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Convergence in the Variation Seminorm by Generalized Kantorovich-Type Szász -Mirakyan Operators Constructed via Appell Polynomials

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Abstract

ABSTRACT. The aim of this study is to investigate the variation detracting property and convergence in variation of the generalized Kantorovich type Szász-Mirakyan operators constructed via Appell polynomials in the space of functions of bounded variation. These problems are examined with respect to the variation seminorm. Additionally, the rate of convergence is analyzed in terms of total variation.
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1 Introduction

In 1950, Szász [1] introduced a well-known sequence of operators defined by the formula:

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{1}$$

where $x \in [0,\infty)$ and $f \in C[0,\infty)$. It may be mentioned that these operators are also examples of positive approximation processes discovered by Korovkin [2]. Numerous researchers have since explored various generalizations of the Szász operators.

Among these efforts, Jakimovski and Leviatan [3] proposed a broader class of Szász-type operators by incorporating Appell polynomials $p_k(x)$, which are defined through the following generating function:

$$g(u)e^{nx} = \sum_{k=0}^{\infty} p_k(x)u^k, \tag{2}$$

where $g(u) = \sum_{k=0}^{\infty} a_k u^k$ is an analytic function within the disc $|u| < R$ for some $R > 1$, and $g(1) \neq 0$. Assuming $p_k(x) \geq 0$ for all $x \in [0,\infty)$, they defined a sequence of positive linear operators as follows:

$$P_n(f;x) = \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right). \tag{3}$$

In the special case where $g(u) = 1$, the generating function (2) yields $p_k(x) = \frac{x^k}{k!}$, and consequently the operator in (3) reduces to the classical Szász operator in (1).

In this study, we present a generalization of the Generalized Kantorovich-type Szász-Mirakyan Operators defined via Appell polynomials. To proceed, we first need the following preliminary results, which will be used in our study [4].

Theorem 1.1. *Let $\{P_k(x)\}_{k=0}^{\infty}$ be a sequence of polynomials. Then the following are equivalent:*

1. *$\{P_k(x)\}_{k=0}^{\infty}$ is a sequence of Appell polynomials.*
2. *$\{P_k(x)\}_{k=0}^{\infty}$ has a generating function of the form*

$$A(t)e^{xt} = \sum_{k=0}^{\infty}\frac{p_k(x)}{k!}t^k, \tag{4}$$

where $A(t)$ is a formal power series independent of x with $A(0) \neq 0$.

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3. *There exists a sequence* $\{a_k\}_{k=0}^{\infty}$ *with* $a_0 \neq 0$ *such that*

$$P_k(x) = \sum_{r=0}^{k} a_{k-r} x^r$$

4. *For every* $k \geq 1$*, we have*

$$P_k'(x) = kP_{k-1}(x).$$

Under the above restrictions, the following positive linear operators have been introduced:

$$(K_n f)(x) = n\frac{e^{-nx}}{A(t)} \sum_{k=0}^{\infty} \frac{P_k(nx)}{k!} \int_{k/n}^{(k+1)/n} f(s)\,ds. \tag{5}$$

For $t = 1$, $x \in [0, \infty)$:

$$(K_n^* f)(x) = n\frac{e^{-nx}}{A(1)} \sum_{k=0}^{\infty} \frac{P_k(nx)}{k!} \int_{k/n}^{(k+1)/n} f(s)\,ds.$$

Note that, Appell polynomials are significant and highly useful due to their special differential property, expressed as in (d) , which simplifies many analytical and computational tasks. This inherent structure makes them excellent candidates for use as basis functions in approximation theory and particularly effective in defining and studying linear positive operators. In addition, Appell polynomials constitute a wide, unifying family that encompasses several well-known classical polynomial systems, including Bernoulli, Euler, and Hermite polynomials, thereby increasing their versatility and scope. Their clear generating functions, straightforward recurrence formulas, and explicit representations further enhance their practicality. Because of these advantages, Appell polynomials play an important role in solving various differential equations, describing the behavior of operators, and constructing efficient approximation methods across both pure and applied mathematical contexts.We can present several well-known operators from the literature as particular cases of the operator family under consideration.

Case 1. Employing the generating functions of the Appell polynomials, Jakimovski and Leviatan formulated the operators in (3) and established their approximation properties in [11].

Case 2. In the particular case $A(t) = 1$, from (4) we deduce $P_n(x) = (nx)^k$ and the operators defined by (5) turn into classical Kantorovich-Type Szász–Mirakjan operators.

The aim of this paper is to establish the variation detracting property and the convergence in variation of the Generalized Kantorovich-type Szász-Mirakyan Operators constructed via Appell polynomials in the space of functions of bounded variation. The rate of approximation is given with respect to the variation seminorm. Our research is mainly based on the ideas developed in [9] and [10].

Let us define the moments as:

$$T_{r,n}(x) = \sum_{k=0}^{\infty} [k - nx]^r G_k(nx), \tag{6}$$

where

$$G_k(nx) = \frac{e^{-nx}}{A(1)} \frac{P_k(nx)}{k!}. \tag{7}$$

Lemma 1.2.

$$\begin{aligned}
T_{0,n}(x) &= 1,\\
T_{1,n}(x) &= \frac{A'(1)}{A(1)},\\
T_{2,n}(x) &= \frac{A''(1)}{A(1)} + \frac{A'(1)}{A(1)} + nx,\\
T_{3,n}(x) &= \frac{A'''(1)}{A(1)} + 3\frac{A''(1)}{A(1)} + (3nx+1)\frac{A'(1)}{A(1)} + nx,\\
T_{4,n}(x) &= \frac{A^{(4)}(1)}{A(1)} + 6\frac{A'''(1)}{A(1)} + (6nx+7)\frac{A''(1)}{A(1)} + (10nx-1)\frac{A'(1)}{A(1)} + 12(nx)^3 + 15(nx)^2 + nx.
\end{aligned}$$

Then there hold the following identities:

Proof. By taking $t = 1$ and nx instead of x in (4), we obtain

$$A(1)e^{nx} = \sum_{k=0}^{\infty} \frac{P_k(nx)}{k!}.$$

Using equation (6), this gives

$$T_{0,n}(x)=\sum_{k=0}^{\infty}\frac{e^{-nx}}{A(1)}\frac{P_k(nx)}{k!}=1.$$

Taking the derivative of both sides of equation (4) with respect to t, and then substituting $t=1$ and x with nx, we arrive at

$$A'(1)e^{nx}+nxA(1)e^{nx}=\sum_{k=1}^{\infty}\frac{P_k(nx)}{(k-1)!}. \tag{8}$$

Thus, we have

$$T_{1,n}(x)=\sum_{k=1}^{\infty}\frac{P_k(nx)}{(k-1)!}\frac{e^{-nx}}{A(1)}-nx\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\frac{e^{-nx}}{A(1)}=\frac{A'(1)}{A(1)}.$$

Taking the second derivative of equation (4) with respect to t yields:

$$[A''(t)+2xA'(t)+x^2A(t)]e^{nx}=\sum_{k=2}^{\infty}\frac{P_k(nx)}{(k-2)!}.$$

Taking $t=1$, substituting nx for x, and referring to equations (6) and (8), we arrive at

$$\begin{aligned}T_{2,n}(x)&=\sum_{k=0}^{\infty}(k-nx)^2\frac{P_k(nx)}{k!}\frac{e^{-nx}}{A(1)}\\&=\sum_{k=2}^{\infty}\frac{P_k(nx)}{(k-2)!}\frac{e^{-nx}}{A(1)}+(1-2nx)\sum_{k=1}^{\infty}\frac{P_k(nx)}{(k-1)!}\frac{e^{-nx}}{A(1)}+(nx)^2\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\frac{e^{-nx}}{A(1)}\\&=\frac{A''(1)}{A(1)}+\frac{A'(1)}{A(1)}+nx.\end{aligned}$$

By using similar techniques, one can obtain the moments for $r=3$ and 4; hence, we omit. □

2 Variation Detracting Property of Kantorovich-Type Generalization of K_n^* Operators

In this section, we present the variation-detracting property and discuss the Kantorovich-type generalization of the K_n^* operators.

Let I be a bounded or unbounded interval. Throughout this work, $V_{[I]}[f]$ stands for the total Jordan variation of the real-valued function f defined on I. We deal with the classes $TV(I)$ and $BV(I)$ of all functions of bounded variation on $I\subset\mathbb{R}$, endowed with the seminorm and norm, respectively,

$$\|f\|_{TV(I)}:=V_{[I]}[f],$$

and

$$\|f\|_{BV(I)}:=V_{[I]}[f]+|f(c)|,$$

where c is any fixed point of I. Some interesting properties of the space $TV(I)$ are presented in [8].

Theorem 2.1. *If $f\in TV[0,\infty)$, then*

$$V_{[0,\infty)}[K_n^*f]\le V_{[0,\infty)}[f] \tag{9}$$

and

$$\|K_n^*f\|_{BV[0,\infty)}\le\|f\|_{BV[0,\infty)}. \tag{10}$$

Proof. For convenience, write the Kantorovich-type generalization of the K_n^* operators as

$$(K_n^*f)(x)=\frac{e^{-nx}}{A(t)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}F_{k,n}, \tag{11}$$

where

$$F_{k,n}:=n\int_{k/n}^{(k+1)/n}f(s)\,ds=\int_0^1 f\left(\frac{v+k}{n}\right)dv.$$

Taking the derivative of (11),

$$\begin{aligned}(K_n^* f)'(x) &= \frac{-ne^{-nx}}{A(t)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}F_{k,n} + \frac{e^{-nx}}{A(t)}\sum_{k=0}^{\infty}\frac{P_k'(nx)}{k!}F_{k,n}\\ &= \frac{-ne^{-nx}}{A(t)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}F_{k,n} + \frac{ne^{-nx}}{A(t)}\sum_{k=1}^{\infty}k\frac{P_{k-1}(nx)}{k!}F_{k,n}\\ &= \frac{-ne^{-nx}}{A(t)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}F_{k,n} + \frac{ne^{-nx}}{A(t)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}F_{k+1,n}\\ &= \frac{ne^{-nx}}{A(t)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}(F_{k+1,n}-F_{k,n})\\ &= \frac{ne^{-nx}}{A(t)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\Delta F_{k,n},\end{aligned}$$

where $\Delta F_{k,n} = F_{k+1,n} - F_{k,n}$. Thus,

$$(K_n^* f)'(x) = \frac{ne^{-nx}}{A(t)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\Delta F_{k,n}. \tag{12}$$

Using representation (12),

$$\begin{aligned}\|K_n^* f\|_{TV[0,\infty)} = V_{[0,\infty)}[K_n^* f] &= \int_0^{\infty} |(K_n^* f)'(x)|\,dx\\ &= \int_0^{\infty}\left|\frac{ne^{-nx}}{A(t)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\Delta F_{k,n}\right|dx\\ &\le \frac{n}{A(t)}\sum_{k=0}^{\infty}|\Delta F_{k,n}|\int_0^{\infty} e^{-nx}\frac{P_k(nx)}{k!}\,dx.\end{aligned}$$

For $t=1$, the following holds $\dfrac{n}{A(t)}\displaystyle\int_0^{\infty} e^{-nx}\frac{P_k(nx)}{k!}\,dx = 1,$, we get

$$\|K_n^* f\|_{TV[0,\infty)} \le \sum_{k=0}^{\infty}|\Delta F_{k,n}|.$$

Now, setting $v_k := \frac{v+k}{n}$ for $0 \le k \le n$, $v_{-1} := 0$ and $v_{n+1} := 1$ (see, e.g., [8], p. 309), we have

$$\begin{aligned}\sum_{k=0}^{\infty}|F_{k+1,n}-F_{k,n}| &= \sum_{k=0}^{\infty}\left|\int_0^1 (f(v_{k+1})-f(v_k))\,dv\right|\\ &\le \int_0^1 \sum_{k=-1}^{\infty}|f(v_{k+1})-f(v_k)|\,dv\\ &\le V_{[0,\infty)}[f].\end{aligned}$$

The desired estimate (9) is now evident.

Since

$$(K_n^* f)(0) = n\int_{k/n}^{(k+1)/n} f(s)\,ds$$

and

$$\|f\|_{BV[0,\infty)} = V_{[0,\infty)}[f] + |f(0)|,$$

relation (10) is a result of (9). Indeed,

$$\begin{aligned}\|K_n^* f\|_{BV[0,\infty)} &= V_{[0,\infty)}[K_n^* f] + |K_n^* f(0)|\\ &\le V_{[0,\infty)}[f] + \left|n\int_{k/n}^{(k+1)/n} f(s)\,ds\right|.\end{aligned}$$

Since $f \in TV[0,\infty)$ and $\left|n\int_{k/n}^{(k+1)/n} f(s)\,ds\right| = |f(0)| \le |f(c)|$, where c is any fixed point of $[0,\infty)$, we get

$$\|K_n^* f\|_{BV[0,\infty)} \le \|f\|_{BV[0,\infty)}.$$

Thus, the proof of the theorem is complete. □

3 Rate of Approximation in TV-norm

This section deals with the rates of approximation K_n^*g to g in the variation seminorm. The space $AC(I)$ of all absolutely continuous real-valued functions on I is a closed subspace of $TV(I)$ with respect to the convergence induced by the seminorm $\|f\|_{TV(I)}$. In addition, it is well known that if $\lim_{n\to\infty} V_{[I]}[g_n - g] = 0$ for a sequence $(g_n)_{n\geq 1}$ in $AC(I)$, then also $g \in AC[0,\infty)$ and

$$V_I[g_n - g] = \int_I |g_n'(t) - g'(t)|\,dt = \|g_n' - g'\|,$$

where

$$\|f\| := \|f\|_{L_1(I)}.$$

So, convergence in variation of $(g_n)_{n\geq 1} \subset AC(I)$ to g exactly means the convergence of the derivatives g_n' to g' in the $L_1(I)$-norm [8].

Theorem 3.1. *Let $g'' \in AC[0,\infty)$. Then*

$$V_{[0,\infty)}[K_n^*g - g] \leq \frac{C}{n}\left(V_{[0,\infty)}[g] + V_{[0,\infty)}[g'']\right), \tag{13}$$

where $C > 1$ is a constant.

Proof. By Taylor's formula with integral remainder term, one has

$$g(t) = g(x) + (t-x)g'(x) + \frac{(t-x)^2}{2}g''(x) + \frac{1}{2}\int_x^t (t-v)^2 g'''(v)\,dv. \tag{14}$$

From (14) we obtain

$$(K_n^*g)'(x) = B_{0,n}(x)g(x) + B_{1,n}(x)g'(x) + B_{2,n}(x)g''(x) + (R_n g)(x), \tag{15}$$

where

$$B_{j,n}(x) = \frac{ne^{-nx}}{A(1)\,j!}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\int_0^1\left[\left(\frac{v+k+1}{n}\right)^j - \left(\frac{v+k}{n}\right)^j\right]dv, \qquad j = 0,1,2.$$

The remainder terms are given by

$$(R_{n,1}g)(x) = \frac{ne^{-nx}}{2A(1)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\int_0^1\left[\int_x^{\frac{v+k+1}{n}}\left(\frac{v+k+1}{n} - u\right)^2 g'''(u)\,du\right]dv, \tag{16}$$

$$(R_{n,2}g)(x) = \frac{ne^{-nx}}{2A(1)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\int_0^1\left[\int_x^{\frac{v+k}{n}}\left(\frac{v+k}{n} - u\right)^2 g'''(u)\,du\right]dv. \tag{17}$$

In view of Lemma 1 we obtain

$$B_{0,n}(x) = 0,$$
$$B_{1,n}(x) = 1,$$
$$B_{2,n}(x) = \frac{1}{n} + \frac{A'(1)}{A(1)}.$$

Hence by (14), (15), (16), and (17),

$$(K_n^*g)'(x) = g'(x) + \left(\frac{1}{n} + \frac{A'(1)}{A(1)}\right)g''(x) + (R_n g)(x).$$

Let's divide $(R_n g)(x)$ into four parts as follows:

$$(R_n g)(x) = \left[(R_{n,1.1}g)(x) + (R_{n,1.2}g)(x)\right] - \left[(R_{n,2.1}g)(x) + (R_{n,2.2}g)(x)\right],$$

where

$$(R_{n,1.1}g)(x) = \frac{ne^{-nx}}{2A(1)}\sum_{|\frac{v+k+1}{n}-x|\leq\delta}^{\infty}\frac{P_k(nx)}{k!}\int_0^1\int_x^{\frac{v+k+1}{n}}\left(\frac{v+k+1}{n} - u\right)^2 g'''(u)\,du\,dv,$$

$$(R_{n,1.2}g)(x) = \frac{ne^{-nx}}{2A(1)}\sum_{|\frac{v+k+1}{n}-x|>\delta}^{\infty}\frac{P_k(nx)}{k!}\int_0^1\int_x^{\frac{v+k+1}{n}}\left(\frac{v+k+1}{n} - u\right)^2 g'''(u)\,du\,dv,$$

and

$$(R_{n,2.1}g)(x) = \frac{ne^{-nx}}{2A(1)} \sum_{|\frac{v+k}{n}-x|\le\delta}^{\infty} \frac{P_k(nx)}{k!} \int_0^1 \int_x^{\frac{v+k}{n}} \left(\frac{v+k}{n} - u\right)^2 g'''(u)\,du\,dv,$$

$$(R_{n,2.2}g)(x) = \frac{ne^{-nx}}{2A(1)} \sum_{|\frac{v+k}{n}-x|>\delta}^{\infty} \frac{P_k(nx)}{k!} \int_0^1 \int_x^{\frac{v+k}{n}} \left(\frac{v+k}{n} - u\right)^2 g'''(u)\,du\,dv.$$

In order to estimate the integration domain of the double integral in the remainder terms (16), we divide the summation into different sums as following;

$$(R_{n,1.1}g)(x) = M_{2,n}g + M_{5,n}g \qquad and \qquad (R_{n,1.2}g)(x) = M_{1,n}g + M_{3,n}g + M_{4,n}g + M_{6,n}g,$$

Here $M_{j,n}g \quad for \quad j = 1, 2, \dots, 6$

$$M_{1,n}g = n\frac{e^{-nx}}{2A(1)} \sum_{\delta<x-\frac{v+k+1}{n}\le x}^{\infty} \frac{P_k(nx)}{k!} \int_0^{\frac{v+k+1}{n}} \left[\int_x^{\frac{v+k+1}{n}} \left(\frac{v+k+1}{n} - u\right)^2 g'''(u)\,du\right] dv$$

$$M_{2,n}g = n\frac{e^{-nx}}{2A(1)} \sum_{0<x-\frac{v+k+1}{n}\le\delta}^{\infty} \frac{P_k(nx)}{k!} \int_{\frac{v+k+1}{n}}^{x} \left[\int_x^{\frac{v+k+1}{n}} \left(\frac{v+k+1}{n} - u\right)^2 g'''(u)\,du\right] dv$$

$$M_{3,n}g = n\frac{e^{-nx}}{2A(1)} \sum_{\delta<x-\frac{v+k+1}{n}\le x}^{\infty} \frac{P_k(nx)}{k!} \int_0^{x} \left[\int_x^{\frac{v+k+1}{n}} \left(\frac{v+k+1}{n} - u\right)^2 g'''(u)\,du\right] dv$$

$$M_{4,n}g = n\frac{e^{-nx}}{2A(1)} \sum_{\delta<\frac{v+k+1}{n}-x\le 1-x}^{\infty} \frac{P_k(nx)}{k!} \int_0^{x} \left[\int_x^{\frac{v+k+1}{n}} \left(\frac{v+k+1}{n} - u\right)^2 g'''(u)\,du\right] dv$$

$$M_{5,n}g = n\frac{e^{-nx}}{2A(1)} \sum_{0<\frac{v+k+1}{n}-x\le\delta}^{\infty} \frac{P_k(nx)}{k!} \int_x^{\frac{v+k+1}{n}} \left[\int_x^{\frac{v+k+1}{n}} \left(\frac{v+k+1}{n} - u\right)^2 g'''(u)\,du\right] dv$$

$$M_{6,n}g = n\frac{e^{-nx}}{2A(1)} \sum_{\delta<\frac{v+k+1}{n}-x\le 1-x}^{\infty} \frac{P_k(nx)}{k!} \int_{\frac{v+k+1}{n}}^{1} \left[\int_x^{\frac{v+k+1}{n}} \left(\frac{v+k+1}{n} - u\right)^2 g'''(u)\,du\right] dv.$$

It is easy to see that, $M_{1,n}g + M_{2,n}g = -M_{4,n}g \quad and \qquad M_{5,n}g + M_{6,n}g = -M_{3,n}g$. So one has,

$$|M_{1,n}g + M_{2,n}g| \le |M_{1,n}g| + |M_{2,n}g|$$

and

$$|M_{5,n}g + M_{6,n}g| \le |M_{5,n}g| + |M_{6,n}g|.$$

Now we only estimate $M_{j,n}g$ for $j = 1, 2, 5, 6$, respectively. We first show the norm of $M_{1,n}g$ in L_1-norm:

$$\begin{aligned}
\|M_{1,n}g\| &\le \int_0^\infty n\frac{e^{-nx}}{2A(1)} \sum_{\delta<x-\frac{v+k+1}{n}\le x}^{\infty} \frac{P_k(nx)}{k!} \int_0^{\frac{v+k+1}{n}} \left|\int_x^{\frac{v+k+1}{n}} \left(\frac{v+k+1}{n} - u\right)^2 g'''(u)\,du\right| dv \\
&\le \int_0^\infty n\frac{e^{-nx}}{2A(1)} \sum_{\delta<x-\frac{v+k+1}{n}\le x}^{\infty} \frac{P_k(nx)}{k!} \int_0^1 \left(\frac{v+k+1}{n} - x\right)^2 \left[\int_x^{\frac{v+k+1}{n}} |g'''(u)|\,du\right] dv \\
&\le \int_0^\infty n\frac{e^{-nx}}{2A(1)} \sum_{k=0}^{\infty} \frac{P_k(nx)}{k!} \frac{3(k-nx)^2 + 9(k-nx) + 7}{3n^2} \left[\int_x^{\frac{v+k+1}{n}} |g'''(u)|\,du\right] dv
\end{aligned}$$

Since $\frac{e^{-nx}}{2A(1)} \sum_{k=0}^{\infty} \frac{P_k(nx)}{k!}\left[3(k-nx)^2 + 9(k-nx) + 7\right] = \frac{3A''(1)}{A(1)} + \frac{12A'(1)}{A(1)} + 3nx + 7$, note that $\frac{3A''(1)}{A(1)} + \frac{12A'(1)}{A(1)} + 3nx \le 3nx + K$ (K>0), and so

$$\|M_{1,n}g\| \le \frac{1}{6n} \int_0^\infty (3nx + K)\,dx \int_x^{(v+k+1)/n} |g'''(u)|\,du.$$

Based on an approach presented by Totik [5] (see also [22, Section 9.3] and Grundmann [12]), it was shown in the case of the $S_n^* f$ that if $f \in L^1(0,\infty)$ and $F(u) = \int_0^x f(u)\,du$, then $\|S_n^* f - f\|_{L^1(0,\infty)} = O(n^{-1}) \quad (n \to \infty)$
$\iff \|x\Delta_\delta^2 F\|_{(TV+L^\infty)[\delta^2,\infty)} = O(\delta^2) \quad (\delta \to 0^+)$
$\iff f \in AC_{[0,\infty)}$ with $xf'(x) = \eta(x) \in TV[0,\infty)$ and $\eta(0) = 0$
This is a result of saturation since $O(1/n)$ is the optimal approximation order for the S_n^*. From the last inequality, we have

$$\|M_{1,n} g\| \le \frac{1}{6n} \int_0^\infty (3nx + K)\,dx \int_x^{\frac{\nu+k+1}{n}} |g'''(u)|\,du = O(1/n) \tag{18}$$

Analogously, $M_{2,n} g$ can be estimated by

$$\|M_{2,n} g\| \le \int_0^\infty n\frac{e^{-nx}}{2A(1)} \sum_{0<x-\frac{\nu+k+1}{n}\le\delta}^{\infty} \frac{P_k(nx)}{k!} \int_{\frac{\nu+k+1}{n}}^{x} \left| \int_x^{\frac{\nu+k+1}{n}} \left(\frac{\nu+k+1}{n} - u\right)^2 g'''(u)\,du \right| d\nu$$

$$\le \int_0^\infty n\frac{e^{-nx}}{2A(1)} \sum_{k=0}^{\infty} \frac{P_k(nx)}{k!} \frac{(k-nx)^2 + 2(1+x)(k-nx) + (1+x)^2}{n^2} \int_x^{\frac{\nu+k+1}{n}} |g'''(u)|\,du$$

as in the proof $M_{1,n} g$

$$\|M_{2,n} g\| = O(1/n). \tag{19}$$

As to term $M_{5,n} g$, we have

$$\|M_{5,n} g\| \le \int_0^\infty n\frac{e^{-nx}}{2A(1)} \sum_{k=0}^{\infty} \frac{P_k(nx)}{k!} \frac{(k-nx)^2 + 2(1+x)(k-nx) + (1+x)^2}{n^2} \int_x^{\frac{\nu+k+1}{n}} |g'''(u)|\,du$$

which yields

$$\|M_{5,n} g\| = O(1/n). \tag{20}$$

Finally to the next term $M_{6,n} g$, we have

$$\|M_{6,n} g\| \le \int_0^\infty n\frac{e^{-nx}}{2A(1)} \sum_{k=0}^{\infty} \frac{P_k(nx)}{k!} \frac{3(k-nx)^2 + 9(k-nx) + 1}{3n^2} \int_x^{\frac{\nu+k+1}{n}} |g'''(u)|\,du$$

this gives

$$\|M_{6,n} g\| = O(1/n). \tag{21}$$

Thus, altogether with the results in (18), (19), (20) and (21), we have

$$\|R_{n,1} g\| = 8O(1/n). \tag{22}$$

Similar to the proof of $R_{n,1} g$, we have

$$(R_{n,2.1} g)(x) = N_{2,n} g + N_{5,n} g \quad \text{and} \quad (R_{n,2.2} g)(x) = N_{1,n} g + N_{3,n} g + N_{4,n} g + N_{6,n} g.$$

Here $N_{j,n} g$ for $j = 1, \ldots, 6$.

$$N_{1,n} g = n\frac{e^{-nx}}{2A(1)} \sum_{\delta<x-\frac{\nu+k}{n}\le x}^{\infty} \frac{P_k(nx)}{k!} \int_0^{\frac{\nu+k}{n}} \left[\int_x^{\frac{\nu+k}{n}} \left(\frac{\nu+k}{n} - u\right)^2 g'''(u)\,du \right] d\nu,$$

$$N_{2,n} g = n\frac{e^{-nx}}{2A(1)} \sum_{0<x-\frac{\nu+k}{n}\le\delta}^{\infty} \frac{P_k(nx)}{k!} \int_{\frac{\nu+k}{n}}^{x} \left[\int_x^{\frac{\nu+k}{n}} \left(\frac{\nu+k}{n} - u\right)^2 g'''(u)\,du \right] d\nu,$$

$$N_{3,n} g = n\frac{e^{-nx}}{2A(1)} \sum_{\delta<x-\frac{\nu+k}{n}\le x}^{\infty} \frac{P_k(nx)}{k!} \int_0^{x} \left[\int_x^{\frac{\nu+k}{n}} \left(\frac{\nu+k}{n} - u\right)^2 g'''(u)\,du \right] d\nu,$$

$$N_{4,n} g = n\frac{e^{-nx}}{2A(1)} \sum_{\delta<\frac{\nu+k}{n}-x\le 1-x}^{\infty} \frac{P_k(nx)}{k!} \int_0^{x} \left[\int_x^{\frac{\nu+k}{n}} \left(\frac{\nu+k}{n} - u\right)^2 g'''(u)\,du \right] d\nu,$$

$$N_{5,n} g = n\frac{e^{-nx}}{2A(1)} \sum_{0<\frac{\nu+k}{n}-x\le\delta}^{\infty} \frac{P_k(nx)}{k!} \int_x^{\frac{\nu+k}{n}} \left[\int_x^{\frac{\nu+k}{n}} \left(\frac{\nu+k}{n} - u\right)^2 g'''(u)\,du \right] d\nu,$$

$$N_{6,n} g = n\frac{e^{-nx}}{2A(1)} \sum_{\delta<\frac{\nu+k}{n}-x\le 1-x}^{\infty} \frac{P_k(nx)}{k!} \int_{\frac{\nu+k}{n}}^{1} \left[\int_x^{\frac{\nu+k}{n}} \left(\frac{\nu+k}{n} - u\right)^2 g'''(u)\,du \right] d\nu.$$

So, we get

$$|(R_{n,2}g)(x)| \le 2\big(|N_{1,n}g| + |N_{2,n}g| + |N_{5,n}g| + |N_{6,n}g|\big).$$

Now we only estimate $N_{j,n}g$ for $j = 1, 2, 5, 6$ respectively. As in the proof of $R_{n,1}g$,

$$\|N_{1,n}g\| = O(1/n) \tag{23}$$

$$\|N_{2,n}g\| = O(1/n) \tag{24}$$

$$\|N_{5,n}g\| = O(1/n) \tag{25}$$

$$\|N_{6,n}g\| = O(1/n) \tag{26}$$

Thus, altogether with the result in (23), (24), (25) and (26), we have

$$\|R_{n,2}g\| = 8\,O(1/n).$$

So, one has

$$(K_n^* g)'(x) \le g'(x) + \Big(\frac{1}{n} + \frac{A'(1)}{A(1)}\Big) g''(x) + O\Big(\frac{1}{n}\Big).$$

According to Stein's inequality (see, e.g., [6],Theorem A10.1), one has

$$\|g''\|_{L_1[0,\infty)} \le C\sqrt{\|g'\|_{L_1[0,\infty)}\,\|g'''\|_{L_1[0,\infty)}}$$
$$\le C\big(\|g'\|_{L_1[0,\infty)} + \|g'''\|_{L_1[0,\infty)}\big).$$

where $C > 1$ is a constant.
Therefore,

$$\|(K_n^* g)' - g'\| \le \frac{C}{n}\big(\|g'\| + \|g'''\|\big).$$

This finally establishes the theorem.

□

Theorem 3.2. *Let f be a bounded function on $\mathbb{R}^+$. If f''' exists at a point $x \in \mathbb{R}^+$, then*

$$\lim_{n\to\infty} n\Big[(K_n^* f)'(x) - f'(x)\Big] = \Big(1 + \frac{A'(1)}{A(1)}\Big) f''(x) + \frac{x}{2} f'''(x).$$

Proof. By Taylor's formula, we have

$$f(v_k) = f(x) + (v_k - x)f'(x) + \frac{(v_k - x)^2}{2} g''(x) + \frac{(v_k - x)^3}{6} f'''(x) + (v_k - x)^3 h(v_k - x),$$

where $h(y) := h_x(y)$ is bounded for all y, $|h(y)| \le M$, and converges to zero with y. So,

$$(K_n^* f)'(x) = B_{0,n}(x)f(x) + B_{1,n}(x)f'(x) + B_{2,n}(x)f''(x) + B_{3,n}(x)f'''(x) + (R_n f)(x) \tag{27}$$

Here

$$B_{j,n}(x) := n\frac{e^{-nx}}{A(1)j!}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\int_0^1 \Big[\Big(\frac{v+k+1}{n} - x\Big)^j - \Big(\frac{v+k}{n} - x\Big)^j\Big]dv$$

and the remainder terms are given by

$$R_1 f(x) = n\frac{e^{-nx}}{A(1)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\int_0^1 \Big(\frac{v+k+1}{n} - x\Big)^3 h\Big(\frac{v+k+1}{n} - x\Big)dv \tag{28}$$

and

$$R_2 f(x) = n\frac{e^{-nx}}{A(1)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\int_0^1 \Big(\frac{v+k}{n} - x\Big)^3 h\Big(\frac{v+k}{n} - x\Big)dv \tag{29}$$

In view of Lemma 1, we obtain

$$B_{0,n}(x)=0,$$
$$B_{1,n}(x)=1,$$
$$B_{2,n}(x)=\frac{1}{n}+\frac{A'(1)}{nA(1)},$$
$$B_{3,n}(x)=\frac{1}{2n^2}\Big[\frac{A''(1)}{A(1)}+\frac{3A'(1)}{A(1)}+nx\Big]+\frac{7}{12n^2}.$$

So by (27), (28) and (29), we have

$$(K_n^*f)'(x)=f'(x)+\Big(\frac{1}{n}+\frac{A'(1)}{nA(1)}\Big)f''(x)+\Big(\frac{1}{2n^2}\Big[\frac{A''(1)}{A(1)}+\frac{3A'(1)}{A(1)}+nx\Big]+\frac{7}{12n^2}\Big)f'''(x)+[R_1(x)-R_2(x)].$$

Let us study only the remainders R_1 and R_2:

$$\begin{aligned}(R_1f)(x)&=n\frac{e^{-nx}}{A(1)}\Bigg\{\sum_{|\frac{v+k+1}{n}-x|<\delta}+\sum_{|\frac{v+k+1}{n}-x|\geq\delta}\Bigg\}\frac{P_k(nx)}{k!}\int_0^1\Big(\frac{v+k+1}{n}-x\Big)^3h\Big(\frac{v+k+1}{n}-x\Big)dv\\&=\sum_1+\sum_2\end{aligned}$$

For any $\varepsilon>0$ there exists a $\delta>0$ such that $|h(y)|<\varepsilon$ for $0\leq y\leq\delta$.

$$\begin{aligned}\sum_1&=n\frac{e^{-nx}}{A(1)}\sum_{|\frac{v+k+1}{n}-x|<\delta}\frac{P_k(nx)}{k!}\int_0^1\Big(\frac{v+k+1}{n}-x\Big)^3h\Big(\frac{v+k+1}{n}-x\Big)dv\\&\leq n\varepsilon\frac{e^{-nx}}{A(1)}\sum_{|\frac{v+k+1}{n}-x|<\delta}\frac{P_k(nx)}{k!}\Big[\frac{(k-nx)^3}{n^3}+\frac{9(k-nx)^2}{2n^3}+\frac{7(k-nx)}{n^3}+\frac{15}{4n^3}\Big]\\&=\frac{\varepsilon}{n^2}\Big[\frac{A'''(1)}{A(1)}+3\frac{A''(1)}{A(1)}+(3nx+1)\frac{A'(1)}{A(1)}+nx\Big]+\frac{9\varepsilon}{2n^2}\Big[\frac{A''(1)}{A(1)}+\frac{A'(1)}{A(1)}+nx\Big]+\frac{7\varepsilon}{n^2}\frac{A'(1)}{A(1)}\\&\leq\ \tilde{\varepsilon}\end{aligned}$$

Secondly, we calculate $\sum_2$:

$$\begin{aligned}\sum_2&=n\frac{e^{-nx}}{A(1)}\sum_{|\frac{v+k+1}{n}-x|\geq\delta}\frac{P_k(nx)}{k!}\int_0^1\Big(\frac{v+k+1}{n}-x\Big)^3h\Big(\frac{v+k+1}{n}-x\Big)dv\\&\leq\frac{n}{\delta^2}\frac{e^{-nx}}{A(1)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\int_0^1\Big(\frac{v+k+1}{n}-x\Big)^5h\Big(\frac{v+k+1}{n}-x\Big)dv\\&\leq n\frac{M}{\delta^2}\frac{e^{-nx}}{A(1)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\int_0^1\Big(\frac{v+k+1}{n}-x\Big)^5dv\\&\leq n\frac{M}{\delta^2}\frac{e^{-nx}}{A(1)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\Bigg[\frac{(k-nx)^5}{n^5}+\frac{25(k-nx)^4}{2n^5}+\frac{130(k-nx)^3}{3n^5}\\&+\frac{135(k-nx)^2}{2n^5}+\frac{51(k-nx)}{n^5}+\frac{93}{n^5}\Bigg]\\&\leq\varepsilon.\end{aligned}$$

(see, e.g.,[7], Theorem 3. Now we can calculate $R_2(x)$.

$$\begin{aligned}(R_2f)(x)&=n\frac{e^{-nx}}{A(1)}\Bigg\{\sum_{|\frac{v+k}{n}-x|<\delta}+\sum_{|\frac{v+k}{n}-x|\geq\delta}\Bigg\}\frac{P_k(nx)}{k!}\int_0^1\Big(\frac{v+k}{n}-x\Big)^3h\Big(\frac{v+k}{n}-x\Big)dv\\&=\sum_3+\sum_4\end{aligned}$$

Here we evaluate $\sum_3$ and $\sum_4$ as:

$$
\begin{aligned}
\sum_3 &= n\frac{e^{-nx}}{A(1)}\sum_{|\frac{v+k}{n}-x|<\delta}\frac{P_k(nx)}{k!}\int_0^1\Big(\frac{v+k}{n}-x\Big)^3 h\Big(\frac{v+k}{n}-x\Big)dv\\
&\le n\tilde{\varepsilon}\frac{e^{-nx}}{A(1)}\sum_{|\frac{v+k}{n}-x|<\delta}\frac{P_k(nx)}{k!}\Big[\frac{(k-nx)^3}{n^3}+\frac{3(k-nx)^2}{2n^3}+\frac{(k-nx)}{n^3}+\frac{1}{4n^3}\Big]\\
&\le \tilde{\varepsilon}
\end{aligned}
$$

$$
\begin{aligned}
\sum_4 &= n\frac{e^{-nx}}{A(1)}\sum_{|\frac{v+k}{n}-x|\ge\delta}\frac{P_k(nx)}{k!}\int_0^1\Big(\frac{v+k}{n}-x\Big)^3 h\Big(\frac{v+k}{n}-x\Big)dv\\
&\le \frac{n}{\delta^2}\frac{e^{-nx}}{A(1)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\int_0^1\Big(\frac{v+k}{n}-x\Big)^5 h\Big(\frac{v+k}{n}-x\Big)dv\\
&\le n\frac{M}{\delta^2}\frac{e^{-nx}}{A(1)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\int_0^1\Big(\frac{v+k}{n}-x\Big)^5 dv\\
&\le n\frac{M}{\delta^2}\frac{e^{-nx}}{A(1)}\sum_{k=0}^{\infty}\frac{P_k(nx)}{k!}\Big[\frac{(k-nx)^5}{n^5}+\frac{5(k-nx)^4}{2n^5}+\frac{10(k-nx)^3}{3n^5}\\
&+\frac{5(k-nx)^2}{2n^5}+\frac{k-nx}{n^5}+\frac{1}{6n^5}\Big]\\
&\le \tilde{\varepsilon}
\end{aligned}
$$

This shows that

$$\lim_{n\to\infty} n\big[(K_n^* f)'(x)-f'(x)\big]=\Big(1+\frac{A'(1)}{A(1)}\Big)f''(x)+\frac{x}{2}f'''(x).$$

□

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