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On the Mellin-Gauss-Weierstrass operators preserving logarithmic functions in the weighted Mellin-Lebesgue spaces

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Abstract

In the present paper, we state a quantitative version of the convergence utilizing the logarithmic weighted modulus of continuity for a new generalization of the Mellin-Gauss-Weierstrass operators which preserve logarithmic functions. Later, we express logarithmic moments of the modified operators, and then we give Voronovskaya-type theorem. Moreover, a rate of convergence is achieved, and onwards the global smoothness preservation feature is stated via the logarithmic weighted modulus of continuity in the weighted Mellin-Lebesgue spaces comprising all Lebesgue measurable functions.

1 Introduction

Mellin transformations perform significant roles not only in mathematics but also in computer science, optical physics, etc. Their significance roots in their implementations to real-life problems. For instance, they can be properly benefitted in problems of signal reconstruction where the samples are not uniformly spaced, as in the classical Shannon Sampling Theorem, but exponentially spaced (see e.g., [1], [7], [8], [10], [12]). The singular integrals of Mellin convolution type were first presented by Kolbe and Nessel in [11]. Butzer and Jansche [9] exhaustively investigated Mellin transform. They defined the Mellin convolution and appeared some significant results. After, Bardaro and Mantellini [6] took into account Mellin convolution operators of type

$$(T_w f)(s) = \int_0^\infty K_w(t) f(ts) \frac{dt}{t}, \; s \in (0,\infty), \tag{1}$$

where f belongs to domain of the operator T_w and $K_w : (0,\infty) \to \mathbb{R}$ is a kernel that provides the condition $\int_0^\infty K_w(t) \frac{dt}{t} = 1$. Compared with the ordinary classical convolution, the translation operator is changed with a dilation operator. A specific kind of Mellin type operators is the Mellin-Gauss-Weierstrass operator, corresponding to the Mellin-Gauss-Weierstrass kernel [9, p. 342 Definition 8]. It is defined by

$$(\mathcal{T}_w f)(s) = \frac{w}{\sqrt{4\pi}} \int_0^\infty f(ts) e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t}, \; s \in \mathbb{R}^+ \text{ and } w \geq 1.$$

It is entitled as Mellin-Gauss-Weierstrass convolution operator.

In literature, various papers are published for similar operators on this subject. For instance, Ozsarac et al. [17] give a new generalization of Mellin-Gauss-Weierstrass operators that preserve logarithmic functions and describe the behavior of the operators in some weighted spaces. Moreover, in [2], a new modulus of continuity for locally integrable function spaces is indicated and the attained outcomes are implemented to the Gauss-Weierstrass operators. Other than these, in [18], Topuz et al. represent a modification of singular integral of Mellin convolution type, and the obtained outputs are restated for the Mellin-Gauss-Weierstrass operator. Very recently, Ozsarac express approximation properties of the Mellin-Gauss-Weierstrass operators in the Mellin-Lebesgue spaces in [16] and presents approximation properties of a modified form of the Mellin Gauss-Weierstrass operator in the space of continuous functions in [15]. In these days, Aral et al. present quantitative approximation theorems for Mellin-Fejer type operators in Mellin-Lebesgue spaces, benefitting a appropriate modulus of continuity in [4], and Ozsarac express a numerical version of the convergence via the logarithmic weighted modulus of continuity for linear combinations of the Mellin convolution operators in the weighted Mellin-Lebesgue spaces in [14].

The logarithmic weighted spaces of functions and a convenient modulus of continuity were presented in the recent study [3]. Additionally, the paper [3] is connected with an operator of Kantorovich type by regarding Mellin-Gauss-Weierstrass kernel.

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The rest of the paper is organized as follows: In the next part, elementary notations and preliminaries concerning with the subject are recalled, and by the motivation of [17], we express a modified Mellin-Gauss-Weierstrass operator $\mathcal{G}_w$ that preserves both the constant function 1 and the function $\log_2$. Later, the properties of $\mathcal{G}_w$ are mentioned, and then we give Voronovskaya-type theorem. Moreover, by the other motivation of [4], a rate of convergence of aforementioned operator is provided, and onwards the global smoothness preservation feature is given via the logarithmic weighted modulus of continuity in the weighted Mellin-Lebesgue spaces comprising all Lebesgue measurable functions.

2 Notations and Preliminary Results

Let us symbolize by $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ the sets of positive integers, positive real numbers and nonnegative real numbers, respectively. By $\mathbb{C}$, we denote the set of complex numbers. Throughout the paper, $C(\mathbb{R}^+)$ stands for the space of all continuous and bounded functions defined on $\mathbb{R}^+$.

Let $\mathbb{R}^+$ be the multiplicative topological group bestowed with the logarithmic (Haar) measure

$$\mu(A) := \int\limits_A \frac{dt}{t},$$

being dt the Lebesgue measure.

By $C(\mathbb{R}^+)$ we symbolize the space of all continuous and bounded functions $f : \mathbb{R}^+ \to \mathbb{C}$, and by $C_{comp}(\mathbb{R}^+)$ the subspace of $C(\mathbb{R}^+)$ containing all functions with compact support in $\mathbb{R}^+$.

For $p \in [1, \infty[$ we symbolize by $X^p(\mathbb{R}^+) \equiv X^p$, the Mellin-Lebesgue spaces with respect to the measure μ and we symbolize by $\|f\|_p$ the corresponding norm of a function $f \in X^p$ i.e.

$$\|f\|_p := \left(\int_0^\infty |f(u)|^p \frac{du}{u} \right)^{1/p}.$$

For $p = \infty$ we define X^∞ as the space of all Lebesgue measurable functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that $\|f\|_\infty := ess\sup_{u \in \mathbb{R}^+} |f(u)| < \infty$.

Moreover, we call that $f \in C^k$ locally at the point $s \in \mathbb{R}^+$ if there is a neighbourhood U_s of the point s such that f is $(k-1)$-times continuously differentiable in U_s and the derivative of order k exists at the point s.

Here, by the motivation [17], for $w \geq 1$, we consider the modified Mellin-Gauss-Weierstrass operator

$$\begin{aligned}(\mathcal{G}_w f)(s) &= \frac{w}{\sqrt{4\pi}} \int\limits_0^\infty f(t) e^{-\left(\frac{w}{2}\log\frac{t}{a_w(s)}\right)^2} \frac{dt}{t}, \quad s > e^{\frac{\sqrt{2}}{w}} \\ &= \frac{w}{\sqrt{4\pi}} \int\limits_0^\infty f(t a_w(s)) e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t}, \quad s > e^{\frac{\sqrt{2}}{w}}, \end{aligned} \tag{2}$$

where

$$a_w(s) = e^{\sqrt{\log^2 s - \frac{2}{w^2}}}, \quad s > e^{\frac{\sqrt{2}}{w}}.$$

Moreover, we define

$$(G_w f)(s) = \begin{cases} (\mathcal{G}_w f)(s), & \text{if } s \in \left(e^{\frac{\sqrt{2}}{w}}, \infty\right) \\ f(s), & \text{if } s \in \left(0, e^{\frac{\sqrt{2}}{w}}\right] \end{cases}.$$

There is a $\overline{w} \in \mathbb{N}$, such that the $\mathcal{G}_w f$, are meaningful for $w \geq \overline{w}$.

Lemma 2.1. *[17] In our calculations, we will use the following well-known equalities:*

1. $\frac{w}{\sqrt{4\pi}} \int\limits_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t} = 1,$
2. $\frac{w}{\sqrt{4\pi}} \int\limits_0^\infty e^{-\left(\frac{w}{2}\log\frac{t}{a_w(s)}\right)^2} \frac{dt}{t} = 1$ for $s > e^{\frac{\sqrt{2}}{w}}$.

For $j \in \mathbb{N}$, the corresponding moments for the kernel are stated by

$$\begin{aligned} m_j &= \frac{w}{\sqrt{4\pi}} \int\limits_0^\infty e^{-\left(\frac{w}{2}\log\frac{t}{a_w(s)}\right)^2} \log^j\left(\frac{t}{s}\right) \frac{dt}{t} \\ &= \frac{w}{\sqrt{4\pi}} \int\limits_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} \log^j\left(\frac{t a_w(s)}{s}\right) \frac{dt}{t}. \end{aligned} \tag{3}$$

Lemma 2.2. *[17] With the aid of Mathematica, for $s > e^{\frac{\sqrt{2}}{w}}$, we can obtain the following moments:*

1. $m_1 = \sqrt{\log^2 s - \frac{2}{w^2}} - \log s$,
2. $m_2 = 2\log^2 s - 2\log s \sqrt{\log^2 s - \frac{2}{w^2}}$,
3. $m_2 \leq \dfrac{4}{w^2}$,
4. $m_4 \leq \dfrac{40}{w^4}$,
5. $m_6 \leq \dfrac{640}{w^6}$.

Corollary 2.3. *From the previous lemma, we have the following limits:*

1. $\lim\limits_{w\to\infty} w^2 m_1 = \log^{-1}\left(\frac{1}{s}\right)$,
2. $\lim\limits_{w\to\infty} w^2 m_2 = 2$.

Lemma 2.4. *[17] The following equalities are valid:*

1. $(\mathcal{G}_w 1)(s) = 1$,
2. $(\mathcal{G}_w \log)(s) = \sqrt{-\frac{2}{w^2} + \log^2 s} \qquad \left(s > e^{\frac{\sqrt{2}}{w}}\right)$,
3. $\left(\mathcal{G}_w \log^2\right)(s) = \log^2 s$.

3 A Voronovskaya formula for $\mathcal{G}_w f$

In this section, we will employ the Mellin differential operator, as expressed by Butzer and Jansche in [9]. The Mellin differential operator Θ or the Mellin derivative Θf of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$\Theta f(s) = s f^{'}(s),\ s \in \mathbb{R}^+,$$

if the classical derivative $f^{'}(s)$ exists. The Mellin differential operator of order $r \in \mathbb{N}$ is defined inductively by setting $\Theta^1 = \Theta$, $\Theta^r = \Theta \circ \Theta^{r-1}$, $\Theta^0 = I$, I being the identity operator. From [9], we get the following formula

$$\Theta^r f(s) = \sum_{k=0}^{r} S(r,k) f^{(k)}(s) s^k,$$

where $S(r,k)$, $r \in \mathbb{N}$, $0 \leq k \leq r$, indicate the Stirling numbers of the second kind.

It is useful to recall the Taylor formula with the remainder in Peano version (see [5], [13]).

Proposition 3.1. *Let $f \in C^n$ locally at a point $s \in \mathbb{R}^+$. Then, there exists $\delta > 1$ such that for $t \in (1/\delta, \delta)$*

$$\begin{aligned} f(t) &= f(s) + \Theta f(s) \log\left(\frac{t}{s}\right) + \frac{\Theta^2 f(s)}{2!}\log^2\left(\frac{t}{s}\right) + \ldots + \frac{\Theta^n f(s)}{n!}\log^n\left(\frac{t}{s}\right) \\ &\quad + h_s(t)\log^n\left(\frac{t}{s}\right), \end{aligned}$$

where $h_s(t) \to 0$ as $t \to s$. Additionally, if $f \in X^\infty$, the above formula valid for every $t \in \mathbb{R}^+$ and the function h_s is bounded on $\mathbb{R}^+$.

We are now ready to explain the asymptotic behaviour of the family $(\mathcal{G}_w f)$ that identifies an order of pointwise approximation.

Theorem 3.2. *Let $f \in X^\infty$ and $s > e^{\frac{\sqrt{2}}{w}}$. We have*

$$\lim_{w\to\infty} w^2 \left[(\mathcal{G}_w f)(s) - f(s)\right] = \log^{-1}\left(\frac{1}{s}\right)\Theta f(s) + \Theta^2 f(s)$$

for $f \in C^2$ locally at the point s.

Proof. From (2) and Lemma 2.1 (2), for $s > e^{\frac{\sqrt{2}}{w}}$,we have

$$\begin{aligned} (G_w f)(s) - f(s) &= (\mathcal{G}_w f)(s) - f(s) \\ &= \frac{w}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{w}{2}\log\frac{t}{a_w(s)}\right)^2} (f(t) - f(s)) \frac{dt}{t} \\ &= \frac{w}{\sqrt{4\pi}} \left(\int_0^{1/\delta} + \int_{1/\delta}^{\delta} + \int_{\delta}^{\infty} \right) e^{-\left(\frac{w}{2}\log\frac{t}{a_w(s)}\right)^2} (f(t) - f(s)) \frac{dt}{t} \\ &: = I_1 + I_2 + I_3. \end{aligned}$$

We take into account first I_2. By Proposition 3.1, we get

$$I_2 = \Theta f(s) m_1 + \frac{\Theta^2 f(s)}{2!} m_2 + \frac{w}{\sqrt{4\pi}} \int_{1/\delta}^{\delta} e^{-\left(\frac{w}{2}\log\frac{t}{a_w(s)}\right)^2} h_s(t) \log^2\left(\frac{t}{s}\right) \frac{dt}{t}.$$

From Corollary 2.3 (1) and (2), we observe that

$$\lim_{w\to\infty} w^2 m_1 = \log^{-1}\left(\frac{1}{s}\right)$$

and

$$\lim_{w\to\infty} w^2 m_2 = 2.$$

As to the last integral, for $\varepsilon > 0$ suppose that $\delta > 1$ with $|h_s(t)| < \varepsilon$ for $t \in (1/\delta, \delta)$. Hence,

$$w^2 \left| \frac{w}{\sqrt{4\pi}} \int_{1/\delta}^{\delta} e^{-\left(\frac{w}{2}\log\frac{t}{a_w(s)}\right)^2} h_s(t) \log^2\left(\frac{t}{s}\right) \frac{dt}{t} \right| \leq \varepsilon w^2 \frac{w}{\sqrt{4\pi}} \int_{0}^{\infty} e^{-\left(\frac{w}{2}\log\frac{t}{a_w(s)}\right)^2} \log^2\left(\frac{t}{s}\right) \frac{dt}{t} = \varepsilon w^2 m_2.$$

As regards I_1 (and analogously for I_3), we obtain

$$|I_1| \leq 2\|f\|_\infty \frac{w}{\sqrt{4\pi}} \int_{0}^{1/\delta} e^{-\left(\frac{w}{2}\log\frac{t}{a_w(s)}\right)^2} \frac{dt}{t} = \|f\|_\infty \frac{1}{\sqrt{\pi}} \int_{0}^{1/(\delta a_w(s))^w} e^{-\left(\frac{1}{2}\log t\right)^2} \frac{dt}{t},$$

so $\lim_{w\to\infty} w^2 |I_1| = 0$. The proof follows by limsup and liminf arguments. □

4 Weighted Approximation

In this chapter, we denote an estimation of approximation of functions including to the weighted Mellin-Lebesgue spaces, by the modified Mellin-Gauss-Weierstrass operators defined in (2) via the logarithmic weighted modulus of continuity.

Definition 4.1. [4] For $1 \leq p < \infty$, we represent by $X_w^p(\mathbb{R}^+) \equiv X_w^p$, the weighted Mellin-Lebesgue spaces containing all the Lebesgue measurable functions f such that

$$\|f\|_{p,w} := \left\{ \int_0^\infty \left| \frac{f(s)}{1+(\log s)^2} \right|^p \frac{ds}{s} \right\}^{1/p} < \infty,$$

(here the letter "w" implies "weighted").

Definition 4.2. [4] We introduce the weighted modulus of continuity of $f \in X_w^p$ as

$$\omega_p^w(f;\delta) = \sup_{|\log h|<\delta} \left\{ \int_0^\infty \left| \frac{f(sh)-f(s)}{1+(|\log s|+|\log h|)^2} \right|^p \frac{ds}{s} \right\}^{1/p} \quad (\delta > 0) \tag{4}$$

and we say it as “logarithmic weighted modulus of continuity in X_w^p”.

ω_p^w has the following necessary features:

Lemma 4.1. *[4] Let $\delta > 0$, $n \in \mathbb{N}$. Then*
(i) for $f \in X_w^p$, the quantity $\omega_p^w(f;\delta)$ is finite,
(ii) for $f \in X_w^p$, $\lim_{\delta\to 0} \omega_p^w(f;\delta) = 0$,
(iii) for all $f \in X_w^p$,

$$\omega_p^w(f;n\delta) \leq n^3 \omega_p^w(f;\delta) \qquad (n \in \mathbb{N})$$

and for any $\lambda \in \mathbb{R}^+$

$$\omega_p^w(f;\lambda\delta) \leq (1+\lambda)^3\, \omega_p^w(f;\delta).$$

Theorem 4.2. *For $1 \leq p < \infty$, if $f \in X_w^p$, then we get*

$$\|G_w f - f\|_{p,w} \leq 354\,\omega_p^w\left(f;\frac{1}{w}\right).$$

Proof. By (2) and Lemma 2.1 (2), for $s > e^{\frac{\sqrt{2}}{w}}$, we can write

$$\begin{aligned} (G_w f)(s)-f(s) &= (\mathcal{G}_w f)(s)-f(s)\\ &= \frac{w}{\sqrt{4\pi}}\int\limits_0^\infty e^{-\left(\frac{w}{2}\log\frac{t}{a_w(s)}\right)^2}\left(f(t)-f(s)\right)\frac{dt}{t}\\ &= \frac{w}{\sqrt{4\pi}}\int\limits_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2}\left(f(ta_w(s))-f(s)\right)\frac{dt}{t}. \end{aligned}$$

Then, for $s > e^{\frac{\sqrt{2}}{w}}$, we deduce

$$\begin{aligned} &\|G_w f-f\|_{p,w}\\ =\ & \left\{\int_0^\infty\left|\int_0^\infty \frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{f(ta_w(s))-f(s)}{1+(\log s)^2}\frac{dt}{t}\right|^p\frac{ds}{s}\right\}^{1/p}\\ \leq\ & \int_0^\infty\left\{\int_0^\infty\left|\frac{f(ta_w(s))-f(s)}{1+\left(|\log s|+\left|\log\frac{ta_w(s)}{s}\right|\right)^2}\,\frac{1+\left(|\log s|+\left|\log\frac{ta_w(s)}{s}\right|\right)^2}{1+(\log s)^2}\right|^p\frac{ds}{s}\right\}^{1/p}\\ & \times\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}\\ \leq\ & 2\int_0^\infty\left\{\int_0^\infty\left|\frac{f(ta_w(s))-f(s)}{1+\left(|\log s|+\left|\log\frac{ta_w(s)}{s}\right|\right)^2}\right|^p\frac{ds}{s}\right\}^{1/p}\left(1+\left(\left|\log\frac{ta_w(s)}{s}\right|\right)^2\right)\\ & \times\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}. \end{aligned}$$

By using the definition of ω_p^w and Lemma 4.1 (iii), and for $\delta > 0$ and $s > e^{\frac{\sqrt{2}}{w}}$, we have

$$\begin{aligned} &\|G_w f-f\|_{p,w}\leq 2\int_0^\infty\omega_p^w\left(f;\left|\log\frac{ta_w(s)}{s}\right|\right)\left(1+\left(\left|\log\frac{ta_w(s)}{s}\right|\right)^2\right)\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}\\ \leq\ & 2\omega_p^w(f;\delta)\int_0^\infty\left(1+\frac{1}{\delta}\left|\log\frac{ta_w(s)}{s}\right|\right)^3\left(1+\left(\left|\log\frac{ta_w(s)}{s}\right|\right)^2\right)\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}\\ \leq\ & 2\omega_p^w(f;\delta)\int_0^\infty\left(1+3\frac{\left|\log\frac{ta_w(s)}{s}\right|}{\delta}+4\frac{\left(\log\frac{ta_w(s)}{s}\right)^2}{\delta^2}+4\frac{\left|\log\frac{ta_w(s)}{s}\right|^3}{\delta^3}\right.\\ & \left.+3\frac{\left(\log\frac{ta_w(s)}{s}\right)^4}{\delta^4}+\frac{\left|\log\frac{ta_w(s)}{s}\right|^5}{\delta^5}\right)\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}. \end{aligned}$$

From (3), we can write

$$\begin{aligned} \|G_w f-f\|_{p,w} \leq\ & 2\omega_p^w(f;\delta)\left(1+\frac{4}{\delta^2}m_2+\frac{3}{\delta^4}m_4+\frac{3}{\delta}\int_0^\infty\left|\log\frac{ta_w(s)}{s}\right|\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}\right.\\ & \left.+\frac{4}{\delta^3}\int_0^\infty\left|\log\frac{ta_w(s)}{s}\right|^3\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}+\frac{1}{\delta^5}\int_0^\infty\left|\log\frac{ta_w(s)}{s}\right|^5\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}\right). \end{aligned}$$

From the Hölder inequality, Lemma 2.2 (3), (4) and (5), we have

$$\begin{aligned} \|G_w f-f\|_{p,w} \leq\ & 2\omega_p^w(f;\delta)\left(1+\frac{4}{\delta^2}m_2+\frac{3}{\delta^4}m_4+\frac{3}{\delta}\sqrt{m_2}+\frac{4}{\delta^3}\sqrt{m_4}\sqrt{m_2}+\frac{1}{\delta^5}\sqrt{m_6}\sqrt{m_4}\right)\\ \leq\ & 2\omega_p^w(f;\delta)\left(1+\frac{4}{\delta^2}\frac{4}{w^2}+\frac{3}{\delta^4}\frac{40}{w^4}+\frac{3}{\delta}\frac{2}{w}+\frac{4}{\delta^3}\frac{\sqrt{40}}{w^2}\frac{2}{w}+\frac{1}{\delta^5}\frac{\sqrt{640}}{w^3}\frac{\sqrt{40}}{w^2}\right). \end{aligned}$$

Choosing $\delta = 1/w$, with $w \geq 1$, from Corollary 2.3 (2), we finally obtain desired result. Furthermore, when $s \in \left(0, e^{\frac{\sqrt{2}}{w}}\right)$, the desired inequality is already valid. □

Corollary 4.3. *Considering Lemma 4.1 (ii),*

$$\lim_{w\to\infty}\|G_w f - f\|_{p,w} = 0$$

valid for all $f \in X_w^p$ *uniformly on* $\mathbb{R}^+$.

Right now, we mention global smoothness preservation feature of the operator (G_w).

Theorem 4.4. *For* $1 \le p < \infty$, *if* $f \in X_w^p$ *and* $\delta > 0$, *then we have*

$$\omega_p^w(G_w f;\delta) \le 4\left(1+\frac{4}{w^2}\right)\omega_p^w(f;\delta).$$

Proof. Performing the generalized Minkowski inequality, for $s > e^{\frac{\sqrt{2}}{w}}$, we gain

$$\begin{aligned}
&\left(\int_0^\infty \left|\frac{(G_w f)(us)-(G_w f)(s)}{\left(1+(|\log s|+|\log u|)^2\right)}\right|^p \frac{ds}{s}\right)^{1/p}\\
=\ &\left(\int_0^\infty \left|\frac{(\mathcal{G}_w f)(us)-(\mathcal{G}_w f)(s)}{\left(1+(|\log s|+|\log u|)^2\right)}\right|^p \frac{ds}{s}\right)^{1/p}\\
=\ &\left(\int_0^\infty \left|\int_0^\infty \frac{f(uta_w(s))-f(ta_w(s))}{\left(1+(|\log s|+|\log u|)^2\right)}\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}\right|^p \frac{ds}{s}\right)^{1/p}\\
\le\ &\int_0^\infty \left(\int_0^\infty \left|\frac{f(uta_w(s))-f(ta_w(s))}{\left(1+(|\log s|+|\log u|)^2\right)}\right|^p \frac{ds}{s}\right)^{1/p}\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}.
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\left(\int_0^\infty \left|\frac{(\mathcal{G}_w f)(us)-(\mathcal{G}_w f)(s)}{1+(|\log s|+|\log u|)^2}\right|^p \frac{ds}{s}\right)^{1/p}\\
\le\ & 4\int_0^\infty \left(\int_0^\infty \left|\frac{f(uta_w(s))-f(ta_w(s))}{\left(1+(|\log a_w(s)|+|\log t|+|\log u|)^2\right)}\right|^p \frac{ds}{s}\right)^{1/p}\left(1+\left|\log\frac{ta_w(s)}{s}\right|^2\right)\\
&\times\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}.
\end{aligned}$$

Taking sup of both sides of the above inequality according to u for $|\log u| < \delta$, we achieve

$$\begin{aligned}
\omega_p^w(\mathcal{G}_w f;\delta) &\le 4\omega_p^w(f;\delta)\int_0^\infty\left(1+\left|\log\frac{ta_w(s)}{s}\right|^2\right)\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}\frac{dt}{t}\\
&= 4(1+m_2)\,\omega_p^w(f;\delta).
\end{aligned}$$

Therefore, from Lemma 2.2 (3), the desired result is achieved.

In the other hand, when $s \in \left(0, e^{\frac{\sqrt{2}}{w}}\right)$, the desired inequality is already valid. Thus, the proof is completed. □

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