



An Alternative Approach to Multivalued F –Contractions in Ultrametric Spaces

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Abstract

This paper establishes new fixed point theorems based on the concept of F -contractions. The results are formulated for multivalued mappings of rational type defined on spherically complete ultrametric spaces. Several special cases are discussed, some of which generalize or improve upon known results. Illustrative examples are provided to support the theoretical findings. Furthermore, an alternative generalization involving almost contractive multivalued mappings of rational type is also presented within the framework of F -contractions.

1 Introduction

Fixed point theory in ultrametric spaces has gained considerable attention in recent years, owing to its applications in p -adic analysis, non-Archimedean functional analysis, and other branches of mathematics. One of the foundational results in this context was established by Gajic [3], who also proved a fixed point theorem for multivalued mappings on ultrametric spaces in [4], and further generalized through various approaches, including implicit relations [9], graph-theoretic structures [6], and topological frameworks [5, 7].

A significant development in the theory of contractions was the introduction of the F -contraction by Wardowski [13], which allowed a broader class of functions to be considered in fixed point results. This approach has been successfully applied to single-valued, multivalued and paired mappings in both metric and ultrametric settings [8, 15]. Moreover, recent studies have explored hybrid contraction types and integral-type generalizations to further extend the applicability of fixed point results [1, 2, 16, 17, 18, 19].

Motivated by these developments, this study aims to establish new fixed point results for generalized multivalued mappings of rational type on ultrametric spaces that are spherically complete. We also propose a broader generalization involving almost F -contractive mappings. The paper begins with essential definitions and properties related to ultrametric spaces and F -contractions, followed by the main results, supporting examples, and several corollaries that demonstrate the flexibility of the proposed framework.

2 Preliminaries

Definition 2.1. ([3]) A metric v on a set Γ is an ultrametric if for all $\eta, \varrho, \omega \in \Gamma$

$$v(\eta, \varrho) \leq \max\{v(\eta, \omega), v(\omega, \varrho)\}.$$

This property is known as the strong triangle inequality, which is a stronger form of the usual triangle inequality required in metric spaces.

A discrete metric on Γ ($\Gamma \neq \emptyset$) is an ultrametric on Γ and also a metric on Q defined via binary expansions also forms an ultrametric.

Remark 1. ([3]) Although every ultrametric is indeed a metric, the converse does not hold; that is, not every metric satisfies the stronger ultrametric inequality. In particular, the standard Euclidean metric on \mathbb{R} provides a clear example: while it fulfills all the axioms of a metric, it fails to satisfy the ultrametric (or strong triangle) inequality.

Definition 2.2. ([14]) Γ be a nonempty set and let (Γ, v) be an ultrametric space. A closed ball centered at $\eta \in \Gamma$ with radius $\varepsilon > 0$ is defined as:

$$\mathfrak{B}(\eta, \varepsilon) = \{\varrho \in \Gamma : v(\eta, \varrho) \leq \varepsilon\},$$

with $\mathfrak{B}(\eta, 0) = \{\eta\}$.

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Remark 2. ([14]) A well-known characteristic property of ultrametric spaces is the following: Let (Γ, v) be an ultrametric space. For any $\eta, \varrho \in \Gamma$ and any radii $0 \leq \varepsilon \leq \delta$, if

$$\mathfrak{B}(\eta, \varepsilon) \cap \mathfrak{B}(\eta, \delta) \neq \emptyset, \text{ then } \mathfrak{B}(\eta, \varepsilon) \subset \mathfrak{B}(\eta, \delta).$$

Definition 2.3. ([14]) Let (Γ, v) be an ultrametric space. The space (Γ, v) is called spherically complete if every decreasing collection of closed balls in Γ , that is, every family $\{\mathfrak{B}(\eta_i, \varepsilon_i)\}_{i \in I}$ ($\eta \in \Gamma, 0 \leq \varepsilon$) satisfying $\mathfrak{B}(\eta_{i+1}, \varepsilon_{i+1}) \subseteq \mathfrak{B}(\eta_i, \varepsilon_i)$ for all $i \in I$, has a nonempty intersection.

All other features related to ultrametric can be found in [14].

An F -contraction is a type of mapping that generalizes the classical contraction condition through a function F . This approach ensures the existence of a fixed point even for mappings that do not satisfy Banach’s contraction condition but meet the F -contraction criterion.

Definition 2.4. ([13]) Let \mathcal{F} denote the class of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ that satisfy the following three assumptions:

- (F1) F is strictly monotone increasing; in other words, for all $\eta, \varrho \in (0, \infty)$ such that $\eta < \varrho, F(\eta) < F(\varrho)$;
- (F2) Whenever $\{\eta_k\}$ is a sequence of positive reals, the limit $\eta_k \rightarrow 0$ if and only if $F(\eta_k) \rightarrow -\infty$ as $k \rightarrow \infty$;
- (F3) $\exists \kappa \in (0, 1) : \lim_{\eta \rightarrow 0^+} \eta^\kappa F(\eta) = 0$.

Any function F belonging to this class will be denoted by $F \in \mathcal{F}$.

Definition 2.5. ([13]) Let (Γ, v) be a metric space. $T : \Gamma \rightarrow \Gamma$ a mapping and $F \in \mathcal{F}$. For $\tau > 0$ if for all $\eta, \varrho \in \Gamma$, whenever $v(T\eta, T\varrho) > 0$ the inequality,

$$\tau + F(v(T\eta, T\varrho)) \leq F(v(\eta, \varrho))$$

holds, then T is called an F -contraction mapping.

Example 2.1. ([13]) Let (Γ, v) be a metric space, $\zeta > 0, \tau > 0$ and define

$$F(\zeta) = \ln(\zeta^2 + \zeta).$$

It is clear that $F \in \mathcal{F}$. Moreover, for all $\eta, \varrho \in \Gamma$ such that $v(T\eta, T\varrho) > 0$, the inequality

$$\frac{v(T\eta, T\varrho)(1 + v(T\eta, T\varrho))}{v(\eta, \varrho)(1 + v(\eta, \varrho))} \leq e^{-\tau}$$

satisfies an F -contraction condition.

Now let us present the fixed point theorem given by Wardowski for an F -contraction.

Theorem 2.1. ([13]) Let (Γ, v) be a complete metric space and let $T : \Gamma \rightarrow \Gamma$ be an F -contraction. Then the mapping T has a unique fixed point.

In the analysis of multivalued mappings, the Hausdorff metric is used to measure the distance between sets.

Definition 2.6. ([8]) Let (Γ, v) be a metric space. Denote by $CB(\Gamma)$ the family of all nonempty, closed and bounded subsets of Γ . For $Z, P \in CB(\Gamma)$, the Hausdorff metric $H(Z, P)$ induced by v is defined by

$$H(Z, P) = \max \left\{ \sup_{\eta \in Z} D(\eta, P), \sup_{\varrho \in P} D(\varrho, Z) \right\},$$

where $D(\eta, Z) = \inf \{v(\eta, \varrho) : \varrho \in Z\}$.

3 Main Results

3.1 A result for generalized multivalued rational type F-contraction

In this section, the definition of the generalized multivalued rational type F -contraction will be given, and some fixed point results will be obtained using this contraction. An illustrative example will also be given.

Definition 3.1. Let (Γ, v) denote an ultrametric space, and let $f : \Gamma \rightarrow CB(\Gamma)$ be a multivalued mapping, where $CB(\Gamma)$ denotes the set of all subsets of Γ that are nonempty, closed and bounded. The mapping f is referred to as a generalized multivalued rational type F -contraction on Γ , whenever the following condition holds: for all $\eta, \varrho \in \Gamma, F \in \mathcal{F}, \tau > 0$,

$$H(f\eta, f\varrho) > 0 \Rightarrow \tau + F(H(f\eta, f\varrho)) \leq F(M(\eta, \varrho)), \tag{1}$$

$$M(\eta, \varrho) = \max \left\{ v(\eta, \varrho), D(\eta, f\eta), D(\varrho, f\varrho), \frac{D(\eta, f\eta)D(\varrho, f\varrho)}{1 + v(\eta, \varrho)}, \frac{D(\eta, f\eta)D(\varrho, f\varrho)}{1 + H(f\eta, f\varrho)} \right\}.$$

Theorem 3.1. Consider a spherically complete ultrametric space (Γ, v) and a generalized multivalued rational type F -contraction $f : \Gamma \rightarrow CB(\Gamma)$. Then there exists a fixed point of f .

Proof. For each $\zeta \in \Gamma$, define a closed ball $\mathfrak{B}_\zeta = B(\zeta, D(\zeta, f\zeta))$ with radius $D(\zeta, f\zeta) = \inf_{s \in f\zeta} v(\zeta, s)$ and let \mathcal{A} be the class of all such balls. Introduce a partial ordering \leq on \mathcal{A} via reverse inclusion:

$$\mathfrak{B}_\zeta \leq \mathfrak{B}_\theta \text{ iff } \mathfrak{B}_\theta \subseteq \mathfrak{B}_\zeta.$$

Let \mathcal{A}_1 be a totally ordered subfamily of \mathcal{A} . Because Γ satisfies spherical completeness, the intersection of all members of \mathcal{A}_1 is nonempty:

$$B = \bigcup \{ \mathfrak{B}_\zeta \mid \mathfrak{B}_\zeta \in \mathcal{A}_1 \} \neq \emptyset.$$

Take $\vartheta \in B$ and an arbitrary $\mathfrak{B}_\zeta \in \mathcal{A}_1$. One can see that, $\vartheta \in \mathfrak{B}_\zeta$ therefore

$$v(\vartheta, \zeta) \leq D(\zeta, f\zeta).$$

By compactness of $f\zeta$ the infimum is attained at some $u \in f\zeta$, i.e., $v(\zeta, u) = D(\zeta, f\zeta)$. In case $\zeta = \vartheta$, the balls coincide: $\mathfrak{B}_\zeta = \mathfrak{B}_\vartheta$. Otherwise, for $\zeta \neq \vartheta$, consider an arbitrary $x \in \mathfrak{B}_\vartheta$.

$$\begin{aligned} v(x, \vartheta) &\leq D(\vartheta, f\vartheta) = \inf_{e \in f\vartheta} v(\vartheta, e) \\ &\leq \max \left\{ v(\vartheta, \zeta), v(\zeta, u), \inf_{e \in f\vartheta} v(u, e) \right\} \\ &\leq \max \{ D(\zeta, f\zeta), H(f\zeta, f\vartheta) \} \\ &< \max \{ D(\zeta, f\zeta), M(\zeta, \vartheta) \}, \end{aligned}$$

where

$$\begin{aligned} \tau + F(H(f\zeta, f\vartheta)) &\leq F(M(\zeta, \vartheta)) \text{ and} \\ M(\zeta, \vartheta) &= \max \left\{ v(\zeta, \vartheta), D(\zeta, f\zeta), D(\vartheta, f\vartheta), \right. \\ &\quad \left. \frac{D(\zeta, f\zeta)D(\vartheta, f\vartheta)}{1+v(\zeta, \vartheta)}, \frac{D(\zeta, f\zeta)D(\vartheta, f\vartheta)}{1+H(f\zeta, f\vartheta)} \right\} \\ &\leq \max \{ D(\zeta, f\zeta), D(\vartheta, f\vartheta) \}. \end{aligned}$$

This step is justified by the monotonicity of F . Now, if $D(\vartheta, f\vartheta) \leq D(\zeta, f\zeta)$, we have

$$v(x, \vartheta) \leq D(\vartheta, f\vartheta) < D(\zeta, f\zeta).$$

Otherwise, if $D(\zeta, f\zeta) \leq D(\vartheta, f\vartheta)$, we have

$$\begin{aligned} v(x, \vartheta) &\leq D(\vartheta, f\vartheta) \\ &\leq \max \{ D(\zeta, f\zeta), H(f\zeta, f\vartheta) \} \\ &< \max \left\{ v(\zeta, \vartheta), D(\zeta, f\zeta), D(\vartheta, f\vartheta), \right. \\ &\quad \left. \frac{D(\zeta, f\zeta)D(\vartheta, f\vartheta)}{1+v(\zeta, \vartheta)}, \frac{D(\zeta, f\zeta)D(\vartheta, f\vartheta)}{1+H(f\zeta, f\vartheta)} \right\} \\ &< \max \{ D(\zeta, f\zeta), D(\vartheta, f\vartheta) \} \\ &< D(\vartheta, f\vartheta), \end{aligned}$$

from $F \in \mathcal{F}$. As a result, we arrive at the clearly contradictory inequality $D(\vartheta, f\vartheta) < D(\vartheta, f\vartheta)$ which invalidates the second assumption. From here, we obtain the following:

$$v(x, \vartheta) \leq D(\vartheta, f\vartheta) \leq D(\zeta, f\zeta)$$

for $x \in \mathfrak{B}_\vartheta$. Now, by the ultrametric property,

$$\begin{aligned} v(x, \zeta) &\leq \max \{ v(x, \vartheta), v(\vartheta, \zeta) \} \\ &\leq \max \{ v(x, \vartheta), D(\zeta, f\zeta) \} = D(\zeta, f\zeta) \end{aligned}$$

holds. Thus $x \in \mathfrak{B}_\zeta$, ensuring the inclusion $\mathfrak{B}_\vartheta \subseteq \mathfrak{B}_\zeta$ for all elements \mathfrak{B}_ζ of \mathcal{A}_1 . Consequently, \mathfrak{B}_ϑ serves as an upper bound of \mathcal{A}_1 within \mathcal{A} . By using Zorn's lemma, we conclude that, \mathcal{A} possesses a maximal element, denoted by $\mathfrak{B}_{\check{s}}$ for some $\check{s} \in \Gamma$.

Our next goal is to establish that $\check{s} \in f\check{s}$. Presume that $\check{s} \notin f\check{s}$, $\rho \in f\check{s}$, $\check{s} \neq \rho$ and $v(\check{s}, \rho) = D(\check{s}, f\check{s})$. From (1), we obtain

$$\begin{aligned} \tau + F(D(\rho, f\rho)) &\leq \tau + F(H(f\check{s}, f\rho)) \\ &\leq F \left(\max \left\{ v(\check{s}, \rho), D(\check{s}, f\check{s}), D(\rho, f\rho), \frac{D(\check{s}, f\check{s})D(\rho, f\rho)}{1+v(\check{s}, \rho)}, \frac{D(\check{s}, f\check{s})D(\rho, f\rho)}{1+H(f\check{s}, f\rho)} \right\} \right) \\ &\leq F(\max \{ D(\check{s}, f\check{s}), D(\rho, f\rho) \}) \end{aligned}$$

and deduce that

$$\begin{aligned} \tau + F(D(\rho, f\rho)) &\leq F(D(\check{s}, f\check{s})) \\ F(D(\rho, f\rho)) &< F(D(\check{s}, f\check{s})). \end{aligned}$$

And, from $F \in \mathcal{F}$, we have $D(\rho, f\rho) < D(\check{s}, f\check{s})$. Otherwise, we would obtain the contradiction $D(\rho, f\rho) < D(\rho, f\rho)$.

Now if $y \in \mathfrak{B}_\rho$, then

$$v(y, \rho) \leq D(\rho, f\rho) < D(\check{s}, f\check{s}) \tag{2}$$

and for $\check{s} \in \Gamma$

$$v(y, \check{s}) \leq \max\{v(y, \rho), v(\rho, \check{s})\} \leq D(\check{s}, f\check{s}),$$

leading to $y \in \mathfrak{B}_{\check{s}}$. Hence $\mathfrak{B}_\rho \subseteq \mathfrak{B}_{\check{s}}$.

Since (2)

$$D(\check{s}, f\check{s}) > D(\rho, f\rho)$$

implies $\check{s} \notin \mathfrak{B}_\rho$. Therefore $\mathfrak{B}_{\check{s}} \subsetneq \mathfrak{B}_\rho$. This contradicts the assumption that \mathfrak{B}_ρ is maximal. Therefore, $\check{s} \in f\check{s}$, indicating that \check{s} remains fixed under f . Hence, the proof is complete. \square

Example 3.1. Assume that \mathbb{N} represents the set of positive integers, and let v be an ultrametric on $\Gamma = \mathbb{N} \cup \{0\}$, specified by the function below:

$$\begin{aligned} v(\gamma, \xi) &= \max\left\{\frac{1}{\gamma}, \frac{1}{\xi}\right\}, \gamma \neq \xi, \\ v(\gamma, \xi) &= 0 \iff \gamma = \xi, \\ v(\gamma, 0) &= \frac{1}{\gamma}, \end{aligned}$$

for all $\gamma, \xi \in \Gamma$. Define $f : \Gamma \rightarrow \Gamma$ by

$$f\gamma = \begin{cases} \{0\} & , \gamma = 0, \gamma = 1 \\ \{\gamma e^{\gamma+2}, 0\} & , \gamma > 1 \end{cases}.$$

Consider $F(\varpi) = \ln \varpi$ for $\varpi > 0$. Hence, f can be classified as an F -contraction of rational type, where τ is taken to be $\frac{1}{4}$. To clarify this point, we analyze the cases below.

First, observe that

$$\gamma, \xi \in \mathbb{N}, v(f\gamma, f\xi) > 0 \iff (\xi > 1 \text{ and } \gamma = 0 \text{ or } \gamma = 1) \text{ or } (\gamma > \xi > 1).$$

Possibility 1. For $\gamma = 0$ and $\xi > 1$, it can be seen that

$$\frac{H(f\gamma, f\xi)}{M(\gamma, \xi)} = \frac{\frac{1}{\xi e^{\xi+2}}}{\frac{1}{\xi}} = \frac{\xi}{\xi e^{\xi+2}} = e^{-\xi-2} \leq e^{-4}.$$

Possibility 2. For $\gamma = 1$ and $\xi > 1$, as a result,

$$\frac{H(f\gamma, f\xi)}{M(\gamma, \xi)} = \frac{\frac{1}{\xi e^{\xi+2}}}{1} = \frac{1}{\xi e^{\xi+2}} = \frac{1}{\xi} e^{-\xi-2} \leq e^{-4}.$$

Possibility 3. For $\gamma > \xi > 1$, we obtain

$$\frac{H(f\gamma, f\xi)}{M(\gamma, \xi)} = \frac{\frac{1}{\xi e^{\xi+2}}}{\frac{1}{\xi}} = \frac{\xi}{\xi e^{\xi+2}} = e^{-\xi-2} \leq e^{-4}.$$

3.2 A result for generalized multivalued almost type F-contraction

As a continuation of the above analysis, we propose a new contraction mapping defined via an almost contractive condition and investigate its fixed point properties within the same ultrametric framework.

Definition 3.2. Let (Γ, ν) be an ultrametric space and $f : \Gamma \rightarrow CB(\Gamma)$ be a multivalued mapping. We say that f is a generalized multivalued almost type F -contraction on Γ if the following satisfies: for all $\eta, \varrho \in \Gamma, F \in \mathcal{F}, L > 0, \tau > 0$,

$$H(f\eta, f\varrho) > 0 \Rightarrow \tau + F(H(f\eta, f\varrho)) \leq F(M(\eta, \varrho) + LN(\eta, \varrho)), \quad (3)$$

where

$$M(\eta, \varrho) = \max \left\{ \nu(\eta, \varrho), D(\eta, f\eta), D(\varrho, f\varrho), \frac{D(\eta, f\varrho) + D(\varrho, f\eta)}{2} \right\}$$

and

$$N(\eta, \varrho) = \min \{D(\eta, f\varrho), D(\varrho, f\eta)\}.$$

Theorem 3.2. Let (Γ, ν) be a spherically complete ultrametric space and $f : \Gamma \rightarrow CB(\Gamma)$ is a generalized multivalued almost type F -contraction. Then f has a fixed point.

Proof. For each $\zeta \in \Gamma$, define a closed ball $\mathfrak{B}_\zeta = B(\zeta, D(\zeta, f\zeta))$ with radius $D(\zeta, f\zeta) = \inf_{s \in f\zeta} \nu(\zeta, s)$ and let \mathcal{A} be the class of all such balls. Introduce a partial ordering \leq on \mathcal{A} via reverse inclusion:

$$\mathfrak{B}_\zeta \leq \mathfrak{B}_\vartheta \text{ iff } \mathfrak{B}_\vartheta \subseteq \mathfrak{B}_\zeta.$$

Let \mathcal{A}_1 be a totally ordered subfamily of \mathcal{A} . Because Γ satisfies spherical completeness, the intersection of all members of \mathcal{A}_1 is nonempty:

$$B = \bigcap \{\mathfrak{B}_\zeta : \mathfrak{B}_\zeta \in \mathcal{A}_1\} \neq \emptyset.$$

Take $\vartheta \in B$ and an arbitrary $\mathfrak{B}_\zeta \in \mathcal{A}_1$. One can see that, $\vartheta \in \mathfrak{B}_\zeta$ therefore

$$\nu(\vartheta, \zeta) \leq D(\zeta, f\zeta).$$

By compactness of $f\zeta$ the infimum is attained at some $u \in f\zeta$, i.e., $\nu(\zeta, u) = D(\zeta, f\zeta)$. In case $\zeta = \vartheta$, the balls coincide: $\mathfrak{B}_\zeta = \mathfrak{B}_\vartheta$. Otherwise, for $\zeta \neq \vartheta$, consider an arbitrary $x \in \mathfrak{B}_\vartheta$.

$$\begin{aligned} \nu(x, \vartheta) &\leq D(\vartheta, f\vartheta) = \inf_{c \in f\vartheta} \nu(\vartheta, c) \\ &\leq \max \left\{ \nu(\vartheta, \zeta), \nu(\zeta, u), \inf_{c \in f\vartheta} \nu(u, c) \right\} \\ &\leq \max \{D(\zeta, f\zeta), H(f\zeta, f\vartheta)\} \\ &< \max \{D(\zeta, f\zeta), M(\zeta, \vartheta) + LN(\zeta, \vartheta)\}, \end{aligned}$$

where

$$\begin{aligned} \tau + F(H(f\zeta, f\vartheta)) &\leq F(M(\zeta, \vartheta) + LN(\zeta, \vartheta)) \text{ and} \\ M(\zeta, \vartheta) + LN(\zeta, \vartheta) &= \max \left\{ \nu(\zeta, \vartheta), D(\zeta, f\zeta), D(\vartheta, f\vartheta), \frac{D(\zeta, f\vartheta) + D(\vartheta, f\zeta)}{2} \right\} \\ &\quad + L \min \{D(\zeta, f\vartheta), D(\vartheta, f\zeta)\} \\ &\leq \max \{D(\zeta, f\zeta), D(\vartheta, f\vartheta)\}. \end{aligned}$$

The expression must be as follows: $D(\vartheta, f\vartheta) \leq D(\zeta, f\zeta)$ and then

$$\nu(x, \vartheta) \leq D(\vartheta, f\vartheta) < D(\zeta, f\zeta).$$

Otherwise, this value $D(\zeta, f\zeta) \leq D(\vartheta, f\vartheta)$,

$$\begin{aligned} \nu(x, \vartheta) &\leq D(\vartheta, f\vartheta) \\ &\leq \max \{D(\zeta, f\zeta), H(f\zeta, f\vartheta)\} \\ &< \max \left\{ \nu(\zeta, \vartheta), D(\zeta, f\zeta), D(\vartheta, f\vartheta), \frac{D(\zeta, f\vartheta) + D(\vartheta, f\zeta)}{2} \right\} \\ &\quad + L \min \{D(\zeta, f\vartheta), D(\vartheta, f\zeta)\} \\ &< \max \{D(\zeta, f\zeta), D(\vartheta, f\vartheta)\} \\ &\leq D(\vartheta, f\vartheta) \end{aligned}$$

leads to a contradiction from $F \in \mathcal{F}$. Consequently,

$$v(x, \vartheta) \leq D(\vartheta, f\vartheta) < D(\zeta, f\zeta)$$

for $x \in \mathfrak{B}_\vartheta$. Hence, it follows that

$$\begin{aligned} v(x, \zeta) &\leq \max\{v(x, \vartheta), v(\vartheta, \zeta)\} \\ &\leq \max\{v(x, \vartheta), D(\zeta, f\zeta)\} = D(\zeta, f\zeta). \end{aligned}$$

Thus $x \in \mathfrak{B}_\zeta$, ensuring the inclusion $\mathfrak{B}_\vartheta \subseteq \mathfrak{B}_\zeta$ for all elements \mathfrak{B}_ζ of \mathcal{A}_1 . As \mathfrak{B}_ϑ bounds the subfamily \mathcal{A}_1 from above in \mathcal{A} the conditions of Zorn's Lemma are satisfied, yielding the presence of a maximal member \mathfrak{B}_ξ indexed by some $\xi \in \Gamma$.

Now, we assert that $\xi \in f\xi$. To continue, we assume the opposite; $\xi \notin f\xi$, $\rho \in f\xi$, $\xi \neq \rho$ and $v(\xi, \rho) = D(\xi, f\xi)$. From (3), we get

$$\begin{aligned} &\tau + F(D(\rho, f\rho)) \leq \tau + F(H(f\xi, f\rho)) \\ &\leq F\left(\max\left\{v(\xi, \rho), D(\xi, f\xi), D(\rho, f\rho), \frac{D(\xi, f\rho) + D(\rho, f\xi)}{2}\right\}\right) + L \min\{D(\xi, f\rho), D(\rho, f\xi)\} \\ &\leq F(\max\{D(\xi, f\xi), D(\rho, f\rho)\}) \end{aligned}$$

which implies that

$$\begin{aligned} \tau + F(D(\rho, f\rho)) &\leq F(D(\xi, f\xi)) \\ F(D(\rho, f\rho)) &< F(D(\xi, f\xi)) \end{aligned}$$

and from $F \in \mathcal{F}$, we have $D(\rho, f\rho) < D(\xi, f\xi)$. Otherwise the $D(\rho, f\rho) > D(\xi, f\xi)$ state would create a $D(\rho, f\rho) < D(\rho, f\rho)$ conflict.

Now if $y \in \mathfrak{B}_\rho$, then

$$v(y, \rho) \leq D(\rho, f\rho) < D(\xi, f\xi) \quad (4)$$

and for $\xi \in \Gamma$

$$v(y, \xi) \leq \max\{v(y, \rho), v(\rho, \xi)\} \leq D(\xi, f\xi)$$

Leading to $y \in \mathfrak{B}_\xi$. Hence $\mathfrak{B}_\rho \subseteq \mathfrak{B}_\xi$.

Since (4)

$$D(\xi, f\xi) > D(\rho, f\rho)$$

implies $\xi \notin \mathfrak{B}_\rho$. Therefore $\mathfrak{B}_\xi \not\subseteq \mathfrak{B}_\rho$. This contradicts the assumption that \mathfrak{B}_ρ is maximal. Therefore, $\xi \in f\xi$, indicating that ξ remains fixed under f . Hence, the proof is complete. \square

We now present the consequences of the above theorem:

Corollary 3.3. Let (Γ, v) be a spherically complete ultrametric space and $f : \Gamma \rightarrow \Gamma$ is a generalized rational type F -contraction on Γ such as: for all $\eta, \varrho \in \Gamma$, $F \in \mathcal{F}$, $\tau > 0$,

$$\begin{aligned} v(f\eta, f\varrho) &> 0 \Rightarrow \tau + F(v(f\eta, f\varrho)) \leq F(M(\eta, \varrho)) \\ M(\eta, \varrho) &= \max\left\{v(\eta, \varrho), v(\eta, f\eta), v(\varrho, f\varrho), \frac{v(\eta, f\eta)v(\varrho, f\varrho)}{1 + v(\eta, \varrho)}\right\}. \end{aligned}$$

Then f has a fixed point.

Corollary 3.4. Let (Γ, v) be a spherically complete ultrametric space, and let $f : \Gamma \rightarrow \Gamma$ be an F -contraction defined as follows, such that for all $\eta, \varrho \in \Gamma$, $F \in \mathcal{F}$ and $\tau > 0$,

$$v(f\eta, f\varrho) > 0 \Rightarrow \tau + F(v(f\eta, f\varrho)) \leq F(\max\{v(\eta, \varrho), v(\eta, f\eta), v(\varrho, f\varrho)\}).$$

Then f has a fixed point.

Based on these given results, we give the following result for single-valued mappings:

Corollary 3.5. Let (Γ, v) be a spherically complete ultrametric space and $f : \Gamma \rightarrow \Gamma$ is a contraction mapping such as: for all $\eta, \varrho \in \Gamma$,

$$v(f\eta, f\varrho) \leq \max\{v(\eta, \varrho), v(\eta, f\eta), v(\varrho, f\varrho)\}.$$

Then f has a fixed point.

We now examine a simple example of the first theorem we have presented:

Example 3.2. Let $\Gamma = \mathbb{N} \cup \{0\}$, and let v be the ultrametric on Γ given by;

$$v(\gamma, \xi) = \begin{cases} \max\left\{\frac{1}{\gamma}, \frac{1}{\xi}\right\} & , \gamma \neq \xi \\ 0 & , \gamma = \xi \\ \frac{1}{\gamma} & , \xi = 0 \end{cases}$$

for all $\gamma, \xi \in \Gamma$. Define $f : \Gamma \rightarrow \Gamma$ by

$$f\gamma = \begin{cases} \{0\} & , \gamma = 0, \gamma = 1 \\ \{\gamma e^{\gamma+2}, 0\} & , \gamma > 1 \end{cases}.$$

Consider $F(\varpi) = -\frac{1}{\varpi^\alpha}$ for $\varpi > 0$, $0 < \alpha < 1$ (and $\alpha < \kappa$). Hence, the mapping f satisfies the F -contractive condition of rational type with $\tau = \frac{1}{4}$. To verify this, we consider the following cases. First, observe that

$$\gamma, \xi \in \mathbb{N}, v(f\gamma, f\xi) > 0 \iff (\xi > 1 \text{ and } \gamma = 0 \text{ or } \gamma = 1) \text{ or } (\gamma > \xi > 1).$$

Possibility 1. For $\gamma = 0$ and $\xi > 1$, it can be seen that

$$H(f\gamma, f\xi) = \frac{1}{\xi e^{\xi+2}} \quad \text{and} \quad M(\gamma, \xi) = \frac{1}{\xi}.$$

Then,

$$\begin{aligned} \tau \leq H(f\gamma, f\xi)^{-\alpha} - M(\gamma, \xi)^{-\alpha} &= \left(\frac{1}{\xi e^{\xi+2}}\right)^{-\alpha} - \left(\frac{1}{\xi}\right)^{-\alpha} \\ &= (\xi e^{\xi+2})^\alpha - (\xi)^\alpha \\ &= \xi^\alpha (e^{\alpha(\xi+2)} - 1). \end{aligned}$$

Hence,

$$\tau \leq \xi^\alpha (e^{\alpha(\xi+2)} - 1).$$

Possibility 2. For $\gamma = 1$ and $\xi > 1$, as a result,

$$H(f\gamma, f\xi) = \frac{1}{\xi e^{\xi+2}} \quad \text{and} \quad M(\gamma, \xi) = 1.$$

Therefore,

$$\begin{aligned} \tau \leq H(f\gamma, f\xi)^{-\alpha} - M(\gamma, \xi)^{-\alpha} &= \left(\frac{1}{\xi e^{\xi+2}}\right)^{-\alpha} - (1)^{-\alpha} \\ &= (\xi e^{\xi+2})^\alpha - 1^\alpha \\ &= \xi^\alpha e^{\alpha(\xi+2)} - 1. \end{aligned}$$

Thereby,

$$\tau \leq \xi^\alpha (e^{\alpha(\xi+2)} - 1) \leq \xi^\alpha e^{\alpha(\xi+2)} - 1.$$

Possibility 3. For $\gamma > \xi > 1$, we obtain

$$H(f\gamma, f\xi) = \frac{1}{\xi e^{\xi+2}} \quad \text{and} \quad M(\gamma, \xi) = \frac{1}{\xi}.$$

It follows that

$$\begin{aligned} \tau \leq H(f\gamma, f\xi)^{-\alpha} - M(\gamma, \xi)^{-\alpha} &= \left(\frac{1}{\xi e^{\xi+2}}\right)^{-\alpha} - \left(\frac{1}{\xi}\right)^{-\alpha} \\ &= (\xi e^{\xi+2})^\alpha - (\xi)^\alpha \\ &= \xi^\alpha (e^{\alpha(\xi+2)} - 1). \end{aligned}$$

Hence,

$$\tau \leq \xi^\alpha (e^{\alpha(\xi+2)} - 1).$$

As a consequence, for $F(\varpi) = -\frac{1}{\varpi^\alpha}$, ($\varpi > 0$, $0 < \alpha < 1$) we obtain $\tau \leq \inf_{\xi \geq 2} \xi^\alpha (e^{\alpha(\xi+2)} - 1) = 2^\alpha (e^{4\alpha} - 1)$; for instance $\alpha = 1/2$, this gives $\tau \leq \sqrt{2}(e^2 - 1) \approx 9.038$.

This example illustrates that the contractive condition is satisfied for a genuinely nonlinear F -function and cannot be reduced to a classical Nadler-type contraction.

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