PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 19 · 2026 · Pages 12–18

On Direct Estimates and Approximation Results by the Kantorovich Operators in Weighted Grand Lebesgue Spaces

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Abstract

We give some new embeddings results for weighted grand Lebesgue spaces $L_{\omega}^{p),\delta}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a open bounded subset. We also obtain the boundedness of the Kantorovich operator K_n in $L_{\omega}^{p),\delta}[0,1]$. In addition, we establish two direct estimates by K-functionals of the rate of approximation in $L_{\omega}^{p),\delta}[0,1]$. We generalize the direct estimate inequality in classical Lebesgue spaces $L^p[0,1]$ to $L_{\omega}^{p),\delta}[0,1]$ using the boundedness of the Hardy-Littlewood maximal operator in $L_{\omega}^{p),\delta}[0,1]$. Finally, we obtain similar results in [8] for the spaces $L_{\omega}^{p),\delta}[0,1]$.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a measurable set of Lebesgue measure $|\Omega| < +\infty$, and $f : \Omega \to \mathbb{R}^d$, $f = \left(f^1, f^2, ..., f^d\right)$ be a mapping defined on Ω. In 1992, T. Iwaniec and C. Sbordone [11] obtained the integrability of the Jacobian determinant, and defined the grand Lebesgue space $L^{p)}(\Omega)$ such that the Jacobian determinant of f is locally integrable for $1 < p < \infty$. The (weighted or nonweighted) grand Lebesgue spaces played an important role in PDEs theory and in Function sapces theory (see [2], [7], [9], [10], [15], [16]). Some authors, such as Danelia and Kokilashvili [5]-[6], Tsanava and Kokilashvili [17] studied on approximation in subspaces of grand Lebesgue spaces. In particular, in [5] the authors obtained direct and inverse theorems of approximation theory in non-weighted grand Lebesgue spaces, and later in [6], in weighted grand Lebesgue spaces. Moreover, Aydın and Akgün [1] investigate trigonometric approximation, and prove the direct and inverse theorems in weighted grand variable exponent Lebesgue spaces.

Definition 1.1. (See [3]) Let $1 < p < \infty$. If a function δ satisfies the following conditions

(i) δ is left continuous on $(0, p-1)$,

(ii) $\delta(0+) = 0$,

(iii) $0 < \delta \leq 1$,

(iv) $\delta(\varepsilon)^{\frac{1}{p-\varepsilon}}$ is increasing in ε,

then it is said to be in class be in the class ß_p. It is easy to check that functions in ß_p are increasing.

Definition 1.2. (See [3]) Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a measurable set of Lebesgue measure $|\Omega| < +\infty$, let $1 < p < \infty$ and let $\delta \in \text{ß}_p$. A generalization of the grand Lebesgue space $L^{p),\delta}(\Omega)$ is the Banach Function space defined on the set of all measurable functions with respect to δ such that

$$\|f\|_{p),\delta} := \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\frac{1}{|\Omega|} \int_{\Omega} |f(y)|^{p-\varepsilon} \, dy \right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

When $\delta(\varepsilon) = \varepsilon^{\theta} = 1$ $(\theta = 0)$ the spaces $L^{p),1}(\Omega)$ reduce to Lebesgue spaces $L^p(\Omega)$ and $\delta(\varepsilon) = \varepsilon^{\theta} = \varepsilon$ $(\theta = 1)$ the spaces $L^{p),\varepsilon}(\Omega)$ reduce to grand Lebesgue spaces $L^{p)}(\Omega)$.

The generalized weighted Grand Lebesgue spaces are the spaces $L_{\omega}^{p),\delta}(\Omega)$, $1 < p < \infty$, equipped with the norm

$$\|f\|_{L_{\omega}^{p),\delta}} := \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\frac{1}{|\Omega|} \int_{\Omega} |f(y)|^{p-\varepsilon} \, \omega(y) dy \right)^{\frac{1}{p-\varepsilon}} < +\infty. \tag{1}$$

Except for the trivial case of ω constant, the space $L_{\omega}^{p),\delta}(\Omega)$, $1 < p < \infty$, is not rearrangement-invariant. The extension of grand Lebesgue spaces to the weighted case has a relevant difference from the case of Lebesgue spaces (To learn more infromation, especailly examples about the space $L_{\omega}^{p),\delta}(\Omega)$, please see [9], [14] and [18]).

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Theorem 1.1. *Let* $\omega^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega)$ *and* $1 < p < \infty$*. Then the following continuous embedding result*

$$L^p_\omega \hookrightarrow L^{p),\delta}_\omega \hookrightarrow L^{p-\varepsilon}_\omega \hookrightarrow L^1 \tag{2}$$

is valid for $0 < \varepsilon < p-1$*.*

Proof. Firstly, we show that $L^{p),\delta}_\omega \hookrightarrow L^{p-\varepsilon}_w$. Using the definition of supremum we obtain

$$\delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_\omega} \le \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_\omega} = \|f\|_{L^{p),\delta}_\omega}$$

and

$$\|f\|_{L^{p-\varepsilon}_\omega} \le \delta(\varepsilon)^{-\frac{1}{p-\varepsilon}} \|f\|_{L^{p),\delta}_\omega},\ L^{p),\delta}_\omega \hookrightarrow L^{p-\varepsilon}_\omega,$$

where $\|.\|_{L^{p-\varepsilon}_\omega}$ denotes the normalized norm. Moreover, we can write

$$\begin{aligned}
\|f\|_{L^{p),\delta}_\omega} &= \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_\omega} \\
&\le C \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^p_\omega} \\
&\le C\delta(p-1)\|f\|_{L^p_\omega},
\end{aligned}$$

where $C > 0$ is the constant for the continuous embedding $L^p_\omega \hookrightarrow L^{p-\varepsilon}_w$ due to $|\Omega| < +\infty$. So we have $L^p_\omega \hookrightarrow L^{p),\delta}_\omega$. Finally, we prove that $L^{p-\varepsilon}_\omega \hookrightarrow L^1$. For this, let $f \in L^p_\omega(\Omega)$ and B be a ball. Then, by Hölder's inequality

$$\int_B |f(x)|\,dx \le \left(\int_B \omega(x)^{-\frac{1}{p-1}}\,dx\right)^{\frac{1}{p'}} \left(\int_B |f(x)|^p\,\omega(x)dx\right)^{\frac{1}{p}} < \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Hence, we get $L^p_\omega \hookrightarrow L^1_{loc}$ for $1 < p < \infty$. Since $0 < \varepsilon < p-1$, $1 < p-\varepsilon$ and $|\Omega| < +\infty$, then we arrive $L^{p-\varepsilon}_\omega \hookrightarrow L^1$. ☐

Theorem 1.2. *The inclusion* $L^{p),\delta}_\omega(\Omega) \subseteq L^{q),\delta}_\omega(\Omega)$ *holds for* $0 < \varepsilon < p-1$ *and* $0 < \varepsilon < q-1$ *if and only if there exists a* $C > 0$ *such that*

$$\|f\|_{L^{q),\delta}_\omega} \le C\,\|f\|_{L^{p),\delta}_\omega} \tag{3}$$

for all $f \in L^{p),\delta}_\omega(\Omega)$*.*

Proof. Define the sum norm on $L^{p),\delta}_\omega(\Omega)$ by

$$\|f\| = \|f\|_{L^{p),\delta}_\omega} + \|f\|_{L^{q),\delta}_\omega}$$

for $f \in L^{p),\delta}_\omega(\Omega)$. Firstly, we show that $\left(L^{p),\delta}_\omega(\Omega), \|.\|\right)$ is a Banach space. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^{p),\delta}_\omega(\Omega)$. Then there exists a $N(\eta) > 0$ such that, whenever $n, m > N(\eta)$ we have

$$\|f_n - f_m\|_{L^{p),\delta}_\omega} = \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f_n - f_m\|_{L^{p-\varepsilon}_\omega} < \eta$$

and

$$\|f_n - f_m\|_{L^{q),\delta}_\omega} = \sup_{0<\varepsilon<q-1} \delta(\varepsilon)^{\frac{1}{q-\varepsilon}} \|f_n - f_m\|_{L^{q-\varepsilon}_\omega} < \eta$$

for all $\eta > 0$ by the definition of the norm $\|.\|$. So the sequence $(f_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $L^{p),\delta}_\omega(\Omega)$ and $L^{q),\delta}_\omega(\Omega)$, and $(f_n)_{n\in\mathbb{N}}$ converges to functions $f \in L^{p),\delta}_\omega(\Omega)$ and $g \in L^{q),\delta}_\omega(\Omega)$ respectively. Using the embedding $L^{p),\delta}_\omega \hookrightarrow L^{p-\varepsilon}_\omega$ for $\varepsilon \in (0, p-1)$ by (2) in Theorem 1.1 we find that there is a subsequence $\left(f_{n_i}\right)_{i\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $f_{n_i} \to f$ (a.e.). Since the sequence $(f_n)_{n\in\mathbb{N}}$ converges to g in $L^{q),\delta}_\omega(\Omega)$, then it is easy to prove that $\left(f_{n_i}\right)_{n\in\mathbb{N}}$ converges to g in $L^{q),\delta}_\omega(\Omega)$ and $f_{n_i} \to g$ (a.e.) due to $L^{q),\delta}_\omega \hookrightarrow L^{q-\varepsilon}_\omega$ for $\varepsilon \in (0, q-1)$. Hence we obtain a subsequence $\left(f_{n_{i_k}}\right)$ of $\left(f_{n_i}\right)$ such that $f_{n_{i_k}} \to g$ (a.e). From $L^{p),\delta}_\omega(\Omega) \subseteq L^{q),\delta}_\omega(\Omega)$ and the following inequality

$$|f(x) - g(x)| \le \left|f(x) - f_{n_{i_k}}(x)\right| + \left|f_{n_{i_k}}(x) - g(x)\right|$$

we have $f = g$ (a.e.) and $f_n \to f$ in $L^{p),\delta}_\omega(\Omega)$ with respect to the norm $\|.\|$. If we define the unit function I from $\left(L^{p),\delta}_\omega(\Omega), \|.\|\right)$ into $\left(L^{p),\delta}_\omega(\Omega), \|.\|_{L^{p),\delta}_\omega}\right)$ then I is continuous by the inequality $\|I(f)\|_{L^{p),\delta}_\omega} = \|f\|_{L^{p),\delta}_\omega} \le \|f\|$. Due to the Banach's theorem, the function I is a homeomorphism [4]. Hence the norms $\|.\|$ and $\|.\|_{L^{p),\delta}_\omega}$ are equivalent. Thus there exists a $C > 0$ such that $\|f\| \le C\,\|f\|_{L^{p),\delta}_\omega}$ for all $f \in L^{p),\delta}_\omega(\Omega)$. Finally, we have $\|f\|_{L^{q),\delta}_\omega} \le \|f\| \le C\,\|f\|_{L^{p),\delta}_\omega}$. Conversely suppose that the inequality (3) holds for $f \in L^{p),\delta}_\omega(\Omega)$. Then it is easy to prove that the inclusion $L^{p),\delta}_\omega(\Omega) \subseteq L^{q),\delta}_\omega(\Omega)$ is obtained. This completes the proof. ☐

Lemma 1.3. *(Lemma 1 in [18]) The continuous embedding* $L^{p_2}_{\omega_2}(\Omega) \hookrightarrow L^{p_1}_{\omega_1}(\Omega)$ *holds for* $1 \le p_1 \le p_2 \le \infty$ *if and only if*

$$\begin{aligned} \int\limits_{\Omega} \left(\frac{\omega_1(x)^{p_2}}{\omega_2(x)^{p_1}}\right)^{\frac{1}{p_2-p_1}} dx &< \infty \text{ for } 1 \le p_1 < p_2 < \infty, \\ \int\limits_{\Omega} \frac{\omega_1(x)}{\omega_2(x)^{p_1}} dx &< \infty \text{ for } 1 \le p_1 < p_2 = \infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} \frac{\omega_1(x)}{\omega_2(x)} &< \infty \text{ for } p_1 = p_2 = \infty, \end{aligned} \tag{4}$$

and the norm of the embedding operator equals $\left\{\int\limits_{\Omega}\left(\frac{\omega_1(x)^{p_2}}{\omega_2(x)^{p_1}}\right)^{\frac{1}{p_2-p_1}} dx\right\}^{\frac{1}{p_1}-\frac{1}{p_2}}$, $\left\{\int\limits_{\Omega}\frac{\omega_1(x)}{\omega_2(x)^{p_1}} dx\right\}^{\frac{1}{p_1}}$ *and* $\operatorname*{ess\,sup}_{x\in\Omega}\frac{\omega_1(x)}{\omega_2(x)}$*, respectively. In particular, the embedding* $L^{p_2}_{\omega}(\Omega) \hookrightarrow L^{p_1}_{\omega}(\Omega)$, $1 \le p_1 \le p_2 \le \infty$*, holds if and only if* $\omega \in L^1(\Omega)$.

Corollary 1.4. *Let* $1 \le p_1 \le p_2 \le \infty$*. Then the inclusion* $L^{p_2),\delta}_{\omega_2}(\Omega) \subseteq L^{p_1),\delta}_{\omega_1}(\Omega)$ *holds with respect to conditions (4).*

Lemma 1.5. *Let* $1 \le p \le q < \infty$ *and* $0 < \omega(x) \le \vartheta(x) < \infty$ *for a.e.* $x \in \Omega$*. Then, the continuous embedding* $L^{q),\delta}_{\vartheta}(\Omega) \hookrightarrow L^{p),\delta}_{\omega}(\Omega)$ *holds for* $0 < \varepsilon < p-1$ *and* $0 < \varepsilon < q-1$.

Proof. It is well known that if $1 \le p \le q < \infty$ and $0 < \omega(x) \le \vartheta(x) < \infty$ for a.e. $x \in \Omega$ and $|\Omega| < \infty$, then $L^q_{\vartheta}(\Omega) \hookrightarrow L^p_{\omega}$ and $\|f\|_{L^p_{\omega}} \le C\|f\|_{L^q_{\vartheta}}$ by Proposition 2 4 in [13]. So we have $L^{q-\varepsilon}_{\vartheta} \hookrightarrow L^{p-\varepsilon}_{\omega}$ and $L^p_{\omega} \hookrightarrow L^{p-\varepsilon}_{\omega}$. By the definition of $\|.\|_{L^{p),\delta}_{\omega}}$ we have $L^{q),\delta}_{\vartheta}(\Omega) \hookrightarrow L^{p),\delta}_{\omega}(\Omega)$. □

Definition 1.3 (Muckenhoupt Condition)**.** Let $1 < p < \infty$. A weight function ω satisfies the Muckenhoupt condition A_p, on the interval $I \subset \mathbb{R}$, if ω is nonnegative and, for all subinterval J the following condition

$$A_p(\omega, I) = \sup_J \left(\frac{1}{|J|}\int_J \omega(x)dx\right)\left(\frac{1}{|J|}\int_J \omega^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty$$

holds. The function ω is said to be a weight of the Muckenhoupt class, $\omega \in A_p(\omega, I)$.

We denote the Hardy-Littlewood maximal operator Mf of f by

$$Mf(x) := \sup_{J\subset I}\frac{1}{|J|}\int\limits_J |f(y)|\,dy,\ x \in I,$$

where the supremum is taken over all intervals I.

The set $AC[a,b]$ denotes the space of the absolute continuous functions on the interval $[a,b]$, $AC^1_{loc}[a,b]$ the space of the functions which are differentiable on any $[a,b] \subset (0,1)$ and their first derivative is in $AC[a,b]$, and $C^2[0,1]$ is the space of the twice continuously differentiable functions on $[0,1]$.

The following Theorem is very important for our study.

Theorem 1.6. *([3], [12]) Let* $I \subset \mathbb{R}$ *be a bounded interval,* $1 < p < \infty$ *and* $\delta \in \text{ß}_p$*. Then the Hardy-Littlewood maximal operator is bounded in* $L^{p),\delta}_{\omega}(\Omega)$*, i.e.* $M : L^{p),\delta}_{\omega}(\Omega) \to L^{p),\delta}_{\omega}(\Omega)$ *if and only if* $\omega \in A_p(\omega, I)$.

Throughout the paper we assume that $I = [0,1]$, $1 < p < \infty$, $\delta \in \text{ß}_p$ and $\omega \in A_p(\omega, I)$. Moreover, we will use the following version of the Hardy-Littlewood maximal operator

$$Mf(x) := \sup_{\substack{t\in[0,1]\\ t\neq x}} \frac{1}{t-x}\int\limits_x^t |f(y)|\,dy,\ x \in [0,1].$$

Definition 1.4. The Kantorovich operator (polynomial) of order $n \in \mathbb{N}_+$ of $f \in L^1[0,1]$ is defined by

$$K_n f(x) := \sum_{k=0}^{n}(n+1)\int\limits_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt\,p_{n,k}(x),\ p_{n,k}(x) := \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}.$$

By (2) in Theorem 1.1 we get $L^{p),\delta}_{\omega}[0,1] \subset L^1[0,1]$. Hence the Kantorovich operators are well-defined for $f \in L^{p),\delta}_{\omega}[0,1]$ and since $K_n f$ is an algebraic polynomial we obtain $K_n f \in L^{p),\delta}_{\omega}[0,1]$ as well. Moreover, by Theorem 1.6 there exists a $C^* > 0$ such that the following norm inequality

$$\|Mf\|_{L^{p),\delta}_{\omega}} \le C^*\|f\|_{L^{p),\delta}_{\omega}} \tag{5}$$

holds for all $f \in L^{p),\delta}_{\omega}[0,1]$. Now we prove that the Kantorovich operators K_n are bounded in $L^{p),\delta}_{\omega}[0,1]$. For this we will use the method in the proof of Proposition 2.2 in [8].

Theorem 1.7. *For all $f\in L^{p),\delta}_{\omega}[0,1]$ and $n\in\mathbb{N}_+$, there holds*

$$\|K_n f\|_{L^{p),\delta}_{\omega}}\le C^*\|f\|_{L^{p),\delta}_{\omega}},$$

where C^* is the constant in (5).

Proof. Let $x\in(0,1)$. Then we have

$$|K_n f(x)|\le\frac{1}{(n+1)\,\varphi^2(x)}\sum_{k=0}^{n+1}(k-(n+1)x)^2\,p_{n+1,k}(x)Mf(x)=Mf(x)$$

for all $x\in(0,1)$ such that $x\ne\frac{k}{n+1}$, $k=1,\dots,n$, where $\varphi(x):=\sqrt{x(1-x)}$. By Theorem 1.6 the Hardy-Littlewood maximal operator Mf is bounded in $L^{p),\delta}_{\omega}[0,1]$. So the Kantorovich operators K_n are bounded in $L^{p),\delta}_{\omega}[0,1]$ due to (5). ☐

We will prove a Jackson-type estimate. For this we use the following Lemma.

Lemma 1.8. *For any $g\in C^2[0,1]$, there holds*

$$\left\|g'\right\|_{L^{p),\delta}_{\omega}}\le 2C^*\|Dg\|_{L^{p),\delta}_{\omega}}$$

and

$$\left\|\varphi^2 g''\right\|_{L^{p),\delta}_{\omega}}\le 3C^*\|Dg\|_{L^{p),\delta}_{\omega}},$$

where C^* is the constant in (5) and $Dg(x):=\left(\varphi^2(x)g'(x)\right)'$.

Proof. Due to

$$\varphi^2(x)g'(x)=\int_0^x Dg(u)du,\ x\in[0,1],$$

then

$$\left|g'(x)\right|\le\frac{2}{x}\int_0^x|Dg(u)|\,du,\ x\in\left(0,\frac{1}{2}\right].$$

As a result,

$$\left|g'(x)\right|\le 2M(Dg)(x),\ x\in\left(0,\frac{1}{2}\right]\text{ and }\left|g'(x)\right|\le 2M(Dg)(x),\ x\in\left[\frac{1}{2},1\right).$$

Hence we get

$$\left|g'(x)\right|\le 2M(Dg)(x),\ x\in(0,1)$$

and

$$\left\|\varphi^2 g''\right\|_{L^{p),\delta}_{\omega}}\le\|Dg\|_{L^{p),\delta}_{\omega}}+\left\|g'\right\|_{L^{p),\delta}_{\omega}}.$$

The proof is completed. ☐

Now we have a Jackson type estimate.

Proposition 1.9. *For any $g\in L^{p),\delta}_{\omega}[0,1]$ such that $g\in AC^1_{loc}[a,b]$, $g'\in L^{p),\delta}_{\omega}[0,1]$ and $g''\in L^{p),\delta}_{\omega}[0,1]$, and all $n\in\mathbb{N}_+$, there holds*

$$\|K_n g-g\|_{L^{p),\delta}_{\omega}}\le\frac{11C^*}{n}\left(\left\|g'\right\|_{L^{p),\delta}_{\omega}}+\left\|\varphi^2 g''\right\|_{L^{p),\delta}_{\omega}}\right),$$

where C^* is the constant in (5).

Proof. By using Taylor's formula we can write

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)du.$$

Hence, by Lemma 2.1 a) in [8], we arrive that

$$K_n g(x)-g(x)=\frac{1-2x}{2(n+1)}g'(x)+\sum_{k=0}^{n}(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}R(x,t)dt\,p_{n,k}(x),$$

where $R(x,t) := \int_x^t (t-u)g''(u)du$. Moreover, we know the following inequality

$$|R(x,t)| \leq \frac{(t-x)^2}{\varphi^2(x)} M(\varphi^2 g'')(x),\ x,t \in (0,1)$$

by Proposition 2 4 in [8]. We will investigate two cases:

$$\text{Case 1: } (n+1)\varphi^2(x) \geq \frac{1}{2} \text{ and Case 2: } (n+1)\varphi^2(x) \leq \frac{1}{2}.$$

For two cases, we have

$$|K_n g(x) - g(x)| \leq \frac{1}{n}\left(3\left|g'(x)\right| + 2M(g')(x) + 11M(\varphi^2 g'')(x)\right),\ x \in [0,1],\ n \in \mathbb{N}_+.$$

This completes the proof. □

From Lemma 1.8 and Proposition 1.9 we obtain the following Jackson-type inequality.

Corollary 1.10. *For all $g \in C^2[0,1]$ and $n \in \mathbb{N}_+$, there holds*

$$\|K_n g - g\|_{L_\omega^{p),\delta}} \leq \frac{55C^{*2}}{n} \|Dg\|_{L_\omega^{p),\delta}},$$

where C^* is the constant in (5).

2 Main Results

In this section, we give three main Theorems about direct estimate and equivalence results in the weighted grand Lebesgue spaces $L_\omega^{p),\delta}[0,1]$. These are the extension results from classical Lebesgue spaces $L^p[0,1]$ to $L_\omega^{p),\delta}[0,1]$. For our aim we need some Sobolev-type spaces and Kfunctorials in $L_\omega^{p),\delta}(\Omega)$

$$W^1_{p),\delta}[0,1] := \left\{g \in L_\omega^{p),\delta}[0,1] : g \in AC[a,b],\ g' \in L_\omega^{p),\delta}[0,1]\right\}$$

and

$$W^2_{p),\delta}(\varphi)[0,1] := \left\{g \in L_\omega^{p),\delta}[0,1] : g \in AC^1_{loc}[a,b],\ \varphi^2 g'' \in L_\omega^{p),\delta}[0,1]\right\}.$$

For $f \in L_\omega^{p),\delta}[0,1]$ and $t > 0$, we define Kfunctorials:

$$K_1(f,t)_{p),\delta} := \inf_{g \in W^1_{p),\delta}[0,1]} \left\{\|f-g\|_{L_\omega^{p),\delta}} + t\left\|g'\right\|_{L_\omega^{p),\delta}}\right\},$$

$$K_{2,\varphi}(f,t)_{p),\delta} := \inf_{g \in W^2_{p),\delta}(\varphi)[0,1]} \left\{\|f-g\|_{L_\omega^{p),\delta}} + t\left\|\varphi^2 g''\right\|_{L_\omega^{p),\delta}}\right\},$$

and

$$K_D(f,t)_{p),\delta} := \inf_{g \in C^2[0,1]} \left\{\|f-g\|_{L_\omega^{p),\delta}} + t\|Dg\|_{L_\omega^{p),\delta}}\right\}.$$

Theorem 2.1. *For all $f \in L_\omega^{p),\delta}[0,1]$ and $n \in \mathbb{N}_+$, there holds*

$$\|K_n f - f\|_{L_\omega^{p),\delta}} \leq 55C^{*2} K_D(f, n^{-1})_{p),\delta},$$

where C^* is the constant in (5).

Proof. By Theorem 1.7 and Corollary 1.10, we obtain

$$\begin{aligned}
\|K_n f - f\|_{L_\omega^{p),\delta}} &\leq \|K_n(f-g)\|_{L_\omega^{p),\delta}} + \|K_n g - g\|_{L_\omega^{p),\delta}} + \|g-f\|_{L_\omega^{p),\delta}} \\
&\leq 2C^*\|f-g\|_{L_\omega^{p),\delta}} + \frac{55C^{*2}}{n}\|Dg\|_{L_\omega^{p),\delta}} \\
&\leq 55C^{*2}\left(\|f-g\|_{L_\omega^{p),\delta}} + \frac{1}{n}\|Dg\|_{L_\omega^{p),\delta}}\right)
\end{aligned}$$

for any $g \in C^2[0,1]$ and $n \in \mathbb{N}_+$. Taking the infimum on $C^2[0,1]$, we get the direct estimate result in the theorem. □

Lemma 2.2. *If $g \in W^2_{p),\delta}(\varphi)[0,1]$ with some $1 < p \leq \infty$, and ω is locally bounded, then $g \in C[0,1]$.*

Proof. We can write

$$g'(x)=g'(\frac{1}{2})+\int_{\frac{1}{2}}^{x} g''(u)du,\ x\in(0,1).$$

By definition of the space $W^2_{p),\delta}(\varphi)[0,1]$ and the continuous embedding $L^{p),\delta}_{\omega}\hookrightarrow L^{p-\varepsilon}_{\omega}$, we have $g,\varphi^2 g''\in L^{p-\varepsilon}_{\omega}[0,1]$ for $1<p-\varepsilon$. Taking $g,\varphi^2 g''\in L^q_{\omega}[0,1]$ $(q=p-\varepsilon>1)$ and using Hölder inequality we get

$$\begin{aligned}\int_{x}^{\frac{1}{2}}\left|g''(u)\right|du &= \int_{x}^{\frac{1}{2}}\left|\varphi^2(u)g''(u)\omega^{\frac{1}{q}}(u)\right|\varphi^{-2}(u)\omega^{-\frac{1}{q}}(u)du\\ &\le C_1\left\|\varphi^2 g''\omega^{\frac{1}{q}}\right\|_{L^q}\left\|\varphi^{-2}\omega^{-\frac{1}{q}}\right\|_{L^r}\\ &= C_1\left\|\varphi^2 g''\right\|_{L^q_{\omega}}\left\|\varphi^{-2}\omega^{-\frac{1}{q}}\right\|_{L^r}\\ &\le C_2\left\|\varphi^2 g''\right\|_{L^q_{\omega}}\left\|\varphi^{-2}\right\|_{L^r}\end{aligned}$$

for $x\in\left(0,\frac{1}{2}\right]$, where $\frac{1}{q}+\frac{1}{r}=1$ and $C_2=C_1\sup\limits_{u\in(0,1)}\omega^{-\frac{1}{q}}(u)<\infty$. Hence, we obtain

$$\int_{x}^{\frac{1}{2}}\left|g''(u)\right|du\le C_2\left\|\varphi^2 g''\right\|_{L^q_{\omega}}\begin{cases} x^{-\frac{1}{q}}, & q<\infty\\ |\log x| & q=\infty\end{cases}$$

for $x\in\left(0,\frac{1}{2}\right]$, whose value is independent of x. Therefore, we arrive at the following inequality

$$\left|g'(x)\right|\le C_3\begin{cases} x^{-\frac{1}{q}}, & q<\infty\\ |\log x| & q=\infty\end{cases}$$

for $x\in\left(0,\frac{1}{2}\right]$, whose value is independent of x. Due to $q>1$, $g'\in L^1\left[0,\frac{1}{2}\right]$ and $g\in C\left[0,\frac{1}{2}\right]$. Similarly, it is easy to prove that $g\in C\left[\frac{1}{2},1\right]$. □

Theorem 2.3. *For all* $f\in L^{p),\delta}_{\omega}[0,1]$ *and* $t\in(0,1]$, *there holds*

$$K_D(f,t)_{p),\delta}\sim K_{2,\varphi}(f,t)+K_1(f,t)_{p),\delta}.$$

Proof. If we follow the method in the proof of Theorem 1.2 in [8], then the proof is completed. □

Theorem 2.4. *For all* $f\in L^{p),\delta}_{\omega}[0,1]$ *and* $n\in\mathbb{N}_+$, *there holds*

$$\|K_n f-f\|_{L^{p),\delta}_{\omega}}\le C\left(K_{2,\varphi}(f,n^{-1})+K_1(f,n^{-1})_{p),\delta}\right),$$

where the constant C *is independent of* f *and* n.

Proof. By Theorem 2.1 and 2.3, the proof is obtained. □

3 Conclusion

In this paper, we obtain approximation properties and boundedness of the Kantorovich operators in the setting of generalized weighted grand Lebesgue spaces via the boundedness of the Hardy-Littlewood maximal operator. Moreover, we proved the new embedding theorems, direct estimates,and Jackson-type inequalities in these spaces.

4 Acknowledgment

We would like to thank the editor and referees for their careful reading, valuable suggestions and corrections that improved the quality and presentation of the paper.

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