PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 19 · 2026 · Pages 288–304

Approximation of Generalized Variational Inequality, Equilibrium, and Fixed Point Problems

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Communicated by Maryam Mohammadi

Abstract

In this paper, we propose and analyze a new algorithm for approximating a common solution to generalized variational inequality problems, equilibrium problems, and fixed point problems involving non-expansive mappings. We establish the strong convergence of the generated sequence, together with explicit linear error estimates, under suitable monotonicity assumptions in Hilbert spaces. Furthermore, we provide numerical examples that illustrate the performance and advantages of our approach compared to other iterative algorithms. Finally, we discuss how the main results can be applied to convex minimization problems and split feasibility equilibrium problems, thereby showing the versatility and applicability of the proposed method.

Keywords: Generalized variational inequality, equilibrium problem, fixed point, iterative method, non-expansive mappings, strong convergence.

2020 Mathematics Subject Classification: Primary: 47H10, 47H09, 47J25; Secondary: 58E35, 65J15.

1 Introduction

Banach's famous contraction principle ensures that every contraction defined on a complete metric space admits a unique fixed point [1]. This fundamental result laid the foundation for the development of fixed point theory, which has become an essential branch of nonlinear functional analysis. Fixed point methods are widely recognized as powerful tools for solving problems arising in optimization, equilibrium theory, signal processing, and applied sciences, where many mathematical models can be reformulated as fixed point problems, thereby motivating the development of various iterative techniques for approximating their solutions [2, 3, 4].

Alongside fixed point theory, equilibrium problems and variational inequality problems are two important frameworks in nonlinear analysis. As introduced by Blum and Oettli [5], the equilibrium problem provides a unifying model that encompasses optimization problems, Nash equilibrium problems, and complementarity problems [6, 7, 8, 9]. This has subsequently led to the development of algorithms for finding solutions of equilibrium problems [10, 11]. Variational inequality problems, introduced by Stampacchia [12], have also been extensively studied due to their broad applicability in economics, engineering, and mechanics [13, 14]. Consequently, numerous iterative methods have been developed to solve variational inequality problems [15, 16]. Furthermore, generalized variational inequality problems introduced by Noor [17] extend the classical framework and allow more flexible modeling of nonlinear systems [18, 19].

In fact, there is a close relationship between these three problems, since variational inequalities and equilibrium problems can both be reformulated as fixed point problems using suitable operators such as projections and resolvents [20, 21]. It is evident that fixed point theory is essential in the interrelation of the three problems. In fixed point theory, contraction mappings ensure strong convergence; however, their restrictive assumptions limit their applicability. This has led to the study of more general classes such as non-expansive mappings. Classical iterative methods for non-expansive mappings such as Mann [22], Ishikawa [23], and Halpern [24] iterations have been widely studied [25, 26]. These methods form the basis for many modern algorithms; however, they often exhibit slow convergence behavior [27]. To address this issue, several accelerated and hybrid iterative schemes have been proposed in the literature, including the methods of Agarwal et al. [28], Gürsoy et al. [29] and Ullah et al. [30]. In particular, Maldar [31] introduced new parallel fixed point algorithms and investigated their applications to a system of variational inequalities, while Atalan [32] studied a new fixed point iterative algorithm for general variational inequalities.

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Despite the above developments, much of the existing work considers generalized variational inequality problems, equilibrium problems, and fixed point problems either separately or in partially combined forms. While some works unify these problems, they still do not focus on generalized variational inequality problems [33, 34, 35]. Furthermore, Dashputre et al. [36] introduced an iterative method for finding a common solution to generalized variational inequality problems and fixed point problems for non-expansive mappings in Hilbert spaces.

Let H be a real Hilbert space and let Ψ be a nonempty, closed, and convex subset of H. Let $f, g : \Psi \to H$ be β_1- and β_2-inverse strongly monotone mappings, respectively, and let $A : H \to H$ be a non-expansive mapping. To find a common element of Fix(A) and GVI(Ψ, f, g) (see Section 2), Dashputre et al. [36] proposed the following algorithm:

$$\begin{cases} t_1 \in \Psi, \\ v_n = (1-\zeta_n)t_n + \zeta_n A\big[t_n - g(t_n) + P_\Psi\{g(t_n) - \lambda f(t_n)\}\big], \\ q_n = A\big[v_n - g(v_n) + P_\Psi\{g(v_n) - \lambda f(v_n)\}\big], \\ s_n = A\big[q_n - g(q_n) + P_\Psi\{g(q_n) - \lambda f(q_n)\}\big], \\ t_{n+1} = A\big[s_n - g(s_n) + P_\Psi\{g(s_n) - \lambda f(s_n)\}\big], \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{1}$$

where $\{\zeta_n\} \subset (0,1)$.

Inspired by the above developments, we construct a new iterative scheme by integrating the resolvent operator associated with the equilibrium problem into (1). This scheme unifies the generalized variational inequality problem, the equilibrium problem, and the fixed point problem for non-expansive mappings within a single framework as follows:

$$\begin{cases} x_1 \in \Psi, \\ u_n = \Phi_{r_n}(x_n), \\ w_n = (1-\alpha_n)u_n + \alpha_n A\big[u_n - g(u_n) + P_\Psi\{g(u_n) - \lambda f(u_n)\}\big], \\ z_n = A\big[w_n - g(w_n) + P_\Psi\{g(w_n) - \lambda f(w_n)\}\big], \\ y_n = A\big[z_n - g(z_n) + P_\Psi\{g(z_n) - \lambda f(z_n)\}\big], \\ x_{n+1} = A\big[y_n - g(y_n) + P_\Psi\{g(y_n) - \lambda f(y_n)\}\big], \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{2}$$

where Φ_{r_n} is the resolvent operator associated with the equilibrium problem and $\{\alpha_n\} \subset (0,1)$ is a sequence satisfying:

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{3}$$

The remainder of the paper is organized as follows. Section 2 introduces preliminary concepts. Section 3 presents the main convergence theorems together with numerical illustrations supporting the theoretical findings. Section 4 covers further applications of the proposed framework to convex optimization problems and split feasibility equilibrium problems. Finally, Section 5 presents concluding remarks and suggests possible directions for future research.

2 Preliminaries

In this paper, H represents a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. For a mapping $T : H \to H$, a point $x^* \in H$ is a fixed point of T whenever $T(x^*) = x^*$. The set of all fixed points of T is denoted by Fix(T).

Definition 2.1 ([37]). A mapping $T : H \to H$ is called non-expansive if it satisfies

$$\|T(x) - T(y)\| \le \|x - y\|, \quad \forall x, y \in H.$$

Definition 2.2 ([38]). A mapping $T : H \to H$ is called firmly non-expansive if it satisfies

$$\|T(x) - T(y)\|^2 \le \langle T(x) - T(y), x - y \rangle, \quad \forall x, y \in H.$$

Equivalently, T is firmly non-expansive if and only if $2T - I$ is non-expansive or $T = \frac{I+S}{2}$ for a certain non-expansive mapping S.

Definition 2.3 ([17]). Let Ψ be a nonempty, closed, and convex subset of a real Hilbert space H. A mapping $f : \Psi \to H$ is called β-inverse strongly monotone if there exists a constant $\beta > 0$ such that

$$\langle f(x) - f(y), x - y \rangle \ge \beta \|f(x) - f(y)\|^2, \quad \forall x, y \in \Psi.$$

Remark 1. Every β-inverse strongly monotone mapping is Lipschitz continuous, with a Lipschitz constant of $1/\beta$.

Definition 2.4 ([17]). Let Ψ be a nonempty, closed, and convex subset of a real Hilbert space H, and let $f, g : \Psi \to H$ be two nonlinear mappings. The generalized nonlinear variational inequality (GVI) problem consists of finding a point $x \in \Psi$ such that

$$\langle f(x), g(y) - g(x) \rangle \ge 0, \quad \forall y \in \Psi. \tag{4}$$

The solution set of (4) is denoted by GVI(Ψ, f, g).

When $g = I$, the generalized nonlinear variational inequality (4) simplifies to the classical variational inequality originally studied by Stampacchia [12], which involves finding a point $x \in \Psi$ such that

$$\langle f(x), y - x\rangle \geq 0, \quad \forall y \in \Psi.$$

Definition 2.5 ([38]). Let Ψ be a nonempty, closed, and convex subset of H. Let $\psi : \Psi \times \Psi \to \mathbb{R}$ be a bifunction. The equilibrium problem (EP) then consists of finding a point $x \in \Psi$ such that

$$\psi(x, y) \geq 0, \quad \forall y \in \Psi. \tag{5}$$

The solution set of the equilibrium problem is denoted by $\mathrm{EP}(\psi) = \{x \in \Psi : (5) \text{ holds}\}$.

Lemma 2.1 ([38]). *Let Ψ be a nonempty, closed, and convex subset of H, and assume that the bifunction $\psi : \Psi \times \Psi \to \mathbb{R}$ satisfies the following conditions:*

(C1) $\psi(x, x) = 0$, $\forall x \in \Psi$ *(reflexivity);*

(C2) $\psi(x, y) + \psi(y, x) \leq 0$, $\forall x, y \in \Psi$ *(monotonicity);*

(C3) For each $x, y, z \in \Psi$, $\limsup_{t \downarrow 0} \psi(tz + (1-t)x, y) \leq \psi(x, y)$ *(upper hemicontinuity);*

(C4) For each $x \in \Psi$, *the function* $y \mapsto \psi(x, y)$ *is convex and weakly lower semicontinuous.*

Then, for every $r > 0$ and $x \in H$, there exists $z \in \Psi$ such that

$$\psi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in \Psi.$$

Lemma 2.2 ([38]). *Let Ψ be a nonempty, closed, and convex subset of H, and let ψ satisfy (C1)–(C4). For $r > 0$, define a mapping*

$$\Phi_r(x) = \left\{ z \in \Psi : \psi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in \Psi \right\}, \quad \forall x \in H.$$

Then, the resolvent operator Φ_r has the following properties:

(i) Φ_r *is single-valued;*

(ii) Φ_r *is firmly non-expansive;*

(iii) $\mathrm{Fix}(\Phi_r) = \mathrm{EP}(\psi)$*;*

(iv) $\mathrm{EP}(\psi)$ *is closed and convex.*

Definition 2.6 ([39]). Let Ψ be a nonempty, closed, convex subset of H. For each $x \in H$, there exists a unique point $P_\Psi(x)$ such that

$$\|x - P_\Psi(x)\| \leq \|x - y\|, \quad \forall y \in \Psi.$$

The mapping $P_\Psi : H \to \Psi$ is called the metric projection of H onto Ψ.

Lemma 2.3 ([40]). *Let $P_\Psi : H \to \Psi$ be the projection mapping of H onto Ψ; then for a given $y \in H$, an element $x \in \Psi$ satisfies the variational inequality*

$$\langle y - x, x - z\rangle \geq 0, \quad \forall z \in \Psi$$

if and only if $x = P_\Psi(y)$. That is,

$$x = P_\Psi(y) \iff \langle y - x, x - z\rangle \geq 0, \quad \forall z \in \Psi.$$

Moreover, the projection operator P_Ψ is non-expansive.

Using Lemma 2.3, the generalized nonlinear variational inequality (4) can be reformulated as a fixed point problem, as stated in the following lemma.

Lemma 2.4. *Let Ψ be a nonempty, closed, and convex subset of a real Hilbert space H, and let $f : \Psi \to H$ and $g : \Psi \to \Psi$ be mappings. Assume that g is surjective, that is, $g(\Psi) = \Psi$. Then, for any $\lambda > 0$, a point $x \in \Psi$ solves the generalized nonlinear variational inequality problem (4) if and only if*

$$g(x) = P_\Psi[g(x) - \lambda f(x)]. \tag{6}$$

Proof. Let $x \in \Psi$. Since $g(x) \in \Psi$, Lemma 2.3 applied with $y = g(x) - \lambda f(x)$ shows that (6) holds if and only if

$$\langle g(x) - \lambda f(x) - g(x), g(x) - z\rangle \geq 0, \quad \forall z \in \Psi,$$

that is, since $\lambda > 0$,

$$\langle f(x), z - g(x)\rangle \geq 0, \quad \forall z \in \Psi. \tag{6a}$$

Since $g(\Psi) = \Psi$, the set of points $z \in \Psi$ coincides with the set of points $g(y)$, $y \in \Psi$. Hence (6a) is equivalent to $\langle f(x), g(y) - g(x)\rangle \geq 0$ for all $y \in \Psi$, which is (4). □

Remark 2. The assumption $g(\Psi)=\Psi$ cannot be omitted in Lemma 2.4. If $H=\Psi=\mathbb{R}$, $f\equiv 1$ and $g\equiv 0$, then every $x\in\mathbb{R}$ solves (4), but $P_{\mathbb{R}}(g(x)-\lambda f(x))=-\lambda\neq 0=g(x)$. Conversely, if $\Psi=[0,1]$, $f\equiv -1$, $g(y)=2y$ and $x=\frac{1}{2}$, then $g(x)=1=P_{[0,1]}(1+\lambda)$, but $\langle f(x),g(1)-g(x)\rangle=-1<0$.

Lemma 2.5 ([41]). *Let $\{\delta_n\}$ and $\{\rho_n\}$ be real sequences converging to δ and ρ, respectively. Assume that the limit*

$$\lim_{n\to\infty}\frac{|\delta_n-\delta|}{|\rho_n-\rho|}=a$$

exists. Then:

(i) $\{\delta_n\}$ is said to converge faster than $\{\rho_n\}$ if $a=0$;

(ii) $\{\delta_n\}$ and $\{\rho_n\}$ are said to converge at the same rate if $0<a<\infty$.

Definition 2.7 ([41]). Let $\{\delta_n\}$ and $\{\rho_n\}$ be two real sequences that converge to the same limit σ. Suppose there exist two sequences of positive real numbers $\{\tau_n\}$ and $\{\mu_n\}$, both converging to 0, such that

$$|\delta_n-\sigma|\leq\tau_n\quad\text{and}\quad|\rho_n-\sigma|\leq\mu_n,\quad\forall n\in\mathbb{N}.$$

Then $\{\delta_n\}$ converges faster to σ than $\{\rho_n\}$ if $\{\tau_n\}$ converges to 0 faster than $\{\mu_n\}$.

Lemma 2.6 ([41]). *Let $\{\delta_n\}$ and $\{\rho_n\}$ be sequences of nonnegative real numbers satisfying*

$$\delta_{n+1}\leq a\delta_n+\rho_n,\quad\forall n\in\mathbb{N},$$

where $0<a<1$ and $\lim_{n\to\infty}\rho_n=0$. Then $\lim_{n\to\infty}\delta_n=0$.

Lemma 2.7 ([42]). *Let $\{\tau_n\}$, $\{\delta_n\}$ and $\{\rho_n\}$ be three sequences of positive real numbers such that*

$$\tau_{n+1}\leq(1-\delta_n)\tau_n+\rho_n,\quad\forall n\in\mathbb{N},$$

where $\delta_n\in(0,1)$, $\sum_{n=1}^{\infty}\delta_n=\infty$ and $\rho_n=o(\delta_n)$, i.e., $\rho_n/\delta_n\to 0$ as $n\to\infty$. Then $\tau_n\to 0$ as $n\to\infty$.

3 Convergence Analysis and Numerical Results

In this section, we present the main convergence results of the proposed algorithm (2).

Before stating these theorems, we briefly explain the construction and motivation behind the iterative scheme (2). As discussed in Lemma 2.4, the generalized variational inequality problem can be reformulated as a fixed point problem. Adding $x-g(x)$ to both sides of (6) yields

$$x=x-g(x)+P_{\Psi}[g(x)-\lambda f(x)].$$

Next, we incorporate an equilibrium problem into this framework. From Lemma 2.2, the resolvent operator Φ_r satisfies $\mathrm{Fix}(\Phi_r)=\mathrm{EP}(\psi)$.

We now seek a point $x\in H$ such that

$$x\in\mathrm{Fix}(A)\cap\mathrm{GVI}(\Psi,f,g)\cap\mathrm{EP}(\psi).$$

Moreover, if x belongs to the intersection defined above, then the following combined fixed point formulation can be obtained from (6) and $x=x-g(x)+P_{\Psi}[g(x)-\lambda f(x)]$:

$$x=A(x)=x-g(x)+P_{\Psi}[g(x)-\lambda f(x)]=\Phi_r(x).$$

More precisely,

$$x=A\big\{x-g(x)+P_{\Psi}[g(x)-\lambda f(x)]\big\}=\Phi_r(x).$$

As a result, finding a common solution is equivalent to determining a fixed point common to the three operators. This unified fixed point structure enables the construction of iterative schemes for approximating such common solutions.

Theorem 3.1. *Let H be a real Hilbert space and let Ψ be a nonempty, closed, and convex subset of H. Let $f:\Psi\to H$ and $g:\Psi\to\Psi$ be β_1- and β_2-inverse strongly monotone mappings, respectively, with $\beta_2\geq\frac{1}{2}$. Assume that f and g are also μ_1- and μ_2-strongly monotone, respectively, that is,*

$$\langle f(x)-f(y),x-y\rangle\geq\mu_1\|x-y\|^2,\qquad\langle g(x)-g(y),x-y\rangle\geq\mu_2\|x-y\|^2,\qquad\forall x,y\in\Psi,$$

for some $\mu_1,\mu_2>0$. Assume, in addition, that g is surjective, that is, $g(\Psi)=\Psi$. Let $A:H\to H$ be a non-expansive mapping, let ψ satisfy (C1)–(C4), let Φ_{r_n} be the resolvent operator of ψ with $r_n>0$, and let $\{\alpha_n\}\subset(0,1)$ satisfy $\sum_{n=1}^{\infty}\alpha_n=\infty$. Assume that $\mathrm{Fix}(A)\cap\mathrm{GVI}(\Psi,f,g)\cap\mathrm{EP}(\psi)\neq\emptyset$. Suppose that the mapping

$$T_{\lambda}(x)=A\big[x-g(x)+P_{\Psi}\{g(x)-\lambda f(x)\}\big]$$

maps Ψ into Ψ, and that the parameter λ satisfies the condition

$$0 < \lambda < 2\beta_1 \quad \textit{and} \quad 0 < \rho < 1, \tag{7}$$

where ρ is given by (8) below. Then the sequence $\{x_n\}$ generated by algorithm (2) converges strongly to $y^* \in \mathrm{Fix}(A) \cap \mathrm{GVI}(\Psi, f, g) \cap \mathrm{EP}(\psi)$. In addition, for all $n \in \mathbb{N}$, the following error estimate holds:

$$\|x_{n+1} - y^*\| \le \rho^{3n} \|x_1 - y^*\| \prod_{i=1}^{n} [1 - \alpha_i(1-\rho)],$$

where the constant ρ is defined by

$$\rho = 2\sqrt{1-(2\beta_2-1)\mu_2^2} + \sqrt{1-\lambda(2\beta_1-\lambda)\mu_1^2}. \tag{8}$$

Remark 3. The radicands in (8) lie in $[0,1]$. Indeed, a β_i-inverse strongly monotone and μ_i-strongly monotone mapping satisfies $\mu_i\|x-y\| \le \|f(x)-f(y)\| \le \frac{1}{\beta_i}\|x-y\|$ (with g in place of f for $i=2$), so that $\beta_i\mu_i \le 1$. Hence $0 \le (2\beta_2-1)\mu_2^2 \le \frac{2\beta_2-1}{\beta_2^2} \le 1$ and $0 < \lambda(2\beta_1-\lambda)\mu_1^2 \le \beta_1^2\mu_1^2 \le 1$. Inverse strong monotonicity alone only implies that the mappings $I-g$ and $I-\lambda f$ are non-expansive, which does not yield a contraction factor smaller than 1 (see Remark 4); this is why strong monotonicity is assumed. The assumption $T_\lambda(\Psi) \subset \Psi$ guarantees that all iterates of (2) remain in Ψ, where f and g are defined: indeed, $u_n = \Phi_{r_n}(x_n) \in \Psi$, w_n is a convex combination of u_n and $T_\lambda(u_n)$, and $z_n = T_\lambda(w_n)$, $y_n = T_\lambda(z_n)$, $x_{n+1} = T_\lambda(y_n)$.

Proof. Let $y^* \in \mathrm{Fix}(A) \cap \mathrm{GVI}(\Psi, f, g) \cap \mathrm{EP}(\psi)$ be a common solution. Since y^* is a solution of the generalized variational inequality and a fixed point of A, we have from Lemma 2.4:

$$y^* = A\big[y^* - g(y^*) + P_\Psi\{g(y^*) - \lambda f(y^*)\}\big]. \tag{9}$$

Additionally, since $y^* \in \mathrm{EP}(\psi)$ and $u_n = \Phi_{r_n}(x_n)$, and since the resolvent operator Φ_{r_n} is firmly non-expansive, and therefore non-expansive, we have

$$\|u_n - y^*\|^2 = \|\Phi_{r_n}(x_n) - \Phi_{r_n}(y^*)\|^2 \le \langle \Phi_{r_n}(x_n) - \Phi_{r_n}(y^*), x_n - y^* \rangle,$$
$$\|u_n - y^*\| \le \|x_n - y^*\|. \tag{10}$$

Recall that f and g are β_1- and β_2-inverse strongly monotone, respectively, and that A and P_Ψ are non-expansive. Using (2) and (9), we get

$$\begin{aligned} \|z_n - y^*\| &= \big\| A\big[w_n - g(w_n) + P_\Psi\{g(w_n) - \lambda f(w_n)\}\big] \\ &\quad - A\big[y^* - g(y^*) + P_\Psi\{g(y^*) - \lambda f(y^*)\}\big] \big\| \\ &\le 2\|w_n - y^* - \{g(w_n) - g(y^*)\}\| + \|w_n - y^* - \lambda\{f(w_n) - f(y^*)\}\|. \end{aligned} \tag{11}$$

Since f is β_1-inverse strongly monotone, we get

$$\begin{aligned} \|w_n - y^* - \lambda\{f(w_n) - f(y^*)\}\|^2 &= \|w_n - y^*\|^2 + \lambda^2\|f(w_n) - f(y^*)\|^2 \\ &\quad - 2\lambda\langle w_n - y^*, f(w_n) - f(y^*)\rangle \\ &\le \|w_n - y^*\|^2 - \lambda(2\beta_1 - \lambda)\|f(w_n) - f(y^*)\|^2. \end{aligned} \tag{12}$$

Similarly, since g is β_2-inverse strongly monotone, we obtain

$$\begin{aligned} \|w_n - y^* - \{g(w_n) - g(y^*)\}\|^2 &= \|w_n - y^*\|^2 + \|g(w_n) - g(y^*)\|^2 \\ &\quad - 2\langle w_n - y^*, g(w_n) - g(y^*)\rangle \\ &\le \|w_n - y^*\|^2 - (2\beta_2 - 1)\|g(w_n) - g(y^*)\|^2. \end{aligned} \tag{13}$$

Therefore, from (11), (12) and (13), we obtain

$$\begin{aligned} \|z_n - y^*\| &\le 2\Big(\|w_n - y^*\|^2 - (2\beta_2 - 1)\|g(w_n) - g(y^*)\|^2\Big)^{1/2} \\ &\quad + \Big(\|w_n - y^*\|^2 - \lambda(2\beta_1 - \lambda)\|f(w_n) - f(y^*)\|^2\Big)^{1/2}. \end{aligned} \tag{14}$$

By the strong monotonicity of f and g and the Cauchy–Schwarz inequality, we have $\|f(w_n) - f(y^*)\| \ge \mu_1\|w_n - y^*\|$ and $\|g(w_n) - g(y^*)\| \ge \mu_2\|w_n - y^*\|$. Since $2\beta_2 - 1 \ge 0$ and $\lambda(2\beta_1 - \lambda) > 0$, it follows from (14) that

$$\|z_n - y^*\| \le \Big(2\sqrt{1-(2\beta_2-1)\mu_2^2} + \sqrt{1-\lambda(2\beta_1-\lambda)\mu_1^2}\Big)\|w_n - y^*\| = \rho\|w_n - y^*\|,$$

where ρ is defined in (8) and $0 < \rho < 1$ by (7). Since w_n was an arbitrary point of Ψ in this argument, we have in fact shown that

$$\|T_\lambda(x) - y^*\| \le \rho\|x - y^*\|, \quad \forall x \in \Psi. \tag{14a}$$

Using algorithm (2) and (14a), we obtain

$$\begin{aligned}\|w_n - y^*\| &= \left\|(1-\alpha_n)u_n + \alpha_n A[u_n - g(u_n) + P_\Psi\{g(u_n) - \lambda f(u_n)\}] - y^*\right\| \\ &\leq (1-\alpha_n)\|u_n - y^*\| + \alpha_n\rho\|u_n - y^*\| \\ &= [1-\alpha_n(1-\rho)]\|u_n - y^*\|. \end{aligned} \tag{15}$$

From (10), the inequality becomes

$$\|w_n - y^*\| \leq [1-\alpha_n(1-\rho)]\|x_n - y^*\|.$$

Also, using the definition of y_n, the non-expansiveness of A and P_Ψ, and the same argument as in (11), we obtain

$$\begin{aligned}\|y_n - y^*\| \leq 2\|z_n - y^* - \{g(z_n) - g(y^*)\}\| \\ + \|z_n - y^* - \lambda\{f(z_n) - f(y^*)\}\|. \end{aligned} \tag{16}$$

Next, using the definition of x_{n+1}, the non-expansiveness of A and P_Ψ, and Lemma 2.4, we obtain

$$\begin{aligned}\|x_{n+1} - y^*\| &\leq \|y_n - y^* - \{g(y_n) - g(y^*)\}\| \\ &\quad + \|g(y_n) - g(y^*) - \lambda\{f(y_n) - f(y^*)\}\| \\ &\leq 2\|y_n - y^* - \{g(y_n) - g(y^*)\}\| \\ &\quad + \|y_n - y^* - \lambda\{f(y_n) - f(y^*)\}\|. \end{aligned} \tag{17}$$

By (16), (17) and the argument leading to (14a), we have $\|y_n - y^*\| \leq \rho\|z_n - y^*\|$ and $\|x_{n+1} - y^*\| \leq \rho\|y_n - y^*\|$. Combining these with $\|z_n - y^*\| \leq \rho\|w_n - y^*\|$ and (15), we get

$$\|x_{n+1} - y^*\| \leq \rho^3[1-\alpha_n(1-\rho)]\|x_n - y^*\|, \quad \forall n \in \mathbb{N}.$$

Since the iteration starts at x_1, applying this inequality successively for $i = n, n-1, \ldots, 1$ yields

$$\|x_{n+1} - y^*\| \leq \rho^{3n}\|x_1 - y^*\| \prod_{i=1}^{n}[1-\alpha_i(1-\rho)], \quad \forall n \in \mathbb{N}.$$

Since $0 < \rho < 1$ and $0 < \alpha_i < 1$, we obtain $0 < 1-\alpha_i(1-\rho) < 1$. Moreover, since $\sum_{i=1}^{\infty} \alpha_i = \infty$ by assumption, we have $\lim_{n\to\infty} \prod_{i=1}^{n}[1-\alpha_i(1-\rho)] = 0$. As a result,

$$\lim_{n\to\infty} \|x_n - y^*\| = 0.$$

Therefore, the sequence $\{x_n\}$ converges strongly to y^*. □

Theorem 3.2. *Let H be a real Hilbert space and let Ψ be a nonempty, closed, and convex subset of H. Assume that f, g, A, Φ_{r_n}, λ and ρ are as in Theorem 3.1. Let $\{t_n\}$ and $\{x_n\}$ be the sequences generated by the iterative schemes (1) and (2), respectively. Assume that condition (7) holds and that*

$$\mathrm{Fix}(A) \cap \mathrm{GVI}(\Psi, f, g) \cap \mathrm{EP}(\psi) \neq \varnothing,$$

Then the following statements hold:

(i) If the sequence $\left\{\frac{1+\rho^3}{\zeta_n}\right\}$ is bounded and $\sum_{n=1}^{\infty} \zeta_n = \infty$, then the difference $\{x_n - t_n\}$ converges to 0 and the following error estimate holds:

$$\|x_{n+1} - t_{n+1}\| \leq [1-\zeta_n(1-\rho)]\|x_n - t_n\| + (1+\rho^3)\|x_n - y^*\|, \quad \forall n \in \mathbb{N}.$$

(ii) If the sequence $\{t_n\}$ converges to $y^* \in \mathrm{Fix}(A) \cap \mathrm{GVI}(\Psi, f, g) \cap \mathrm{EP}(\psi)$, then the difference $\{t_n - x_n\}$ converges strongly to 0 and, consequently, $\{x_n\}$ also converges strongly to y^*. Moreover, the following error estimate holds:

$$\|t_{n+1} - x_{n+1}\| \leq \rho^3\|t_n - x_n\| + (1+\rho^3)\|t_n - y^*\|, \quad \forall n \in \mathbb{N}.$$

Proof. (i) From Theorem 3.1, we already have $\lim_{n\to\infty} \|x_n - y^*\| = 0$, where $y^* \in \mathrm{Fix}(A) \cap \mathrm{GVI}(\Psi, f, g) \cap \mathrm{EP}(\psi)$.

Employing arguments as in the proof of Theorem 3.1 and the iterative schemes (1) and (2), we obtain the following estimate:

$$\|t_{n+1} - y^*\| \leq \rho^3[1-\zeta_n(1-\rho)]\|t_n - y^*\|. \tag{18}$$

Moreover, as in the proof of Theorem 3.1, we have

$$\|x_{n+1} - y^*\| \leq \rho^3[1-\alpha_n(1-\rho)]\|x_n - y^*\|. \tag{19}$$

Now, using the triangle inequality $\|x_{n+1} - t_{n+1}\| \leq \|x_{n+1} - y^*\| + \|t_{n+1} - y^*\|$, from (18) and (19), we obtain

$$\|x_{n+1} - t_{n+1}\| \leq \rho^3[1-\alpha_n(1-\rho)]\|x_n - y^*\| + \rho^3[1-\zeta_n(1-\rho)]\|t_n - y^*\|.$$

Applying the triangle inequality, $\|t_n - y^*\| \le \|x_n - t_n\| + \|x_n - y^*\|$, we obtain

$$\begin{aligned}\|x_{n+1} - t_{n+1}\| &\le \rho^3[1-\alpha_n(1-\rho)]\|x_n - y^*\| \\ &\quad + \rho^3[1-\zeta_n(1-\rho)](\|x_n - t_n\| + \|x_n - y^*\|) \\ &\le \rho^3[1-\zeta_n(1-\rho)]\|x_n - t_n\| \\ &\quad + \rho^3([1-\alpha_n(1-\rho)] + [1-\zeta_n(1-\rho)])\|x_n - y^*\|. \end{aligned} \tag{20}$$

Since $0 < \rho < 1$ and $0 < \alpha_n, \zeta_n < 1$, we obtain $[1-\alpha_n(1-\rho)] \le 1$ and $[1-\zeta_n(1-\rho)] \le 1$. Furthermore, $\rho^3+\rho^3 = 2\rho^3 \le 1+\rho^3$. Therefore, from (20) we obtain:

$$\|x_{n+1} - t_{n+1}\| \le [1-\zeta_n(1-\rho)]\|x_n - t_n\| + (1+\rho^3)\|x_n - y^*\|. \tag{21}$$

Let

$$\tau_n = \|x_n - t_n\|, \quad \delta_n = \zeta_n(1-\rho), \quad \sigma_n = (1+\rho^3)\|x_n - y^*\|.$$

Then inequality (21) can be rewritten as

$$\tau_{n+1} \le (1-\delta_n)\tau_n + \sigma_n.$$

By hypothesis, since $\sum_{n=1}^{\infty} \zeta_n = \infty$, we have $\sum_{n=1}^{\infty} \delta_n = \infty$. Still by hypothesis, the sequence $\left\{\frac{1+\rho^3}{\zeta_n}\right\}$ is bounded, implying that $\left\{\frac{1+\rho^3}{\zeta_n(1-\rho)}\right\}$ is also bounded. Therefore, we can choose a constant $L > 0$ such that $\frac{1+\rho^3}{\zeta_n(1-\rho)} \le L$ for all $n \in \mathbb{N}$. Thus,

$$\sigma_n = (1+\rho^3)\|x_n - y^*\| = \zeta_n(1-\rho)\left[\frac{1+\rho^3}{\zeta_n(1-\rho)}\right]\|x_n - y^*\| \le L\delta_n\|x_n - y^*\|.$$

Since by Theorem 3.1 $\lim_{n\to\infty} \|x_n - y^*\| = 0$, we have $\frac{\sigma_n}{\delta_n} \to 0$ as $n \to \infty$, that is, $\sigma_n = o(\delta_n)$. Therefore, all assumptions of Lemma 2.7 are satisfied. Thus,

$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \|x_n - t_n\| = 0,$$

which completes the proof of part (i).

(ii) Suppose that $\{t_n\}$ converges strongly to $y^* \in \mathrm{Fix}(A) \cap \mathrm{GVI}(\Psi, f, g) \cap \mathrm{EP}(\psi)$.

From (18) and (19), we have

$$\|t_{n+1} - y^*\| \le \rho^3[1-\zeta_n(1-\rho)]\|t_n - y^*\|,$$
$$\|x_{n+1} - y^*\| \le \rho^3[1-\alpha_n(1-\rho)]\|x_n - y^*\|.$$

Using the triangle inequality, $\|t_{n+1} - x_{n+1}\| \le \|t_{n+1} - y^*\| + \|x_{n+1} - y^*\|$, we get

$$\|t_{n+1} - x_{n+1}\| \le \rho^3[1-\zeta_n(1-\rho)]\|t_n - y^*\| + \rho^3[1-\alpha_n(1-\rho)]\|x_n - y^*\|.$$

Also, employing the triangle inequality $\|x_n - y^*\| \le \|t_n - x_n\| + \|t_n - y^*\|$, we get

$$\begin{aligned}\|t_{n+1} - x_{n+1}\| &\le \rho^3[1-\zeta_n(1-\rho)]\|t_n - y^*\| \\ &\quad + \rho^3[1-\alpha_n(1-\rho)](\|t_n - x_n\| + \|t_n - y^*\|) \\ &\le \rho^3[1-\alpha_n(1-\rho)]\|t_n - x_n\| \\ &\quad + \rho^3([1-\zeta_n(1-\rho)] + [1-\alpha_n(1-\rho)])\|t_n - y^*\|. \end{aligned} \tag{22}$$

Since $0 < \rho < 1$, $0 < \alpha_n, \zeta_n < 1$, we obtain $[1-\alpha_n(1-\rho)] \le 1$, $[1-\zeta_n(1-\rho)] \le 1$ and $\rho^3 + \rho^3 = 2\rho^3 \le 1 + \rho^3$. Thus, (22) becomes

$$\|t_{n+1} - x_{n+1}\| \le \rho^3\|t_n - x_n\| + (1+\rho^3)\|t_n - y^*\|. \tag{23}$$

Now we define

$$\varepsilon_n = \|t_n - x_n\|, \quad \tilde{\sigma}_n = (1+\rho^3)\|t_n - y^*\|.$$

Then inequality (23) becomes

$$\varepsilon_{n+1} \le \rho^3\varepsilon_n + \tilde{\sigma}_n, \quad \text{where } 0 < \rho^3 < 1. \tag{24}$$

Since $\{t_n\}$ converges to y^*, we have $\lim_{n\to\infty} \|t_n - y^*\| = 0$, and hence $\lim_{n\to\infty} \tilde{\sigma}_n = 0$. Therefore, all the conditions of Lemma 2.6 are satisfied. As a result,

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|t_n - x_n\| = 0.$$

Finally, using the triangle inequality,

$$\|x_n - y^*\| \le \|t_n - x_n\| + \|t_n - y^*\| \to 0 \quad \text{as } n \to \infty.$$

Hence, the sequence $\{x_n\}$ also converges strongly to y^*, which concludes the proof of part (ii). □

Theorem 3.3. *Let H be a real Hilbert space and let Ψ be a nonempty, closed, and convex subset of H. Assume that the mappings and parameters satisfy the hypotheses of Theorem 3.1. Let $\{t_n\}$ and $\{x_n\}$ be the sequences generated by the iterative schemes (1) and (2), respectively. Assume that condition (7) holds and*

$$\mathrm{Fix}(A)\cap \mathrm{GVI}(\Psi,f,g)\cap \mathrm{EP}(\psi)\neq\varnothing.$$

Assume, in addition, that there exist constants $0<\zeta<\alpha<1$ such that $\alpha\le\alpha_n<1$ and $\zeta\le\zeta_n<1$ for every $n\in\mathbb{N}$. In this theorem, these bounds replace the requirement $\lim_{n\to\infty}\alpha_n=0$ in (3), which is incompatible with $\alpha_n\ge\alpha>0$; note that the proof of Theorem 3.1 uses only $\{\alpha_n\}\subset(0,1)$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$, and the latter holds automatically since $\alpha_n\ge\alpha>0$. If $t_1=x_1\neq y^*$, then the geometric error upper bound for $\{x_n\}$ converges to zero faster than the corresponding geometric error upper bound for $\{t_n\}$, where $y^*\in\mathrm{Fix}(A)\cap\mathrm{GVI}(\Psi,f,g)\cap\mathrm{EP}(\psi)$.

Proof. From (17), we have

$$\|x_{n+1}-y^*\|\le\rho^3[1-\alpha_n(1-\rho)]\|x_n-y^*\|. \tag{25}$$

By the additional parameter assumptions, $0<\zeta<\alpha\le\alpha_n<1$ for every $n\in\mathbb{N}$. Then from (25) we obtain

$$\|x_{n+1}-y^*\|\le\rho^3[1-\alpha(1-\rho)]\|x_n-y^*\|. \tag{26}$$

Applying (26) recursively yields

$$\|x_{n+1}-y^*\|\le\big(\rho^3[1-\alpha(1-\rho)]\big)^n\|x_1-y^*\|\quad\forall n\in\mathbb{N}. \tag{27}$$

Now, we estimate the convergence behavior of the sequence $\{t_n\}$. Applying the same arguments in Theorem 3.1 for ζ_n, we obtain

$$\|t_{n+1}-y^*\|\le\rho^3[1-\zeta_n(1-\rho)]\|t_n-y^*\|. \tag{28}$$

By the additional parameter assumptions, $0<\zeta\le\zeta_n<1$ for every $n\in\mathbb{N}$. Then from (28) we obtain

$$\|t_{n+1}-y^*\|\le\rho^3[1-\zeta(1-\rho)]\|t_n-y^*\|. \tag{29}$$

By applying (29) recursively we obtain,

$$\|t_{n+1}-y^*\|\le\big(\rho^3[1-\zeta(1-\rho)]\big)^n\|t_1-y^*\|\quad\forall n\in\mathbb{N}. \tag{30}$$

Now from (27) and (30), we define

$$a_n=\big(\rho^3[1-\alpha(1-\rho)]\big)^n\|x_1-y^*\|,\quad b_n=\big(\rho^3[1-\zeta(1-\rho)]\big)^n\|t_1-y^*\|.$$

Since $t_1=x_1$ by hypothesis, we have $\|t_1-y^*\|=\|x_1-y^*\|$. Thus,

$$\frac{a_n}{b_n}=\frac{\big(\rho^3[1-\alpha(1-\rho)]\big)^n\|x_1-y^*\|}{\big(\rho^3[1-\zeta(1-\rho)]\big)^n\|t_1-y^*\|}=\left(\frac{1-\alpha(1-\rho)}{1-\zeta(1-\rho)}\right)^n.$$

Since $0<\rho<1$ and $0<\zeta<\alpha<1$, we have

$$0<\frac{1-\alpha(1-\rho)}{1-\zeta(1-\rho)}<1.$$

Consequently,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=0.$$

Thus, the geometric error upper bound a_n decays faster than the corresponding geometric error upper bound b_n. It should be emphasized that this comparison concerns the derived error upper bounds and does not necessarily imply

$$\lim_{n\to\infty}\frac{\|x_n-y^*\|}{\|t_n-y^*\|}=0.$$

Therefore, the faster decay conclusion is established for the geometric error upper bounds under the stated parameter conditions. □

Now, we provide numerical examples showing the effectiveness and convergence behavior of the proposed algorithm (2) in comparison with various existing iterative methods, that is, Mann iteration, Ishikawa iteration, Halpern iteration, S-iteration, and Dashputre's iterative algorithm (1).

Example 3.1. Let $H=\mathbb{R}^2$ with the standard inner product and norm. Let $\Psi=\{x\in\mathbb{R}^2 : x_1\geq 0, x_2\geq 0\}$ be a nonempty, closed, and convex subset of H. The mappings $f,g:\Psi\to H$ and $A:H\to H$ are defined by:

$$f(x)=(2x_1,2x_2),\quad g(x)=(x_1,x_2),\quad A(x)=\left(\frac{x_1}{2},\frac{x_2}{2}\right).$$

The bifunction $\psi:\Psi\times\Psi\to\mathbb{R}$ is given by $\psi(u,v)=u_1(v_1-u_1)+u_2(v_2-u_2)$, which satisfies conditions (C1)–(C4), and its resolvent is

$$\Phi_r(x)=\left(\frac{x_1}{1+r},\frac{x_2}{1+r}\right),\quad r>0.$$

Clearly, $y^*=(0,0)$ is the unique solution satisfying:

- $A(y^*)=y^*$, thus $y^*\in \mathrm{Fix}(A)$;
- $\langle f(y^*), g(x)-g(y^*)\rangle = 0\geq 0$, thus $y^*\in \mathrm{GVI}(\Psi,f,g)$;
- $\psi(y^*,x)=0\geq 0$, thus $y^*\in \mathrm{EP}(\psi)$.

Moreover, f is $\frac{1}{2}$-inverse strongly monotone and 2-strongly monotone, while $g=I$ is 1-inverse strongly monotone, 1-strongly monotone and satisfies $g(\Psi)=\Psi$; A is non-expansive. For $\lambda=0.4\in(0,2\beta_1)=(0,1)$, the constant in (8) is

$$\rho=2\sqrt{1-(2-1)\cdot 1}+\sqrt{1-0.4(1-0.4)\cdot 4}=0+\sqrt{0.04}=0.2\in(0,1),$$

and $T_\lambda(x)=A\big[P_\Psi(0.2x)\big]=0.1x\in\Psi$ for all $x\in\Psi$. Hence all assumptions of Theorem 3.1 are satisfied.

The numerical parameters are chosen as $\alpha_n=\zeta_n=0.5$. Therefore, the strict parameter condition $0<\zeta<\alpha<1$ required in Theorem 3.3 is not satisfied.

Consequently, the numerical results illustrate the observed convergence behavior of the considered algorithms for the selected parameters, rather than a direct application of the faster decay result for the geometric error upper bounds established in Theorem 3.3.

For the numerical test, we choose the initial point $x_1=t_1=(5,5)$, parameters $\alpha_n=0.5$, $\zeta_n=0.5$, $\lambda=0.4$, $r=1$ (resolvent parameter) and $n=10$ iterations. The absolute error is computed as $\|x_n-y^*\|$.

n	Mann	Ishikawa	Halpern	S-iteration	Dashputre	Proposed
1	7.07e+00	7.07e+00	7.07e+00	7.07e+00	7.07e+00	7.07e+00
2	5.30e+00	4.86e+00	1.77e+00	2.65e+00	3.89e-03	1.94e-03
3	3.98e+00	3.34e+00	5.89e-01	9.94e-01	2.14e-06	5.35e-07
4	2.98e+00	2.30e+00	2.21e-01	3.73e-01	1.18e-09	1.47e-10
5	2.24e+00	1.58e+00	8.84e-02	1.40e-01	6.47e-13	4.04e-14
6	1.68e+00	1.09e+00	3.68e-02	5.24e-02	3.56e-16	1.11e-17
7	1.26e+00	7.47e-01	1.58e-02	1.97e-02	1.96e-19	3.06e-21
8	9.44e-01	5.13e-01	6.91e-03	7.37e-03	1.08e-22	8.41e-25
9	7.08e-01	3.53e-01	3.07e-03	2.77e-03	5.92e-26	2.31e-28
10	5.31e-01	2.43e-01	1.38e-03	1.04e-03	3.26e-29	6.36e-32

Table 1: Comparison of convergence errors $\|x_n-y^*\|$ for different iterative algorithms.

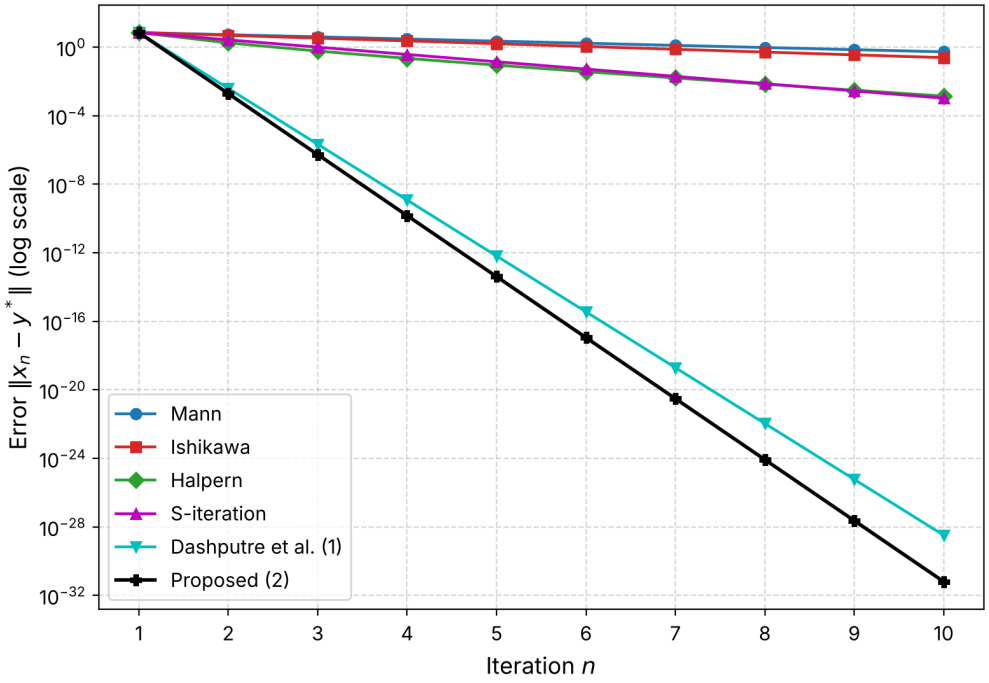

Figure 1: Convergence behavior (logarithmic scale) of the considered iterative schemes in Example 3.1.

From Table 1 and Figure 1, it is observed that all six iterative algorithms converge to the common solution $y^* = (0,0)$. However, their convergence rates differ significantly. In this example, the proposed algorithm (2) converges faster than the algorithm (1) of Dashputre et al. and all other methods considered. Here, the improvement can be attributed to the additional resolvent step $u_n = \Phi_{r_n}(x_n)$, which in this example is itself a contraction, since $\Phi_1(x) = \frac{x}{2}$.

Example 3.2. Let $H = \mathbb{R}$ and $\Psi = [1,3]$, equipped with the usual norm $\|x\| = |x|$ and inner product $\langle x, y\rangle = xy$. Define the mappings $f : \Psi \to H$, $g : \Psi \to \Psi$, and $A : H \to H$ by

$$f(x) = 2x, \qquad g(x) = x, \qquad A(x) = \frac{x+1}{2}.$$

Let the bifunction $\psi : \Psi \times \Psi \to \mathbb{R}$ be defined by

$$\psi(x,y) = (y-x)(x-1).$$

First, for all $x, y \in \Psi$, we have

$$\begin{aligned}\langle f(x) - f(y), x - y\rangle &= 2|x-y|^2 \\ &= \frac{1}{2}\|f(x) - f(y)\|^2.\end{aligned}$$

Thus, f is $\frac{1}{2}$-inverse strongly monotone. Similarly,

$$\begin{aligned}\langle g(x) - g(y), x - y\rangle &= |x-y|^2 \\ &= \|g(x) - g(y)\|^2.\end{aligned}$$

Hence, g is 1-inverse strongly monotone. Moreover, since $g(x) = x$ for all $x \in \Psi$, we obtain

$$g(\Psi) = \Psi.$$

Therefore, g is surjective onto Ψ, as required by Lemma 2.4. Furthermore,

$$|A(x) - A(y)| = \frac{1}{2}|x-y| \le |x-y|,$$

so A is non-expansive. Next, we verify conditions (C1)–(C4) for the equilibrium bifunction ψ. For every $x \in \Psi$,

$$\psi(x,x) = 0,$$

and hence (C1) holds. For all $x, y \in \Psi$,

$$\begin{aligned}\psi(x,y)+\psi(y,x) &= (y-x)(x-1)+(x-y)(y-1)\\ &= -(x-y)^2 \le 0.\end{aligned}$$

Thus, ψ is monotone and (C2) holds.

Since ψ is continuous in its first variable, for any $x, y, z \in \Psi$,

$$\limsup_{t\downarrow 0} \psi(tz+(1-t)x, y) = \psi(x,y).$$

Therefore, condition (C3) is satisfied. For each fixed $x \in \Psi$, the function

$$y \mapsto \psi(x,y) = (x-1)(y-x)$$

is affine and continuous. Consequently, it is convex and weakly lower semicontinuous, so (C4) holds.

We now determine the common solution. Solving $A(y^*) = y^*$ gives

$$y^* = 1.$$

Moreover, for every $y \in \Psi$,

$$\psi(1,y) = 0 \ge 0.$$

Therefore, $1 \in \mathrm{EP}(\psi)$. Also,

$$\begin{aligned}\langle f(1), g(y)-g(1)\rangle &= 2(y-1)\\ &\ge 0, \qquad \forall y \in [1,3].\end{aligned}$$

Hence,

$$1 \in \mathrm{GVI}(\Psi, f, g).$$

Consequently,

$$y^* = 1 \in \mathrm{Fix}(A) \cap \mathrm{GVI}(\Psi, f, g) \cap \mathrm{EP}(\psi).$$

Finally, f is also 2-strongly monotone and g is 1-strongly monotone, so that $\beta_1 = \frac{1}{2}$, $\mu_1 = 2$, $\beta_2 = \mu_2 = 1$. Choose $\lambda = \frac{1}{4} \in (0, 2\beta_1) = (0,1)$. Then the contraction factor appearing in Theorem 3.1 is

$$\rho = 2\sqrt{1-(2\beta_2-1)\mu_2^2} + \sqrt{1-\lambda(2\beta_1-\lambda)\mu_1^2} = 0 + \sqrt{1-\frac{1}{4}\cdot\frac{3}{4}\cdot 4} = \frac{1}{2} \in (0,1).$$

Moreover, for $x \in \Psi$ we have $x - g(x) + P_\Psi\{g(x) - \lambda f(x)\} = P_{[1,3]}(x/2) = \max\{1, x/2\}$, hence

$$T_\lambda(x) = \frac{\max\{1, x/2\}+1}{2} \in \Big[1, \frac{7}{4}\Big] \subset \Psi,$$

so T_λ maps Ψ into Ψ. For instance, the control sequence $\alpha_n = 1/(n+2)$ satisfies the conditions in (3), and we may take $r_n = 1$. Thus, all assumptions of Theorem 3.1 are satisfied, and the proposed iterative scheme converges strongly to the common solution $y^* = 1$.

Moreover, to illustrate Theorem 3.3, choose the constant parameters $\alpha_n = \alpha = \frac{3}{4}$ and $\zeta_n = \zeta = \frac{1}{4}$ for all $n \in \mathbb{N}$, so that $0 < \zeta < \alpha < 1$. Since $\rho = \frac{1}{2}$, we have $\rho^3[1-\alpha(1-\rho)] = \frac{5}{64}$ and $\rho^3[1-\zeta(1-\rho)] = \frac{7}{64}$. Hence, the geometric error upper bounds in (27) and (30) become

$$a_n = \Big(\frac{5}{64}\Big)^n \|x_1 - y^*\|, \qquad b_n = \Big(\frac{7}{64}\Big)^n \|t_1 - y^*\|.$$

Therefore, if $t_1 = x_1 \neq y^*$, then

$$\frac{a_n}{b_n} = \Big(\frac{5}{7}\Big)^n \to 0 \quad \text{as } n \to \infty,$$

that is, in addition to strong convergence, the geometric error upper bound of the proposed algorithm (2) decays faster than that of the algorithm (1).

4 Applications

4.1 Convex Optimization Problems

In this section, we adapt the structure of the proposed algorithm (2) to solve a convex minimization problem.

Let H be a real Hilbert space and let Ψ be a nonempty closed convex subset of H. Let $P_\Psi : H \to \Psi$ denote the metric projection onto Ψ. Assume that $F : \Psi \to \mathbb{R}$ is a convex and Fréchet differentiable function, and consider the convex minimization problem [43]

$$\min_{x\in\Psi} F(x). \tag{31}$$

A point $x^* \in \Psi$ solves problem (31) precisely when it satisfies the variational inequality

$$\langle \nabla F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Psi.$$

Equivalently, x^* solves problem (31) precisely when it is a fixed point of the projection operator $P_\Psi(I - \lambda\nabla F)$, that is,

$$x^* = P_\Psi[x^* - \lambda\nabla F(x^*)]. \tag{32}$$

The fixed point formulation (32) underlies the gradient projection method, which is a fundamental tool for solving constrained convex optimization problems [44]. Projection-type and gradient-based methods have been thoroughly studied for solving problem (31) [45, 46, 47].

Motivated by the proposed iterative framework, we consider the following modified gradient projection algorithm for solving problem (31). Its strong convergence and error estimates are established independently under the additional strong monotonicity assumption on ∇F:

$$\begin{cases} x_1 \in \Psi, \\ w_n = (1-\alpha_n)x_n + \alpha_n P_\Psi[x_n - \lambda\nabla F(x_n)], \\ z_n = P_\Psi[w_n - \lambda\nabla F(w_n)], \\ y_n = P_\Psi[z_n - \lambda\nabla F(z_n)], \\ x_{n+1} = P_\Psi[y_n - \lambda\nabla F(y_n)], \end{cases} \tag{33}$$

where $\{\alpha_n\} \subset (0,1)$.

For the iterative scheme (33), we obtain the following result.

Theorem 4.1. *Let H be a real Hilbert space and let Ψ be a nonempty, closed, and convex subset of H. Let $F : \Psi \to \mathbb{R}$ be a convex and Fréchet differentiable mapping whose gradient $\nabla F : \Psi \to H$ is β_1-inverse strongly monotone and μ-strongly monotone, that is,*

$$\langle \nabla F(x) - \nabla F(y), x - y \rangle \geq \mu\|x - y\|^2, \quad \forall x, y \in \Psi,$$

for some $\mu > 0$. Assume that the convex minimization problem (31) has a solution x^*, let $\{\alpha_n\} \subset (0,1)$, and let $\lambda \in (0, 2\beta_1)$. Define

$$\rho = \sqrt{1 - \lambda(2\beta_1 - \lambda)\mu^2}. \tag{33a}$$

Then $0 \leq \rho < 1$, x^* is the unique solution of (31), and the sequence $\{x_n\}$ generated by (33) satisfies

$$\|x_{n+1} - x^*\| \leq \rho^3[1 - \alpha_n(1-\rho)]\|x_n - x^*\|, \quad \forall n \in \mathbb{N}, \tag{33b}$$

and consequently

$$\|x_{n+1} - x^*\| \leq \rho^{3n}\prod_{k=1}^{n}[1 - \alpha_k(1-\rho)]\,\|x_1 - x^*\| \leq \rho^{3n}\|x_1 - x^*\|, \quad \forall n \in \mathbb{N}. \tag{33c}$$

In particular, $\{x_n\}$ converges strongly and linearly to x^*.

Proof. Since ∇F is β_1-inverse strongly monotone, it is $\frac{1}{\beta_1}$-Lipschitz continuous. Combining this with the μ-strong monotonicity and the Cauchy–Schwarz inequality gives, for all $x, y \in \Psi$,

$$\mu\|x - y\| \leq \|\nabla F(x) - \nabla F(y)\| \leq \frac{1}{\beta_1}\|x - y\|. \tag{33d}$$

In particular, $\beta_1\mu \leq 1$. Since $0 < \lambda(2\beta_1 - \lambda) \leq \beta_1^2$ for $\lambda \in (0, 2\beta_1)$, we get $0 < \lambda(2\beta_1 - \lambda)\mu^2 \leq \beta_1^2\mu^2 \leq 1$, hence $0 \leq \rho < 1$.

Let $T = P_\Psi(I - \lambda\nabla F)$. For $x, y \in \Psi$, by the β_1-inverse strong monotonicity of ∇F (as in (12)) and the left inequality in (33d), we have

$$\begin{aligned} \|x - y - \lambda\{\nabla F(x) - \nabla F(y)\}\|^2 &\leq \|x - y\|^2 - \lambda(2\beta_1 - \lambda)\|\nabla F(x) - \nabla F(y)\|^2 \\ &\leq \big(1 - \lambda(2\beta_1 - \lambda)\mu^2\big)\|x - y\|^2 = \rho^2\|x - y\|^2, \end{aligned}$$

where we used $\lambda(2\beta_1-\lambda)>0$. As P_Ψ is non-expansive (Lemma 2.3), it follows that

$$\|T(x)-T(y)\|\le\rho\|x-y\|,\quad\forall x,y\in\Psi,\tag{33e}$$

that is, T is a contraction on Ψ. Every solution of (31) is a fixed point of T by (32); since a contraction has at most one fixed point, x^* is the unique solution of (31) and $T(x^*)=x^*$.

All iterates of (33) lie in Ψ, because $x_1\in\Psi$, P_Ψ maps into Ψ, and Ψ is convex. Using (33) and (33e), we obtain

$$\begin{aligned}\|w_n-x^*\|&\le(1-\alpha_n)\|x_n-x^*\|+\alpha_n\|T(x_n)-T(x^*)\|\le[1-\alpha_n(1-\rho)]\|x_n-x^*\|,\\ \|z_n-x^*\|&=\|T(w_n)-T(x^*)\|\le\rho\|w_n-x^*\|,\\ \|y_n-x^*\|&=\|T(z_n)-T(x^*)\|\le\rho\|z_n-x^*\|,\\ \|x_{n+1}-x^*\|&=\|T(y_n)-T(x^*)\|\le\rho\|y_n-x^*\|.\end{aligned}$$

Combining these four inequalities yields (33b). Applying (33b) successively for $k=n,n-1,\dots,1$ gives the first inequality in (33c), and the second follows from $0<1-\alpha_k(1-\rho)\le1$. Since $0\le\rho<1$, we conclude $\|x_n-x^*\|\to0$ at a linear rate. □

Remark 4. (i) Inverse strong monotonicity of ∇F alone does not yield a contraction factor smaller than 1: for $\lambda\in(0,2\beta_1]$ it only guarantees that $P_\Psi(I-\lambda\nabla F)$ is non-expansive. For instance, take $H=\Psi=\mathbb{R}^2$ and $F(x)=\frac{x_1^2}{2}+\frac{x_2^2}{4}$, so that $\nabla F=\operatorname{diag}(1,\frac12)$ is 1-inverse strongly monotone and $\frac12$-strongly monotone, and let $\lambda=\frac12$. Then $P_\Psi(I-\lambda\nabla F)=\operatorname{diag}(\frac12,\frac34)$ has Lipschitz constant $\frac34$, which exceeds the value $\left|\frac{\beta_1-\lambda}{\beta_1}\right|=\frac12$. In contrast, (33a) gives $\rho=\frac{\sqrt{13}}{4}\approx0.901\ge\frac34$, in agreement with (33e). This is why the strong monotonicity assumption on ∇F is required for the linear error estimate (33c).

(ii) The iteration starts at x_1, so the error estimate (33c) is expressed in terms of $\|x_1-x^*\|$ and the product runs over $k=1,\dots,n$.

(iii) Theorem 4.1 is consistent with Theorem 3.1: taking $f=\nabla F$, $g=I$, $A=I$ and $\psi\equiv0$ (so that $\Phi_r=P_\Psi$ is the identity on Ψ), we have $\beta_2=\mu_2=1$, the first term in (8) vanishes, and (8) reduces to (33a).

Example 4.1. Let $H=L^2[0,1]=\{h:[0,1]\to\mathbb{R}:\int_0^1h^2(x)dx<\infty\}$. Then $(H,\|\cdot\|_2)$ forms a Hilbert space, where the norm and inner product are given by

$$\|h\|_2^2=\langle h,h\rangle=\int_0^1h^2(x)dx.$$

Consider the closed convex subset,

$$\Psi=\{h\in L^2[0,1]:\|h\|_2\le1\}.$$

Define the functional $F:\Psi\to\mathbb{R}$ by

$$F(h)=\|h\|_2^2=\int_0^1h^2(x)dx.$$

The functional F is Fréchet differentiable on Ψ and its gradient $\nabla F:\Psi\to H$ is given by

$$\nabla F(h)=2h,\quad\forall h\in\Psi.$$

We now demonstrate the inverse strong monotonicity of ∇F. Let $h_1,h_2\in\Psi$, we get

$$\langle\nabla F(h_1)-\nabla F(h_2),h_1-h_2\rangle=\langle2h_1-2h_2,h_1-h_2\rangle=2\|h_1-h_2\|_2^2.$$

On the other hand,

$$\|\nabla F(h_1)-\nabla F(h_2)\|_2^2=\|2(h_1-h_2)\|_2^2=4\|h_1-h_2\|_2^2.$$

Combining the two equalities, we get

$$\langle\nabla F(h_1)-\nabla F(h_2),h_1-h_2\rangle=\frac12\|\nabla F(h_1)-\nabla F(h_2)\|_2^2.$$

Thus, ∇F is $\frac12$-inverse strongly monotone, that is $\beta_1=\frac12$. Moreover, the first equality shows that ∇F is μ-strongly monotone with $\mu=2$.

Taking $\lambda=\frac14\in(0,2\beta_1)=(0,1)$, the constant in (33a) is

$$\rho=\sqrt{1-\frac14\left(1-\frac14\right)\cdot4}=\sqrt{\frac14}=\frac12.$$

Thus, all assumptions of Theorem 4.1 hold. In this example the constant $\rho=\frac12$ is sharp, since $h-\lambda\nabla F(h)=\frac12h$ for every $h\in\Psi$.

Clearly, the convex minimization problem

$$\min_{h\in\Psi}F(h)$$

admits the unique solution $h^*(x) = 0$.

Choosing the parameter sequence $\alpha_n = \frac{1}{n+1} \in (0,1)$, $n \in \mathbb{N}$, which satisfies $\sum_{n=1}^{\infty} \alpha_n = \infty$, the iterative algorithm (33) becomes

$$\begin{cases} x_1 \in \Psi, \\ w_n = \left(1 - \dfrac{1}{n+1}\right) x_n + \dfrac{1}{n+1} P_\Psi[x_n - \lambda \nabla F(x_n)], \\ z_n = P_\Psi[w_n - \lambda \nabla F(w_n)], \\ y_n = P_\Psi[z_n - \lambda \nabla F(z_n)], \\ x_{n+1} = P_\Psi[y_n - \lambda \nabla F(y_n)], \end{cases} \tag{34}$$

where $P_\Psi : H \to \Psi$ is the metric projection onto the closed unit ball of $L^2[0,1]$, given by:

$$P_\Psi(h) = \begin{cases} h, & \|h\|_2 \leq 1, \\ \dfrac{h}{\|h\|_2}, & \|h\|_2 > 1. \end{cases}$$

Since all conditions of Theorem 4.1 are met, the sequence $\{x_n\}$ from (34) converges strongly in $L^2[0,1]$ to the unique solution $h^*(x) = 0$, and by (33c),

$$\|x_{n+1}\|_2 \leq \left(\frac{1}{8}\right)^n \prod_{k=1}^{n} \left[1 - \frac{1}{2(k+1)}\right] \|x_1\|_2 \leq \left(\frac{1}{8}\right)^n \|x_1\|_2, \quad \forall n \in \mathbb{N}.$$

4.2 Split Feasibility Equilibrium Problems

Here, we extend the split feasibility problem (SFP) by including an equilibrium problem and applying Theorem 3.1 to solve the combined problem.

Let H_1 and H_2 be real Hilbert spaces. Consider two nonempty, closed, and convex sets $\Psi_1 \subset H_1$ and $\Psi_2 \subset H_2$. Let $B : H_1 \to H_2$ be a bounded linear operator, and let $B^* : H_2 \to H_1$ denote its adjoint.

The split feasibility problem (SFP) consists of finding a point x^* such that

$$x^* \in \Psi_1 \quad \text{and} \quad Bx^* \in \Psi_2. \tag{35}$$

Let Υ be the solution set of problem (35); then,

$$\Upsilon = \{x^* \in \Psi_1 : Bx^* \in \Psi_2\} = \Psi_1 \cap B^{-1}\Psi_2.$$

The split feasibility problem is widely applied in numerous fields, including inverse problems, signal processing, and image reconstruction [48, 49, 50].

In [51], Xu established a relationship between the split feasibility problem (35) and the fixed point problem associated with the operator

$$S = P_{\Psi_1}[I - \lambda B^*(I - P_{\Psi_2})B],$$

where B^* denotes the adjoint operator of B and P_{Ψ_1} and P_{Ψ_2} are the metric projection operators onto Ψ_1 and Ψ_2 respectively.

More precisely, for $\lambda > 0$, a point $x^* \in \Psi_1$ is a solution of problem (35) if and only if

$$x^* = P_{\Psi_1}[I - \lambda B^*(I - P_{\Psi_2})B](x^*).$$

Consequently, solving the split feasibility problem reduces to finding a fixed point of the operator

$$S = P_{\Psi_1}[I - \lambda B^*(I - P_{\Psi_2})B].$$

Byrne [44] proposed the following iterative scheme, known as the CQ algorithm, to solve the split feasibility problem (35):

$$x_{n+1} = P_{\Psi_1}[I - \lambda B^*(I - P_{\Psi_2})B](x_n), \quad \forall n \geq 0,$$

where $0 < \lambda < \frac{2}{\|B\|^2}$.

Recall that $\mathrm{EP}(\psi)$ denotes the solution set of the equilibrium problem (5). We therefore aim to find a common solution

$$x^* \in \Upsilon \cap \mathrm{EP}(\psi),$$

that is,

$$x^* \in \Psi_1, \quad Bx^* \in \Psi_2, \quad x^* \in \mathrm{EP}(\psi).$$

From Lemma 2.2, the resolvent operator Φ_r satisfies $\mathrm{Fix}(\Phi_r) = \mathrm{EP}(\psi)$.

Based on the above results and applying Theorem 3.1, we propose a new iterative scheme for approximating a common solution of both the split feasibility problem and the equilibrium problem as follows:

$$\begin{cases} x_1 \in \Psi_1, \\ u_n = \Phi_{r_n}(x_n), \\ w_n = (1-\alpha_n)u_n + \alpha_n S(u_n), \\ z_n = S(w_n), \\ y_n = S(z_n), \\ x_{n+1} = S(y_n), \end{cases} \tag{36}$$

where $\{\alpha_n\} \subset (0,1)$, $r_n > 0$ and $S = P_{\Psi_1}[I - \lambda B^*(I - P_{\Psi_2})B]$ is non-expansive whenever $0 < \lambda < \frac{2}{\|B\|^2}$.

Theorem 4.2. *Assume that $\Upsilon \cap \mathrm{EP}(\psi) \neq \varnothing$, that $0 < \lambda < \frac{2}{\|B\|^2}$, and that the mapping $f = B^*(I - P_{\Psi_2})B$ is μ-strongly monotone on Ψ_1 for some $\mu > 0$. Let $\{\alpha_n\} \subset (0,1)$ and $r_n > 0$, and define*

$$\rho = \sqrt{1 - \lambda\Big(\frac{2}{\|B\|^2} - \lambda\Big)\mu^2}. \tag{36a}$$

If $\rho > 0$, then $\Upsilon \cap \mathrm{EP}(\psi) = \{x^*\}$ and the sequence $\{x_n\}$ from (36) converges strongly to x^*, that is,

$$x^* \in \Psi_1, \quad Bx^* \in \Psi_2, \quad x^* \in \mathrm{EP}(\psi).$$

In addition, the error estimate below holds:

$$\|x_{n+1} - x^*\| \leq \rho^{3n}\|x_1 - x^*\| \prod_{k=1}^{n}[1 - \alpha_k(1-\rho)], \quad \forall n \in \mathbb{N}.$$

Proof. It is well known that $f = B^*(I - P_{\Psi_2})B$ is the gradient of the convex function $\frac{1}{2}\|(I - P_{\Psi_2})Bx\|^2$ and is $\frac{1}{\|B\|^2}$-inverse strongly monotone [44, 51]. We apply Theorem 3.1 with $\Psi = \Psi_1$, $f = B^*(I - P_{\Psi_2})B$, $g = I$ and $A = I$, so that $\beta_1 = \frac{1}{\|B\|^2}$, $\mu_1 = \mu$ and $\beta_2 = \mu_2 = 1$. Then $g(\Psi_1) = \Psi_1$, $\mathrm{Fix}(A) = H_1$ and

$$T_\lambda(x) = x - x + P_{\Psi_1}[x - \lambda f(x)] = S(x),$$

which maps Ψ_1 into Ψ_1. Hence algorithm (2) reduces to (36), the constant (8) reduces to (36a), and condition (7) holds since $0 < \lambda < 2\beta_1$ and $0 < \rho < 1$ (Remark 3). By Lemma 2.4 and [51], a point $x \in \Psi_1$ belongs to $\mathrm{GVI}(\Psi_1, f, I)$ if and only if $x = S(x)$, and, since $\Upsilon \neq \varnothing$, this holds if and only if $x \in \Upsilon$. Therefore,

$$\mathrm{Fix}(A) \cap \mathrm{GVI}(\Psi_1, f, I) \cap \mathrm{EP}(\psi) = \Upsilon \cap \mathrm{EP}(\psi) \neq \varnothing.$$

Let $x^* \in \Upsilon \cap \mathrm{EP}(\psi)$. If $x \in \Upsilon$, then $S(x) = x$, and (14a) (with $y^* = x^*$) gives $\|x - x^*\| \leq \rho\|x - x^*\|$; since $\rho < 1$, $x = x^*$. Hence $\Upsilon \cap \mathrm{EP}(\psi) = \{x^*\}$. The conclusion and the error estimate now follow from Theorem 3.1. □

Remark 5. Since $f(x) = 0$ for every $x \in \Upsilon$, the strong monotonicity of f on Ψ_1 forces Υ to contain at most one point: if $x, y \in \Upsilon$, then $0 = \langle f(x) - f(y), x - y\rangle \geq \mu\|x - y\|^2$. This assumption holds, for instance, when $\Psi_2 = \{b\}$ and $B^*B \geq \mu I$, because then $f(x) = B^*(Bx - b)$ and $\langle f(x) - f(y), x - y\rangle = \|B(x - y)\|^2 \geq \mu\|x - y\|^2$. Without strong monotonicity, S is only non-expansive for $0 < \lambda < \frac{2}{\|B\|^2}$, and, as shown in Remark 4, a linear error estimate of the above form cannot be expected in general.

5 Conclusion

In this paper, we proposed and analyzed a new iterative scheme for approximating a common solution of three important classes of problems in nonlinear analysis, namely generalized variational inequality problems, equilibrium problems, and fixed point problems for non-expansive mappings.

The proposed algorithm (2) extends the iterative method (1) of Dashputre et al. [36] by incorporating the resolvent operator of an equilibrium problem, which allows these classes of problems to be treated within a single iterative structure.

Under suitable assumptions on the control parameters and the involved operators, in particular the inverse strong monotonicity and strong monotonicity of f and g, we established in Theorem 3.1 that the sequence $\{x_n\}$ generated by (2) converges strongly to a point $y^* \in \mathrm{Fix}(A) \cap \mathrm{GVI}(\Psi, f, g) \cap \mathrm{EP}(\psi)$, and we obtained explicit error estimates showing linear convergence. We also compared the proposed algorithm with the method of Dashputre et al. (Theorems 3.2 and 3.3).

Numerical examples were presented to support the theoretical results and to illustrate the performance of the proposed algorithm in comparison with existing methods.

Furthermore, we applied the theoretical results to convex minimization problems and to a combined split feasibility and equilibrium problem, for which we formulated a new iterative method and established linear convergence under a strong monotonicity assumption.

The main contribution of this work is a unified iterative framework that addresses generalized variational inequality problems, equilibrium problems, and fixed point problems simultaneously, whereas many existing algorithms address only one or two of these problems.

Future research could focus on extending the proposed algorithm to more general settings, such as Banach spaces and stochastic frameworks, on relaxing the strong monotonicity assumptions, and on inertial-type and other acceleration strategies to further improve the convergence speed.

Acknowledgments

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions. Also, the authors would like to thank the Gümüşhane University Scientific Research Projects Coordination Unit (project number 26.E0120.07.01) for the financial support provided during the preparation of this article.

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