbreviated to $\mathfrak{sf}$ and $\mathfrak{scf}$, respectively :

$$\mathfrak{sf} := \mathfrak{sf}(Y^X) = \left\{(f_n) \mid \forall n \in \mathbb{N},\, f_n \in Y^X\right\}, \qquad \mathfrak{scf} := \mathfrak{scf}(Y^X) = \left\{(f_n) \mid \forall n \in \mathbb{N},\, f_n \in C(Y^X)\right\}.$$

We will use the following notations to describe various properties of function sequences. As $\mathfrak{c}_{\mathfrak{sf}} := \mathfrak{c}_{\mathfrak{sf}}(Y^X)$, it denotes the set of all pointwise convergent sequences of functions on X to Y , while $\mathfrak{c}^u_{\mathfrak{sf}} := \mathfrak{c}^u_{\mathfrak{sf}}(Y^X)$ represents sequences that are uniformly convergent. The subscript $\mathfrak{scf}$ denotes that the sequences consist of continuous functions, applied consistently across all related notations throughout this paper. We use the standard notation "$f_n \to f$" for the pointwise convergence and "$f_n \rightrightarrows f$" for the uniform convergence of the sequence (f_n) to f.

Definition 2.1. [12] The sequence $(f_n) \in \mathfrak{sf}(Y^X)$ *alpha convergent* to $f \in Y^X$ on X and denoted by $f_n \to_\alpha f$, if for every $x \in X$ and for every sequence (x_n) of points of X converging to x, the sequence $(f_n(x_n))$ convergent to $f(x)$.

The set of alpha convergent sequences of functions on X to Y will be denoted by $\mathfrak{c}^\alpha_{\mathfrak{sf}}(Y^X)$:

$$\mathfrak{c}^\alpha_{\mathfrak{sf}} := \mathfrak{c}^\alpha_{\mathfrak{sf}}(Y^X) = \left\{(f_n) \in \mathfrak{sf} \mid \exists f \in Y^X : f_n \to_\alpha f\right\}.$$

It is proved in [2] that the alpha convergence of the sequence (f_n) at $x_0 \in X$ to f is equivalent with the following condition:

$$\forall \varepsilon > 0, \exists \delta = \delta(x_0, \varepsilon) > 0, \exists n_0 = n_0(x_0, \varepsilon) \in \mathbb{N} : \forall x \in B_d(x_0, \delta),\, \forall n \geq n_0 \implies \rho(f_n(x), f(x_0)) < \varepsilon.$$

Definition 2.2. [21] The sequence $(f_n) \in \mathfrak{sf}(Y^X)$ is called *semi alpha convergent* to $f \in Y^X$ at $x_0 \in X$ if

(i) $f_n(x_0) \to f(x_0)$

(ii) For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, x_0) > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $m \geq n$ such that for all $x \in B(x_0, \delta)$ we have $\rho(f_m(x), f(x_0)) < \varepsilon$

hold.

If (f_n) is semi alpha convergent to f for all $x \in X$ we say that (f_n) is semi alpha convergent to f on X and denoted by $f_n \to_{s\alpha} f$. The set of semi alpha convergent sequences of functions on X to Y will be denoted by $\mathfrak{c}^{s\alpha}_{\mathfrak{sf}}(Y^X)$:

$$\mathfrak{c}^{s\alpha}_{\mathfrak{sf}} := \mathfrak{c}^{s\alpha}_{\mathfrak{sf}}(Y^X) = \left\{(f_n) \in \mathfrak{sf} \mid \exists f \in Y^X : f_n \to_{s\alpha} f\right\}.$$

Remark 1. Semi alpha convergence lies between alpha convergence and pointwise convergence. Therefore, there exist examples such that

(i) A function sequence is semi alpha convergent but not alpha convergent.

(ii) A function sequence is pointwise convergent but not semi alpha convergent.

One may refer to Example 3.4 in [21], that is

(i) For all $n \in \mathbb{N}$, $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} \frac{1}{n}, & n \text{ is odd} \\ 0, & n \text{ is even and } x \in \left(-\infty, -\frac{1}{n}\right) \cup (0, \infty) \\ 2nx + 2, & n \text{ is even and } x \in \left[-\frac{1}{n}, -\frac{1}{2n}\right] \\ -2nx, & n \text{ is even and } x \in \left(-\frac{1}{2n}, 0\right] \end{cases}$$

which is semi alpha convergent but not alpha convergent.

(ii) For all $n \in \mathbb{N}$, $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} 0, & x \in \left(-\infty, -\frac{1}{n}\right) \cup \left(\frac{1}{n}, \infty\right) \\ 1, & x \in \left[-\frac{1}{n}, \frac{1}{n}\right] \end{cases}$$

which is pointwise convergent but not semi alpha convergent.

Definition 2.3. [16] The sequence $(f_n) \in \mathfrak{sf}(Y^X)$ is called *uniform alpha convergent* to the function $f \in Y^X$ on X and denoted by $f_n \rightrightarrows_\alpha f$, if for every sequence (x_n) and (y_n) of points of X with $d(x_n, y_n) \to 0$, the sequence $\rho(f_n(x_n), f(y_n)) \to 0$.

We will use the notation $\mathfrak{c}^{u\alpha}_{\mathfrak{sf}}(Y^X)$ to denote set of all uniform alpha convergent sequences of functions in Y^X:

$$\mathfrak{c}^{u\alpha}_{\mathfrak{sf}} := \mathfrak{c}^{u\alpha}_{\mathfrak{sf}}(Y^X) = \left\{(f_n) \in \mathfrak{sf} \mid \exists f \in Y^X : f_n \rightrightarrows_\alpha f\right\}.$$

It is proved in [16] that the uniform alpha convergence of the sequence (f_n) to f on X is equivalent with the following condition:

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, \exists n_0 = n_0(\varepsilon) \in \mathbb{N} : \forall x, y \in X \ (d(x, y) < \delta), \forall n \geq n_0 \implies \rho(f_n(x), f(y)) < \varepsilon.$

Definition 2.4. [16] The sequence $(f_n) \in \mathfrak{sf}(Y^X)$ is called *Cauchy alpha convergent* to the function $f \in Y^X$ on X and denoted by $f_n \to_{c\alpha} f$, if for every Cauchy sequence (x_n) and (y_n) of points of X with $d(x_n, y_n) \to 0$, the sequence $\rho(f_n(x_n), f(y_n)) \to 0$.

We will use the notation $\mathfrak{c}^{c\alpha}_{\mathfrak{sf}}(Y^X)$ to denote set of all Cauchy alpha convergent sequences of functions in Y^X:

$$\mathfrak{c}^{c\alpha}_{\mathfrak{sf}} := \mathfrak{c}^{c\alpha}_{\mathfrak{sf}}(Y^X) = \left\{(f_n) \in \mathfrak{sf} \mid \exists f \in Y^X : f_n \to_{c\alpha} f\right\}.$$

Here are some facts related with alpha and uniform alpha convergence from [16, 18, 21].

Lemma 2.1. *Let $(f_n) \in \mathfrak{sf}(Y^X)$ and $f \in Y^X$ be given.*

(i) *If X is compact set and $f_n \to_\alpha f$, then $f_n \rightrightarrows f$.*

(ii) *If $f_n \to_\alpha f$, then $f_n \to_{s\alpha} f$.*

(iii) *If $f_n \rightrightarrows f$ and $f \in C$, then $f_n \to_\alpha f$.*

(iv) *If $f_n \rightrightarrows_\alpha f$, then $f \in UC$.*

(v) *If $f_n \rightrightarrows_\alpha f$, then $f_n \to_\alpha f$.*

(vi) *If $f_n \rightrightarrows_\alpha f$, then $f_n \rightrightarrows f$.*

(vii) *If $f_n \rightrightarrows f$ and $f \in UC$, then $f_n \rightrightarrows_\alpha f$.*

(viii) *If $f_n \in UC$ for all $n \in \mathbb{N}$ and $f_n \rightrightarrows f$, then $f_n \rightrightarrows_\alpha f$.*

(ix) *If X is compact set, then $\mathfrak{c}^{\alpha}_{\mathfrak{sf}} = \mathfrak{c}^{c\alpha}_{\mathfrak{sf}} = \mathfrak{c}^{u\alpha}_{\mathfrak{sf}}$ and $\mathfrak{c}^{\alpha}_{\mathfrak{scf}} = \mathfrak{c}^{c\alpha}_{\mathfrak{scf}} = \mathfrak{c}^{u}_{\mathfrak{scf}} = \mathfrak{c}^{u\alpha}_{\mathfrak{scf}}$.*

Definition 2.5. [9] The function sequence $(f_n) \in \mathfrak{scf}(Y^X)$ is called *uniformly equicontinuous* on X if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$ and for all $x, y \in X$ with $d(x, y) < \delta$ we have $\rho(f_n(x), f_n(y)) < \varepsilon$.

Definition 2.6. [18] The function sequence $(f_n) \in \mathfrak{sf}(Y^X)$ is called *(pointwise) exhaustive* at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x_0) > 0$ and $n_0 = n_0(\varepsilon, x_0) \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x \in B_d(x_0, \delta)$ we have $\rho(f_n(x), f_n(x_0)) < \varepsilon$.

The set of exhaustive sequences of functions on X to Y will be denoted by $\mathfrak{e}_{\mathfrak{sf}} := \mathfrak{e}_{\mathfrak{sf}}(Y^X)$.

3 Main Results

3.1 A New Type of Convergence

In this section, we define a new mode of convergence that guarantees the limit function is uniformly continuous and is weaker than uniform alpha convergence.

Definition 3.1. The sequence $(f_n) \in \mathfrak{sf}(Y^X)$ is called *semi uniform alpha convergent* to $f \in Y^X$ on X and denoted by $f_n \rightrightarrows_{s\alpha} f$ if

(i) $f_n \to f$

(ii) For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $m \geq n$ such that for all $x, y \in X$ with $d(x, y) < \delta$ we have $\rho(f_m(x), f(y)) < \varepsilon$

hold.

The set of semi uniform alpha convergent sequences of functions on X to Y will be denoted by $\mathfrak{c}^{su\alpha}_{\mathfrak{sf}}(Y^X)$:

$$\mathfrak{c}^{su\alpha}_{\mathfrak{sf}} := \mathfrak{c}^{su\alpha}_{\mathfrak{sf}}(Y^X) = \left\{(f_n) \in \mathfrak{sf} \mid \exists f \in Y^X : f_n \rightrightarrows_{s\alpha} f\right\}.$$

If (f_n) satisfies the condition (ii) in Definition 3.1, then we call (f_n) has the semi uniform alpha property with respect to the function f.

Remark 2. Even if (f_n) has semi uniform alpha property with respect to the function $f \in Y^X$, it may not be semi uniform alpha convergent to f. Indeed, let for all $n \in \mathbb{N}$, the functions $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} 1, & n \text{ even} \\ \frac{1}{n}, & n \text{ odd} \end{cases}$$

are given. The sequence (f_n) has the semi uniform alpha property with respect to the function $f \equiv 0$ but it lacks semi uniform alpha convergence to f.

From their definitions, it is clear that the uniform alpha convergence is stronger than the semi uniform alpha convergence:

Proposition 3.1. *If $f_n \rightrightarrows_\alpha f$, then $f_n \rightrightarrows_{s\alpha} f$ that is $\mathfrak{c}^{u\alpha}_{\mathfrak{sf}} \subset \mathfrak{c}^{su\alpha}_{\mathfrak{sf}}$.*

Just as in pointwise and uniform convergence, there is the following relationship between semi uniform alpha and semi alpha convergence:

Proposition 3.2. *If $f_n \rightrightarrows_{s\alpha} f$, then $f_n \to_{s\alpha} f$ that is $\mathfrak{c}^{su\alpha}_{\mathfrak{sf}} \subset \mathfrak{c}^{s\alpha}_{\mathfrak{sf}}$.*

Just like uniform alpha convergence, semi uniform alpha convergence ensures the uniform continuity of the limit function. More specifically, the semi uniform alpha property guarantees uniform continuity of related function.

Proposition 3.3. *If $(f_n) \in \mathfrak{sf}(Y^X)$ has the semi uniform alpha property with respect to the function $f \in Y^X$, then $f \in UC$.*

Proof. Let $\varepsilon > 0$ be given. Assume that (f_n) has semi uniform alpha property with respect to the function f, that is, there exists $\delta = \delta(\varepsilon) > 0$ such that for every $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ with $m \geq n$ such that for every $x, y \in X$ with $d(x, y) < \delta$ we have $\rho\,(f_m(x), f(y)) < \varepsilon/2$. Then every $x, y \in X$ with $d(x, y) < \delta$ implies $\rho\,(f(x), f(y)) \leq \rho\,(f(x), f_m(x)) + \rho\,(f_m(x), f(y)) \leq \varepsilon$. □

Corollary 3.4. *If $(f_n) \in \mathfrak{sf}(Y^X)$ is semi uniform alpha convergent to $f \in Y^X$, then $f \in UC$.*

Remark 3. From Definition 3.1, one can see semi uniform alpha convergence implies semi alpha convergence while reverse implication may fail. Indeed, for every $n \in \mathbb{N}$, the functions $f_n : (0, 1) \to \mathbb{R}$, $f_n(x) = \frac{1}{x}$ are given. $f_n \to_{s\alpha} \frac{1}{x}$ while $f_n \not\rightrightarrows_{s\alpha} \frac{1}{x}$.

Remark 4. Semi uniform alpha convergence lies between pointwise convergence and uniform alpha convergence. So there exist examples such that

(i) A function sequence is semi uniform alpha convergent but not uniform alpha convergent.

(ii) A function sequence is pointwise convergent but not semi uniform alpha convergent.

For (i) let for all $n \in \mathbb{N}$, the functions $f_n : \mathbb{R} \to \mathbb{R}$, If n is odd, $f_n(x) = \frac{x}{n}$ and if n is even, $f_n(x) = 0$ are given. It is clear that $f_n \rightrightarrows_{s\alpha} 0$ but $f_n \not\rightrightarrows_\alpha 0$ because $f_n \not\rightrightarrows 0$.

For (ii) one check the example in Remark 1 (ii) which is pointwise convergent but not semi uniform alpha convergent, as it fails to be semi alpha convergent.

Proposition 3.5. *If $(f_n) \in \mathfrak{sf}(Y^X)$ has semi uniform alpha property with respect to the function $f \in Y^X$ and $f_n \rightrightarrows f$, then $f_n \rightrightarrows_\alpha f$.*

Proof. Let $\varepsilon > 0$ be given. From uniform convergence of (f_n) to f there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x \in X$ we have $\rho\,(f_n(x), f(x)) < \varepsilon/3$. From semi uniform alpha property, there exists $\delta = \delta(\varepsilon) > 0$ such that for $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $m \geq n_0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ we have $\rho\,(f_m(x), f(y)) < \varepsilon/3$. Now for all $n \geq n_0$ and for all $x, y \in X$ with $d(x, y) < \delta$ we have $\rho\,(f_n(x), f(y)) \leq \rho\,(f_n(x), f(x)) + \rho\,(f_m(y), f(x)) + \rho\,(f_m(y), f(y)) < \varepsilon$. □

The diagram in Figure 1 illustrates the relations between semi uniform alpha convergence and some types of convergences on any set X.

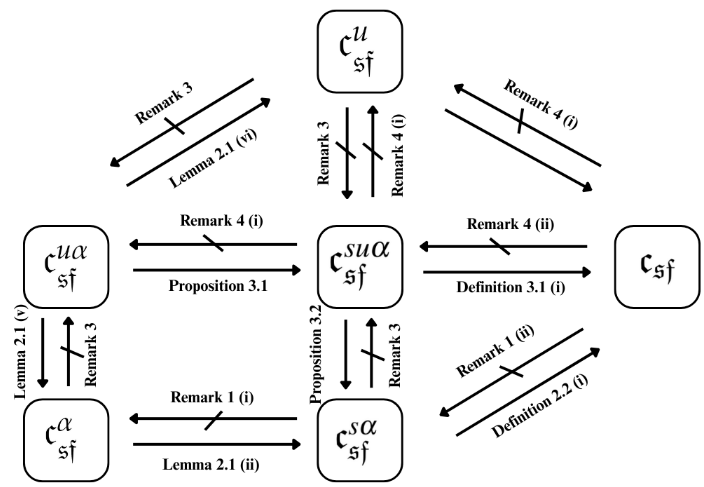

Figure 1: Relations between some types of convergences

Definition 3.2. [14] The sequence $(f_n) \in \mathfrak{sf}(Y^X)$ is called *uniformly exhaustive* on X if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n_0$ and for all points $x, y \in X$ with $d(x, y) < \delta$ we have $\rho(f_n(x), f_n(y)) < \varepsilon$.

The set of uniform exhaustive sequences of functions on X to Y will be denoted by $\mathfrak{e}^u_{\mathfrak{sf}} := \mathfrak{e}^u_{\mathfrak{sf}}(Y^X)$.

Remark 5. From Definition 3.2, one can see uniformly exhaustiveness implies exhaustiveness while reverse implication is may fail. For example, the function sequence (f_n) in Remark 3 is exhaustive while not uniformly exhaustive. Indeed, δ cannot be chosen depending only on ε. Let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$ with $|x_n - y_n| \to 0$ while $|f_n(x_n) - f_n(y_n)| = |n + 1 - n| \to 1 \neq 0$.

Proposition 3.6. *The function sequence $(f_n) \in \mathfrak{sf}(Y^X)$ is uniformly equicontinuous if and only if for all $n \in \mathbb{N}$, $f_n \in UC$ and $(f_n) \in \mathfrak{e}^u_{\mathfrak{sf}}$.*

Proof. ($\Longrightarrow$) Assume that (f_n) is a uniformly equicontinuous function sequence. From the definition for every $n \in \mathbb{N}$, we have $f_n \in UC$ and $(f_n) \in \mathfrak{e}^u_{\mathfrak{sf}}$.

($\Longleftarrow$) On the other hand let $\varepsilon > 0$ be given. By uniform exhaustiveness of (f_n), there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $\delta_0 = \delta_0(\varepsilon) > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta_0$ and for all $n \geq n_0$ we have $\rho(f_n(x), f_n(y)) < \varepsilon$. Moreover for all $n \in \mathbb{N}$, the functions $f_n \in UC$ then there exists $\delta_n = \delta_n(\varepsilon) > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta_n$ we have $\rho(f_n(x), f_n(y)) < \varepsilon$. One can choose $\delta = \min\left\{\delta_0, \delta_1, \cdots, \delta_{n_0 - 1}\right\}$ then for all $x, y \in X$ with $d(x, y) < \delta$ and for all $n \in \mathbb{N}$ we have $\rho(f_n(x), f_n(y)) < \varepsilon$. □

Proposition 3.7. *If $f_n \to f$ and $(f_n) \in \mathfrak{e}^u_{\mathfrak{sf}}(Y^X)$ then $f \in UC$.*

Proof. Let $\varepsilon > 0$ be given. From $f_n \to f$ there exists $n_x \in \mathbb{N}$ such that for all $n \geq n_x$ we have $\rho(f_n(x), f(x)) < \varepsilon/3$. Similarly, there exists $n_y \in \mathbb{N}$ such that for all $n \geq n_y$ we have $\rho(f_n(y), f(y)) < \varepsilon/3$. Also from $(f_n) \in \mathfrak{e}^u_{\mathfrak{sf}}$ there exists $n_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \geq n_0$ and for all $x, y \in X$ with $d(x, y) < \delta$ we have $\rho(f_n(x), f_n(y)) < \varepsilon/3$.From here, for all $x, y \in X$ with $d(x, y) < \delta$ and for all $n \geq \max\{n_0, n_x, n_y\}$ we have

$$\rho(f(x), f(y)) \leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) + \rho(f_n(y), f(y)) < \varepsilon$$

which completes the proof. □

Proposition 3.8. *$f_n \rightrightarrows_\alpha f$ if and only if $f_n \rightrightarrows f$ and $(f_n) \in \mathfrak{e}^u_{\mathfrak{sf}}(Y^X)$.*

Proof. Let $\varepsilon > 0$ be given. Assume that $f_n \rightrightarrows_\alpha f$. From Lemma 2.1 (vi) we have $f_n \rightrightarrows f$. From uniform alpha convergence there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $\delta = \delta(\varepsilon) > 0$ such that for all $n \geq n_0$ and for all $x, y \in X$ with $d(x, y) < \delta$ we have $\rho(f_n(x), f(y)) < \varepsilon/2$. Then for all $n \geq n_0$ for all $x, y \in X$ with $d(x, y) < \delta$ we have

$$\rho(f_n(x), f_n(y)) \leq \rho(f_n(x), f(y)) + \rho(f(y), f_n(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

that is $(f_n) \in \mathfrak{e}^u(Y^X)$.

On the other hand from $(f_n) \in \mathfrak{e}^u_{\mathfrak{sf}}$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ and for all $n \geq n_0$, we have $\rho(f_n(x), f_n(y)) < \varepsilon/2$. Moreover from $f_n \rightrightarrows f$, there exists $n_1 = n_1(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n_1$ and for all $x \in X$ we have $\rho(f_n(x), f(x)) < \varepsilon/2$. Therefore for all $n \geq n_2 = \max\{n_0, n_1\}$ and for all $x, y \in X$ with $d(x, y) < \delta$ we have

$$\rho(f_n(x), f(y)) < \rho(f_n(x), f_n(y)) + \rho(f_n(y), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which is desired. □

Remark 6. Uniform convergence of (f_n) to f in Proposition 3.8 can not be replaced with pointwise convergence or alpha convergence. Indeed for every $n \in \mathbb{N}$, the functions $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{n}$ are given. Its clear that $(f_n) \in \mathfrak{e}^u_{\mathfrak{sf}}$ and $f_n \to 0$ ($f_n \to_\alpha 0$) while $f_n \not\rightrightarrows_\alpha 0$ because of $f_n \not\rightrightarrows 0$.

Definition 3.3. [14] The sequence $(f_n) \in \mathfrak{sf}(Y^X)$ is called *semi uniform exhaustive* on X if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $m \geq n$ such that for all $x, y \in X$ with $d(x, y) < \delta$ we have $\rho(f_m(x), f_m(y)) < \varepsilon$.

The set of semi uniform exhaustive sequences of functions on X to Y will be denoted by $\mathfrak{e}^{su}_{\mathfrak{sf}} := \mathfrak{e}^{su}_{\mathfrak{sf}}(Y^X)$.

Remark 7. By definition 3.3, uniform exhaustiveness implies semi uniform exhaustiveness. But reverse implication may fail. Let for all $n \in \mathbb{N}$, the functions $f_n : (0,1) \to \mathbb{R}$,

$$f_n(x) = \begin{cases} \frac{1}{x}, & n \, \text{even} \\ 0, & n \, \text{odd} \end{cases}$$

are given. It is clear that (f_n) is not uniform exhaustive while (f_n) is semi uniform exhaustive.

Proposition 3.9. *Let $(f_n) \in \mathfrak{sf}(Y^X)$ be given. If (f_n) has semi uniform alpha property, then $(f_n) \in \mathfrak{e}_{\mathfrak{sf}}^{su}$.*

Proof. Let $\varepsilon > 0$ be given and assume that (f_n) has semi uniform alpha property with respect to a function $f \in Y^X$, that is there exists $\delta = \delta(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $m \geq n$ such that for all $x, y \in X$ with $d(x,y) < \delta$ we have $\rho(f_m(x), f(y)) < \varepsilon/2$. Then for every $n \in \mathbb{N}$ one can choose the same $m \in \mathbb{N}$ with $m \geq n$ before and for every $x, y \in X$ with $d(x,y) < \delta$ we have

$$\rho(f_m(x), f_m(y)) \leq (f_m(x), f(y)) + (f(y), f_m(y)) < \varepsilon$$

which completes the proof. □

Proposition 3.10. *If $(f_n) \in \mathfrak{sf}(Y^X)$ is semi uniform exhaustive and $f_n \rightrightarrows f$, then $f \in UC$.*

Proof. Let $\varepsilon > 0$ be given. From uniform convergence of (f_n) to f there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x \in X$ we have $\rho(f_n(x), f(x)) < \varepsilon/3$. By semi uniform exhaustiveness of (f_n), there exists $\delta = \delta(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $m \geq n$ such that for all $x, y \in X$ with $d(x,y) < \delta$ we have $\rho(f_m(x), f_m(y)) < \varepsilon/3$. From here, for all $n \geq n_0$ and for all $x, y \in X$ with $d(x,y) < \delta$ we have

$$\rho(f(x), f(y)) \leq \rho(f(x), f_m(x)) + \rho(f_m(x), f_m(y)) + \rho(f_m(y), f(y)) < \varepsilon$$

that is $f \in UC$. □

3.2 A Korovkin-Type Theorem

We will be concerned with positive and linear operators defined $C_b(X)$. The positivity of an operator L is understood as the condition that for every positive function f, the function $L(f)$ is also positive. Let $e_k(t) = t^k$, where $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $t \in \mathbb{R}$. For $X = [a, b]$, we present the Korovkin's Theorem to discuss an approximation property of positive and linear operator sequences on $C_b(X)$:

Theorem 3.11. *[19] Let $X = [a, b]$ and (L_n) be a sequence of positive linear operators on $C(X)$. If $L_n(e_k) \rightrightarrows e_k$ for $k = 0, 1, 2$ then $L_n(f) \rightrightarrows f$ for all $f \in C(X)$.*

We now state the Korovkin-type theorem for uniform exhaustiveness for metric spaces with using similar method in [1].

Theorem 3.12. *Let (L_n) be a sequence of positive linear operators on $C_b(X)$. If $(L_n(e_0))$ is uniformly exhaustive and bounded on X then $(L_n(f))$ is uniformly exhaustive on X for all $f \in UC_b(X)$.*

Proof. Let $f \in UC_b(X)$ and $\varepsilon > 0$ be given. From boundedness of f let define $M_1 := \sup_{x \in X} |f(x)|$. By uniform exhaustiveness of $(L_n(e_0))$ on X, there exists $\delta_0 = \delta(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for all $x, y \in X$ with $d(x,y) < \delta_0$ and for all $n \geq n_0$ we have

$$|L_n(e_0; x) - L_n(e_0; y)| < \frac{\varepsilon}{3(M_1 + 1)} := A_1(\varepsilon).$$

From boundedness of the sequence $(L_n(e_0; x))$ there exists $M_2 > 0$ such that $L_n(e_0; x) \leq M_2$.
By uniform continuity of f on X there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that for all $x, y \in X$ with $d(x,y) < \delta_1$ we have

$$|f(x) - f(y)| < \frac{\varepsilon}{3(A_1(\varepsilon) + M_2)} := A_2(\varepsilon).$$

From properties of positive linear operators we have

$$L_n(|f - f(y)|; x) < A_2(\varepsilon) |L_n(e_0; x) - L_n(e_0; y)| + A_2(\varepsilon) |L_n(e_0; y)|.$$

Now if we choose $\delta = \min\{\delta_0, \delta_1\}$ then for all $x, y \in X$ with $d(x, y) < \delta$ and $n \geq n_0$ we have

$$\begin{aligned} |L_n(f;x) - L_n(f;y)| &\leq |L_n(f;x) - L_n(f(y);x)| + |L_n(f(y);x) - L_n(f(y);y)| + |L_n(f(y);y) - L_n(f;y)| \\ &\leq L_n(|f - f(y)|;x) + |f(y)| \cdot |L_n(e_0;x) - L_n(e_0;y)| + L_n(|f - f(y)|;y) \\ &< 2A_2(\varepsilon)(A_1(\varepsilon) + M_2) + M_1 \cdot A_1(\varepsilon) \\ &< \varepsilon, \end{aligned}$$

that is $(L_n(f))$ is uniformly exhaustive. □

4 Conclusion

In this paper, we introduced the concept of semi uniform alpha convergence for sequences of functions between metric spaces. We demonstrated that this mode of convergence, although weaker than uniform alpha convergence, still guarantees the uniform continuity of the limit function. We also gave several relationships between semi uniform alpha convergence and existing convergence notions, including semi alpha convergence, uniform exhaustiveness, and semi uniform exhaustiveness. Additionally, we proved a Korovkin-type approximation theorem for uniformly exhaustive operator sequences, which generalizes the classical Korovkin theorem to settings involving semi-uniform behavior. One advantage of the Korovkin-type theorem for uniform exhaustiveness, similar to exhaustiveness, is that one can use only one test function instead of three.

For future work, one may consider statistical and ideal versions of semi uniform alpha convergence and exhaustiveness and their relations in different types of spaces.

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References

[1] F. Altomare. *Korovkin-type theorems and local approximation problems*, Expositiones Mathematicae, 40(4):1229-1243, 2022.

[2] E. Athanassiadou, C. Papachristodoulos and N. Papanastassiou. *α and hyper α-convergence in function spaces*, Questions and Answers in General Topology, 33:1–16, 2015.

[3] A. Aral. *Weighted approximation: Korovkin and quantitative type theorems*. Modern Mathematical Methods, 1(1): 1–21, 2023.

[4] A. Aral, M.L. Limmam and F. Ozsarac. *Approximation properties of Szász-Mirakyan-Kantorovich type operators*. Mathematical Methods in the Applied Sciences, 42(16): 5233–5240, 2019.

[5] A. Aral, T. Acar and F. Ozsarac. *Differentiated Bernstein type operators*. Dolomites Research Notes on Approximation, 13(1): 47–54, 2020.

[6] J. Bustamante. *Approximation of bounded functions by positive linear operators*. Modern Mathematical Methods, 2(3): 117–131, 2024.

[7] M. Campiti *On the Korovkin-type approximation of set-valued continuous functions*. Constructive Mathematical Analysis, 4(1): 119–134, 2021.

[8] C. Carathéodory. Stetige konvergenz und normale Familien von Funktionen. Mathematische Annalen, 101(1): 515-533, 1929.

[9] N. L. Carothers. Real analysis, Cambridge University Press, 2000.

[10] A. Caserta, A., G. Di Maio and L. D. Kočinac. Statistical convergence in function spaces. In Abstract and Applied Analysis. 1:420419, 2011.

[11] A. Caserta and L. D. Kočinac. On statistical exhaustiveness. Applied Mathematics Letters, 25(10): 1447-1451, 2012.

[12] R. Courant. Ueber eine Eigenschaft der Abbildungsfunktioen bei konformer Abbildung. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 101-109, 1914.

[13] R. Das and N. Papanastassiou. Some types of convergence of sequences of real valued functions, Real An. Exch., 29(1):43–58, 2004.

[14] S. Das and A. Ghosh. A Study on statistical versions of convergence of sequences of functions. Mathematica Slovaca, 72(2):443-458, 2022.

[15] A. Erdem, T. Tunç. Some Convergence Types of Function Sequences and Their Relations, Journal of Universal Mathematics, DOI:10.33773/jum.1504000, 2025.

[16] A. Erdem and T. Tunç. On Cauchy Alpha and Uniform Alpha Convergence, Filomat, 39(20):xx–xx (accepted), 2025.

[17] Z. Finta. *King operators which preserve x^j*. Constructive Mathematical Analysis, 6(2): 90-101, 2023.

[18] V. Gregoriades and N. Papanastassiou. The notion of exhaustiveness and Ascoli-type theorems, Top. and its App., 155(10):1111–1128, 2008.

[19] P. P. Korovkin. Linear Operators and Approximation Theory, Hindustan Publishing Corp., 1960.

[20] I. Mantellini, A. Boccuto and C. Bardaro. A survey on recent results in Korovkin's approximation theory in modular spaces. Constructive Mathematical Analysis, 4(1): 48–60, 2021.

[21] N. Papanastassiou. A note on convergence of sequences of functions. Topology and its Applications, 275, 107017, 2020.

[22] H. Poppe. Stetige Konvergenz und der Satz von Ascoli und Arzelá . V. Proceedings of the Japan Academy, 44(5):318-321, 1968.

[23] H. Schaefer. Stetige Konvergenz in allgemeinen topologischen Räumen. Archiv der Mathematik, 6(6):423-427, 1955.