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On mixed-type generalized degenerate Lucas–Bernoulli/Euler polynomials

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Abstract

In this paper, we define and study the mixed-type generalized degenerate Lucas–Bernoulli and generalized degenerate Lucas–Euler polynomials. We obtain a wide range of novel and meaningful identities and relationships within these classes of polynomials. In addition, we present numerical approximations of zeros and computational visualizations that illustrate the structure and distribution of the zeros of these mixed-type polynomials.

1 Introduction

Special polynomials such as the Fibonacci, Lucas, Bernoulli, and Euler families play a fundamental role in mathematical analysis, number theory, and combinatorics. These classical sequences exhibit elegant algebraic properties and are deeply related to generating functions, recurrence relations, and various applications in both theoretical and applied disciplines, including cryptography, coding theory, and mathematical physics [4, 9, 23, 24].

In addition to their intrinsic algebraic interest, special polynomial systems and their degenerate/generalized variants appear naturally in integral evaluations involving hypergeometric- and Bessel-type functions and in the construction of computational schemes for differential and fractional differential equations. Representative examples include integral relations in [1, 2, 15, 16], polynomial-based discretizations for variable-order fractional models in [10], and model-oriented uses of generalized Fibonacci-type polynomials in [3]; see also the Apostol-type generalizations and related constructions in [7].

In particular, Fibonacci and Lucas polynomials have been widely studied due to their structural richness and links to combinatorics. Fibonacci polynomials are defined by a recurrence relation analogous to the classical Fibonacci sequence and have applications ranging from number theory to algorithm design. Lucas polynomials, defined similarly, are related to Lucas numbers and show intriguing properties associated with prime numbers and quadratic residues [11, 17, 18].

Simsek [28] conducted a comprehensive study of generating functions for special numbers and polynomials, including multivariate generalizations. Carlitz [5, 6] introduced degenerate versions of the Bernoulli and Euler polynomials, revealing new arithmetic and combinatorial structures. These degenerate families, along with other mixed-type extensions, have become a significant area of exploration [8, 12, 13, 14, 19, 25, 26, 27, 29].

The main objective of this paper is to define and analyze new families of polynomials by combining Lucas polynomials with degenerate Bernoulli and Euler polynomials. In Section 2, we introduce the mixed-type generalized degenerate Lucas–Bernoulli and generalized degenerate Lucas–Euler, derived from generating functions and functional equations. We then establish new identities, recurrence relations, and present specific zero values and computational visualizations that illustrate the structure and distribution of the zeros of these mixed-type polynomials.

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2 Notation and classical background

Throughout this article, we use the standard notation: $\mathbb{N}$ for the set of natural numbers, $\mathbb{N}_0$ for non-negative integers, $\mathbb{Z}$ for integers, $\mathbb{R}$ for real numbers, and $\mathbb{C}$ for complex numbers. The principal branch is consistently applied for complex powers, with the conventions $1^\alpha = 1$ for all $\alpha \in \mathbb{C}$, and $0^0 = 1$.

The Fibonacci polynomials $F_n(\xi)$ are defined for $n \geq 2$ by the recurrence:

$$F_n(\xi) = \xi F_{n-1}(\xi) + F_{n-2}(\xi),$$

with initial conditions $F_0(\xi) = 0$, $F_1(\xi) = 1$. Lucas polynomials $L_n(\xi)$ satisfy a similar recurrence:

$$L_n(\xi) = \xi L_{n-1}(\xi) + L_{n-2}(\xi),$$

with $L_0(\xi) = 2$, $L_1(\xi) = \xi$. For $\xi = 1$, these polynomials yield the classical Fibonacci and Lucas numbers: $F_n(1) = F_n$, $L_n(1) = L_n$.

Their closed-form expressions involve the roots $\alpha(\xi) = \frac{\xi+\sqrt{\xi^2+4}}{2}$ and $\beta(\xi) = \frac{\xi-\sqrt{\xi^2+4}}{2}$:

$$F_n(\xi) = \frac{\alpha^n(\xi) - \beta^n(\xi)}{\alpha(\xi) - \beta(\xi)} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} \xi^{n-2k-1},$$

$$L_n(\xi) = \alpha^n(\xi) + \beta^n(\xi) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} \xi^{n-2k}.$$

Their generating functions are:

$$\frac{t}{1-\xi t - t^2} = \sum_{n=0}^{\infty} F_n(\xi) t^n, \quad |t| < \frac{1}{\alpha(\xi)}$$

and

$$\frac{2-\xi t}{1-\xi t - t^2} = \sum_{n=0}^{\infty} L_n(\xi) t^n, \quad |t| < \frac{1}{\alpha(\xi)}. \tag{1}$$

Simsek introduced multivariable polynomial families defined by generating functions:

$$\sum_{n=0}^{\infty} \mathbb{Y}_n(P(\vec{\xi}_m)) w^n = \frac{1}{1 + \sum_{j=1}^{m} P_j(\xi_j) w^j},$$

$$\sum_{n=0}^{\infty} \mathbb{S}_n(P(\vec{\xi}_m); Q(\vec{\xi}_k)) w^n = \frac{\sum_{j=0}^{k} Q_j(\xi_j) w^j}{1 + \sum_{j=1}^{m} P_j(\xi_j) w^j},$$

where $P_j(\xi_j) = \sum_{v=0}^{d} a_v \xi_j^v$ and $Q_l(\xi_l) = \sum_{v=0}^{c} b_v \xi_l^v$.

For $s \in \mathbb{C}$, the falling factorial is defined as:

$$(s)_0 = 1, \quad (s)_i = \prod_{j=1}^{i} (s-j+1), \quad i \geq 1.$$

The Bernoulli and Euler polynomials of order α are defined via generating functions:

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{\xi t} = \sum_{n=0}^{\infty} \mathscr{B}_n^{(\alpha)}(\xi) \frac{t^n}{n!}, \quad |t| < 2\pi,$$

$$\left(\frac{2}{e^t+1}\right)^\alpha e^{\xi t} = \sum_{n=0}^{\infty} \mathscr{E}_n^{(\alpha)}(\xi) \frac{t^n}{n!}, \quad |t| < \pi.$$

Degenerate versions are based on the degenerate exponential function (cf. [8, 26]):

$$e_h^\xi(t) = (1+ht)^{\xi/h}, \quad h \neq 0,$$
$$e_h^\xi(t) = e^{\xi t}, \quad h = 0,$$

with the corresponding expansions involving:

$$(\xi)_{n,h} = \prod_{i=1}^{n} (\xi - (i-1)h).$$

Carlitz defined degenerate Bernoulli and Euler polynomials by:

$$\left[\frac{t}{e_h(t)-1}\right]^\alpha e_h^\xi(t) = \sum_{n=0}^{\infty} \mathscr{B}_{n,h}^{(\alpha)}(\xi) \frac{t^n}{n!}, \tag{2}$$

$$\left[\frac{2}{e_h(t)+1}\right]^\alpha e_h^\xi(t) = \sum_{n=0}^{\infty} \mathscr{E}_{n,h}^{(\alpha)}(\xi) \frac{t^n}{n!}.$$

For further developments, see [12, 14, 19, 25, 27, 29] and references therein.

3 Mixed-type generalized degenerate Lucas–Bernoulli/Euler polynomials

This section presents the mixed-type generalized degenerate Lucas–Bernoulli and generalized degenerate Lucas–Euler polynomials. An examination of their generating functions yields insights into the characteristics of these polynomial types.

Definition 3.1. For all $n \in \mathbb{N}_0$, $h \in \mathbb{R}$ and $\alpha \in \mathbb{C}$, the mixed-type generalized degenerate Lucas–Bernoulli and generalized degenerate Lucas–Euler polynomials of degree n are defined by means of the following generating functions:

$$\left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{t}{e_h(t)-1}\right]^{\alpha} e_h^{\xi}(t)=\sum_{n=0}^{\infty} {}_L\mathscr{B}_{n,h}^{(\alpha)}(\xi)\frac{t^n}{n!}, \tag{3}$$

where $|t|<\min\left\{\dfrac{2}{\xi+\sqrt{\xi^2+4}},2\pi\right\}$

and

$$\left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{2}{e_h(t)+1}\right]^{\alpha} e_h^{\xi}(t)=\sum_{n=0}^{\infty} {}_L\mathscr{E}_{n,h}^{(\alpha)}(\xi)\frac{t^n}{n!}, \tag{4}$$

where $|t|<\min\left\{\dfrac{2}{\xi+\sqrt{\xi^2+4}},\pi\right\}$.

Remark 1. The two mixed-type families $\{{}_L\mathscr{B}_{n,h}^{(\alpha)}(\xi)\}_{n\geq 0}$ and $\{{}_L\mathscr{E}_{n,h}^{(\alpha)}(\xi)\}_{n\geq 0}$ share the same Lucas prefactor $\dfrac{2-\xi t}{1-\xi t-t^2}$ in (3)–(4). Hence the *basis difference* is entirely governed by the kernel that generates the "Appell/Sheffer part": $\left(\frac{t}{e_h(t)-1}\right)^{\alpha}$ (Bernoulli-type) versus $\left(\frac{2}{e_h(t)+1}\right)^{\alpha}$ (Euler-type). Equivalently, both sequences are Sheffer-type for the same delta series $f(t)=\dfrac{1}{h}\log(1+ht)$ (so the same lowering operator), but with different invertible series $g_B(t)=\left(\frac{e_h(t)-1}{t}\right)^{\alpha}\frac{1-\xi t-t^2}{2-\xi t}$ and $g_E(t)=\left(\frac{e_h(t)+1}{2}\right)^{\alpha}\frac{1-\xi t-t^2}{2-\xi t}$. Therefore, $\{{}_L\mathscr{B}_{n,h}^{(\alpha)}\}$ and $\{{}_L\mathscr{E}_{n,h}^{(\alpha)}\}$ form genuinely different polynomial bases: they are not connected by a scalar prefactor, and they inherit distinct normalization/parity properties from the Bernoulli and Euler kernels.

Proposition 3.1. *The mixed-type generalized degenerate Lucas–Bernoulli $\left\{{}_L\mathscr{B}_{n,h}(\xi)\right\}_{n\geq 0}$ and generalized degenerate Lucas–Euler $\left\{{}_L\mathscr{E}_{n,h}(\xi)\right\}_{n\geq 0}$ polynomials satisfy the following identities:*

$${}_L\mathscr{B}_{n,h}^{(\alpha)}(\xi)=\sum_{k=0}^{n}\frac{n!}{(n-k)!}L_k(\xi)\mathscr{B}_{n-k,h}^{(\alpha)}(\xi), \quad n\geq 0,\ h\in\mathbb{R}, \tag{5}$$

and

$${}_L\mathscr{E}_{n,h}^{(\alpha)}(\xi)=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}L_k(\xi)\mathscr{E}_{n-k,h}^{(\alpha)}(\xi), \quad n\geq 0. \tag{6}$$

Proof. Due to the generating functions (3)

$$\sum_{n=0}^{\infty}{}_L\mathscr{B}_{n,h}^{(\alpha)}(\xi)\frac{t^n}{n!}=\left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{t}{e_h(t)-1}\right]^{\alpha}e_h^{\xi}(t). \tag{7}$$

Substituting (1) and (2) into (7) and using the Cauchy product, we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_L\mathscr{B}_{n,h}^{(\alpha)}(\xi)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}L_n(\xi)t^n\sum_{n=0}^{\infty}\mathscr{B}_{n,h}^{(\alpha)}(\xi)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}L_k(\xi)t^k\mathscr{B}_{n-k,h}^{(\alpha)}(\xi)\frac{t^{n-k}}{(n-k)!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}L_k(\xi)\mathscr{B}_{n-k,h}^{(\alpha)}(\xi)\frac{t^{n}}{(n-k)!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{(n-k)!}L_k(\xi)\mathscr{B}_{n-k,h}^{(\alpha)}(\xi)\frac{t^{n}}{n!}.\end{aligned} \tag{8}$$

Comparing the coefficients on both sides of (8), we obtain (5).

Analogously, (4) implies that

$$\sum_{n=0}^{\infty}{}_L\mathscr{E}_{n,h}^{(\alpha)}(\xi)\frac{t^n}{n!}=\sum_{n=0}^{\infty}L_n(\xi)t^n\sum_{n=0}^{\infty}\mathscr{E}_{n,h}^{(\alpha)}(\xi)\frac{t^n}{n!}.$$

Therefore, we have

$$\sum_{n=0}^{\infty}{}_L\mathscr{E}_{n,h}^{(\alpha)}(\xi)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{(n-k)!}L_k(\xi)\mathscr{E}_{n-k,h}^{(\alpha)}(\xi)\frac{t^n}{n!}, \tag{9}$$

comparing the coefficients on both sides of (9), equality (6) follows. □

Theorem 3.2. *For $n \geq 0$, $\xi, y \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, the mixed-type generalized degenerate Lucas–Bernoulli $\left\{{}_{L}\mathscr{B}_{n,h}^{(\alpha)}(\xi)\right\}_{n\geq 0}$ and generalized degenerate Lucas–Euler $\left\{{}_{L}\mathscr{E}_{n,h}^{(\alpha)}(\xi)\right\}_{n\geq 0}$ polynomials, satisfy the following addition formulas:*

$$\begin{aligned}{}_{L}\mathscr{B}_{n,h}^{(\alpha)}(\xi+y) &= \sum_{k=0}^{n}\frac{n!}{(n-k)!}L_k(\xi+y)\mathscr{B}_{n-k,h}^{(\alpha)}(\xi+y) \\ &= \sum_{k=0}^{n}\sum_{\ell=0}^{k}\frac{k!}{(k-\ell)!}\binom{n}{k}L_\ell(\xi+y)\mathscr{B}_{k-\ell,h}^{(\alpha)}(\xi)(y)_{n-k,h}\end{aligned} \tag{10}$$

and

$$\begin{aligned}{}_{L}\mathscr{E}_{n,h}^{(\alpha)}(\xi+y) &= \sum_{k=0}^{n}\frac{n!}{(n-k)!}L_k(\xi+y)\mathscr{E}_{n-k,h}^{(\alpha)}(\xi+y) \\ &= \sum_{k=0}^{n}\sum_{\ell=0}^{k}\frac{k!}{(k-\ell)!}\binom{n}{k}L_\ell(\xi+y)\mathscr{E}_{k-\ell,h}^{(\alpha)}(\xi)(y)_{n-k,h}.\end{aligned} \tag{11}$$

Proof.

Substituting ξ in (3) by $\xi+y$, we find that

$$\sum_{n=0}^{\infty}{}_{L}\mathscr{B}_{n,h}^{(\alpha)}(\xi+y)\frac{t^n}{n!} = \left[\frac{2-(\xi+y)t}{1-(\xi+y)t-t^2}\right]\left[\frac{t}{e_h(t)-1}\right]^{\alpha}e_h^{(\xi+y)}(t). \tag{12}$$

So, by using the Cauchy product, we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{L}\mathscr{B}_{n,h}^{(\alpha)}(\xi+y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}L_n(\xi+y)t^n\sum_{n=0}^{\infty}\mathscr{B}_{n,h}^{(\alpha)}(\xi+y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}L_k(\xi+y)t^k\mathscr{B}_{n-k,h}^{(\alpha)}(\xi+y)\frac{t^{n-k}}{(n-k)!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{(n-k)!}L_k(\xi+y)\mathscr{B}_{n-k,h}^{(\alpha)}(\xi+y)\frac{t^n}{n!}.\end{aligned} \tag{13}$$

Comparing the coefficients on both sides of (13), equality (10) follows. The Equation (11) is obtained by rewriting the equation (12) as:

$$\sum_{n=0}^{\infty}{}_{L}\mathscr{B}_{n,h}^{(\alpha)}(\xi+y)\frac{t^n}{n!} = \left[\frac{2-(\xi+y)t}{1-(\xi+y)t-t^2}\right]\left[\frac{t}{e_h(t)-1}\right]^{\alpha}e_h^{\xi}(t)e_h^{y}(t),$$

and applying Cauchy product series. □

Corollary 3.3. *For $n \geq 0$, $\xi \in \mathbb{R}$, the mixed-type generalized degenerate Lucas–Bernoulli $\left\{{}_{L}\mathscr{B}_{n,h}^{(\alpha)}(\xi)\right\}_{n\geq 0}$ and degenerate Lucas–Euler $\left\{{}_{L}\mathscr{E}_{n,h}^{(\alpha)}(\xi)\right\}_{n\geq 0}$ polynomials, satisfy the following formulas:*

$${}_{L}\mathscr{B}_{n,h}^{(\alpha)}(\xi) = \sum_{k=0}^{n}\sum_{\ell=0}^{k}\frac{k!}{(k-\ell)!}\binom{n}{k}\mathscr{B}_{k-\ell,h}^{(\alpha)}\ L_\ell(\xi)\ (\xi)_{n-k,h}$$

and

$${}_{L}\mathscr{E}_{n,h}^{(\alpha)}(\xi) = \sum_{k=0}^{n}\sum_{\ell=0}^{k}\frac{k!}{(k-\ell)!}\binom{n}{k}\mathscr{E}_{k-\ell,h}^{(\alpha)}\ L_\ell(\xi)\ (\xi)_{n-k,h},$$

where $\mathscr{B}_{n,h}^{(\alpha)} = \mathscr{B}_{n,h}^{(\alpha)}(0)$ and $\mathscr{E}_{n,h}^{(\alpha)} = \mathscr{E}_{n,h}^{(\alpha)}(0)$ is the generalized degenerate Bernoulli (Euler) numbers of order α.

Theorem 3.4. *For $h \in \mathbb{R}$, $n \in \mathbb{N}_0$ and $\alpha = 1$, the mixed-type generalized degenerate Lucas–Bernoulli and degenerate Lucas–Euler polynomials satisfy the following identities:*

$$\begin{aligned}&(n-1)_L\mathcal{B}_{n,h}(\xi) + n[h(n-2)+\xi]_L\mathcal{B}_{n-1,h}(\xi) - \sum_{k=0}^{n}\big[(n)_kF_k(\xi)\big(\xi_L\mathcal{B}_{n-k,h}(\xi) \\ &+\ (n-k)(2+h\xi)_L\mathcal{B}_{n-(k+1),h}(\xi)\ +\ 2h(n-k)(n-k-1)_L\mathcal{B}_{n-(k+2),h}(\xi)\ -\xi\mathcal{B}_{n-k,h}(\xi) \\ &-h\xi(n-k)\mathcal{B}_{n-(k+1),h}(\xi)\big)\big] + \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{k,h}(1)_L\mathcal{B}_{n-k,h}(\xi) = 0.\end{aligned} \tag{14}$$

and

$$(n-\xi)_L\mathcal{E}_{n,h}(\xi)+n(n-1)h_L\mathcal{E}_{n-1,h}(\xi)-\sum_{k=0}^{n}\Big[(n)_kF_k(\xi)\big(\xi_L\mathcal{E}_{n-k,h}(\xi)$$
$$+\ (n-k)(2+h\xi)_L\mathcal{E}_{n-(k+1),h}(\xi)\ +\ 2h(n-k)(n-k-2)_L\mathcal{E}_{n-(k+2),h}(\xi)\ -\xi\mathcal{E}_{n-k,h}(\xi)$$
$$-h\xi(n-k)\mathcal{E}_{n-(k+1),h}(\xi)\big)\Big]+\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k,h}(1)_L\mathcal{E}_{n-k,h}(\xi)=0, \tag{15}$$

where $\left\{\mathcal{B}_{n,h}(\xi)\right\}_{n\geq 0}$, $\left\{\mathcal{B}_{n,h}\right\}_{n\geq 0}$ *and,* $\left\{\mathcal{E}_{n,h}\right\}_{n\geq 0}$, $\left\{\mathcal{E}_{n,h}(\xi)\right\}_{n\geq 0}$, $\{F_n(\xi)\}_{n\geq 0}$ *be the sequences of degenerate Bernoulli polynomials, degenerate Bernoulli numbers, degenerate Euler numbers, degenerate Euler polynomials, and Fibonacci polynomials, respectively.*

Proof. Partially differentiating with respect to t in (3), we have

$$\begin{aligned}\frac{\partial}{\partial t}\sum_{n=0}^{\infty}{}_L\mathcal{B}_{n,h}(\xi)\frac{t^n}{n!} &= \frac{\partial}{\partial t}\left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{te_h^{\xi}(t)}{e_h(t)-1}\right]\\
&= \left[\frac{\xi+2t}{1-\xi t-t^2}\frac{2-\xi t}{1-\xi t-t^2}-\frac{\xi}{1-\xi t-t^2}\right]\left[\frac{te_h^{\xi}(t)}{e_h(t)-1}\right]\\
&\quad+\left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{e_h^{\xi}(t)}{e_h(t)-1}+\frac{\xi t}{1+ht}\frac{e_h^{\xi}(t)}{e_h(t)-1}-\frac{1}{1+ht}\frac{e_h(t)}{e_h(t)-1}\frac{te_h^{\xi}(t)}{e_h(t)-1}\right].\end{aligned}$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty}n\ {}_L\mathcal{B}_{n,h}(\xi)\frac{t^n}{n!} &= -\xi\sum_{n=0}^{\infty}F_n(\xi)t^n\sum_{n=0}^{\infty}\mathcal{B}_{n,h}(\xi)\frac{t^n}{n!}+(2t+\xi)\sum_{n=0}^{\infty}F_n(\xi)t^n\sum_{n=0}^{\infty}{}_L\mathcal{B}_{n,h}(\xi)\frac{t^n}{n!}\\
&\quad+\sum_{n=0}^{\infty}{}_L\mathcal{B}_{n,h}(\xi)\frac{t^n}{n!}+\frac{\xi t}{1+ht}\sum_{n=0}^{\infty}{}_L\mathcal{B}_{n,h}(\xi)\frac{t^n}{n!}\\
&\quad-\frac{1}{1+ht}\sum_{n=0}^{\infty}\mathcal{B}_{n,h}(1)\frac{t^n}{n!}\sum_{n=0}^{\infty}{}_L\mathcal{B}_{n,h}(\xi)\frac{t^n}{n!}.\end{aligned}$$

By rewriting, simplifying, and applying the Cauchy product for series to the previous expression, we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty}\left[n\ {}_L\mathcal{B}_{n,h}(\xi)+nh(n-1)\ {}_L\mathcal{B}_{n-1,h}(\xi)\right]\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\Bigg[\sum_{k=0}^{n}-\xi(n)_kF_k(\xi)\mathcal{B}_{n-k,h}(\xi)\\
&\quad-\sum_{k=0}^{n}h\xi(n)_k(n-k)F_k(\xi)\mathcal{B}_{n-(k+1),h}(\xi)\\
&\quad+\sum_{k=0}^{n}2(n)_k(n-k)F_k(\xi)_L\mathcal{B}_{n-(k+1),h}(\xi)\\
&\quad+\sum_{k=0}^{n}\xi(n)_kF_k(\xi)_L\mathcal{B}_{n-k,h}(\xi)\\
&\quad+\sum_{k=0}^{n}2h(n)_k(n-k)(n-k-1)F_k(\xi)_L\mathcal{B}_{n-(k+2),h}(\xi)\\
&\quad+\sum_{k=0}^{n}h\xi(n)_k(n-k)F_k(\xi)_L\mathcal{B}_{n-(k+1),h}(\xi)\\
&\quad+{}_L\mathcal{B}_{n,h}(\xi)+nh_L\mathcal{B}_{n-1,h}(\xi)+\xi n_L\mathcal{B}_{n-1,h}(\xi)\\
&\quad-\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{k,h}(1)_L\mathcal{B}_{n-k,h}(\xi)\Bigg]\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\dfrac{t^n}{n!}$ on both sides of the above expression, we have

$$
\begin{aligned}
(n-1)_L\mathcal{B}_{n,h}(\xi)+n[h(n-2)+\xi]_L\mathcal{B}_{n-1,h}(\xi) \quad = \quad & \sum_{k=0}^{n}\Big[(n)_k F_k(\xi)\big(\xi_L\mathcal{B}_{n-k,h}(\xi)\\
&+\ (n-k)(2+h\xi)_L\mathcal{B}_{n-(k+1),h}(\xi)\\
&+\ 2h(n-k)(n-k-1)_L\mathcal{B}_{n-(k+2),h}(\xi)\\
&+-\xi\mathcal{B}_{n-k,h}(\xi)-h\xi(n-k)\mathcal{B}_{n-(k+1),h}(\xi)\big)\\
&-\binom{n}{k}\mathcal{B}_{k,h}(1)_L\mathcal{B}_{n-k,h}(\xi)\Big].
\end{aligned}
$$

This confirms what we aimed to demonstrate. Similarly, taking the derivative with respect to t in (4), and after some straightforward calculations, we obtain (15). □

Remark 2. The proof of Theorem 3.4 is based on term-by-term differentiation and coefficient comparison in the generating functions (3)–(4). Hence it is valid in the common domain where the involved power series converge absolutely. In particular, we require

$$|t| < \min\left\{\frac{2}{\xi+\sqrt{\xi^2+4}},\, 2\pi\right\} \quad \text{(Lucas–Bernoulli)}$$

and

$$|t| < \min\left\{\frac{2}{\xi+\sqrt{\xi^2+4}},\, \pi\right\} \quad \text{(Lucas–Euler).}$$

Moreover, the expansion $\dfrac{1}{1+ht} = \sum_{m\geq 0}(-h)^m t^m$ used in the derivation requires the additional condition $|ht| < 1$ when $h \neq 0$. Under these analyticity constraints, the coefficient extraction is stable. From a computational viewpoint, recurrences obtained from Theorem 3.4 are numerically more robust when evaluated within this disc of analyticity, since Cauchy-type estimates control coefficient growth.

Theorem 3.5. *For $\alpha = 1$, $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$, the mixed-type degenerate Lucas–Euler and mixed-type degenerate Lucas–Bernoulli polynomials satisfy the following relations:*

$$
{}_L\mathcal{E}_{n-1,h}(\xi) \quad = \quad \frac{1}{n}\left[\sum_{k=0}^{n}\binom{n}{k}{}_L\mathcal{B}_{n-k,h}(\xi)\big(\mathcal{E}_{k,h}(1)-\mathcal{E}_{k,h}(0)\big)\right], \tag{16}
$$

and

$$
{}_L\mathcal{B}_{n,h}(\xi) = \sum_{k=0}^{n}\frac{1}{2}\binom{n}{k}{}_L\mathcal{E}_{n-k,h}(\xi)\big(\mathcal{B}_{k,h}(1)-\mathcal{B}_{k,h}(0)\big). \tag{17}
$$

Proof. From (3) and (4), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}{}_L\mathscr{E}_{n,h}(\xi)\frac{t^n}{n!} \quad &= \quad \left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{2}{e_h(t)+1}\right]e_h^{\xi}(t)\\
&= \quad 2\left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{e_h(t)-1}{e_h^2(t)-1}\right]e_h^{\xi}(t),
\end{aligned}
$$

then, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} n_L\mathscr{E}_{n-1,h}(\xi)\frac{t^n}{n!} \quad &= \quad 2\left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{te_h(t)}{e_h^2(t)-1}e_h^{\xi}(t)-\frac{t}{e_h^2(t)-1}e_h^{\xi}(t)\right]\\
&= \quad \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}{}_L\mathscr{B}_{n-k,h}(\xi)\big(\mathscr{E}_{k,h}(1)-\mathscr{E}_{k,h}(0)\big)\right]\frac{t^n}{n!},
\end{aligned}
\tag{18}
$$

hence, comparing the coefficients of $\dfrac{t^n}{n!}$ on both sides of (18), we obtain (16). □

From (3) and (4), we have

$$\sum_{n=0}^{\infty} {}_L\mathscr{B}_{n,h}(\xi)\frac{t^n}{n!} = \left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{t}{e_h(t)-1}\right]e_h^{\xi}(t)$$
$$= \frac{t}{2}\left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{2(e_h(t)+1)}{e_h^2(t)-1}\right]e_h^{\xi}(t).$$

Then, we find that

$$\sum_{n=0}^{\infty} {}_L\mathscr{B}_{n,h}(\xi)\frac{t^n}{n!} = \frac{1}{2}\left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{2te_h(t)}{e_h^2(t)-1}e_h^{\xi}(t)-\frac{2t}{e_h^2(t)-1}e_h^{\xi}(t)\right]$$
$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{1}{2}\binom{n}{k}{}_L\mathscr{E}_{n-k,h}(\xi)\left(\mathscr{B}_{k,h}(1)-\mathscr{B}_{k,h}(0)\right)\right]\frac{t^n}{n!}. \tag{19}$$

Hence, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of (19), we obtain (17).

4 Structure and distribution of zeros: illustrative visualizations

In this section, we present graphical visualizations of the zeros of the mixed-type generalized degenerate Lucas–Bernoulli and Lucas–Euler polynomials.

For any $n \in \mathbb{N}_0$, the first few mixed-type generalized degenerate Lucas–Bernoulli polynomials for $\alpha = 1$, are given by

$$\begin{aligned}
{}_L\mathscr{B}^{(1)}_{0,h}(\xi) &= 2,\\
{}_L\mathscr{B}^{(1)}_{1,h}(\xi) &= 3\xi + h - 1,\\
{}_L\mathscr{B}^{(1)}_{2,h}(\xi) &= 6\xi^2 + (h-3)\xi + \frac{13-h^2}{3},\\
{}_L\mathscr{B}^{(1)}_{3,h}(\xi) &= 17\xi^3 - 9\xi^2 + \left(\frac{63+6h-h^2}{2}\right)\xi + (h^3+11h-12),\\
{}_L\mathscr{B}^{(1)}_{4,h}(\xi) &= 66\xi^4 - 2\xi^3(17+h) + 6\xi^2\left(33+3h+h^2\right) + \xi\left(-60+31h-8h^2+h^3\right)\\
&\quad + \frac{1}{15}\left(779-40h^2-19h^4\right),\\
{}_L\mathscr{B}^{(1)}_{5,h}(\xi) &= 327\xi^5 - 5\xi^4(33+h) + \frac{5}{3}\xi^3\left(797+54h+28h^2\right) - 5\xi^2\left(96-30h+15h^2+5h^3\right)\\
&\quad + \frac{1}{6}\xi\left(5337+360h-50h^2+180h^3-19h^4\right) + \frac{221h}{2} + 5h^3 - 120,
\end{aligned}$$

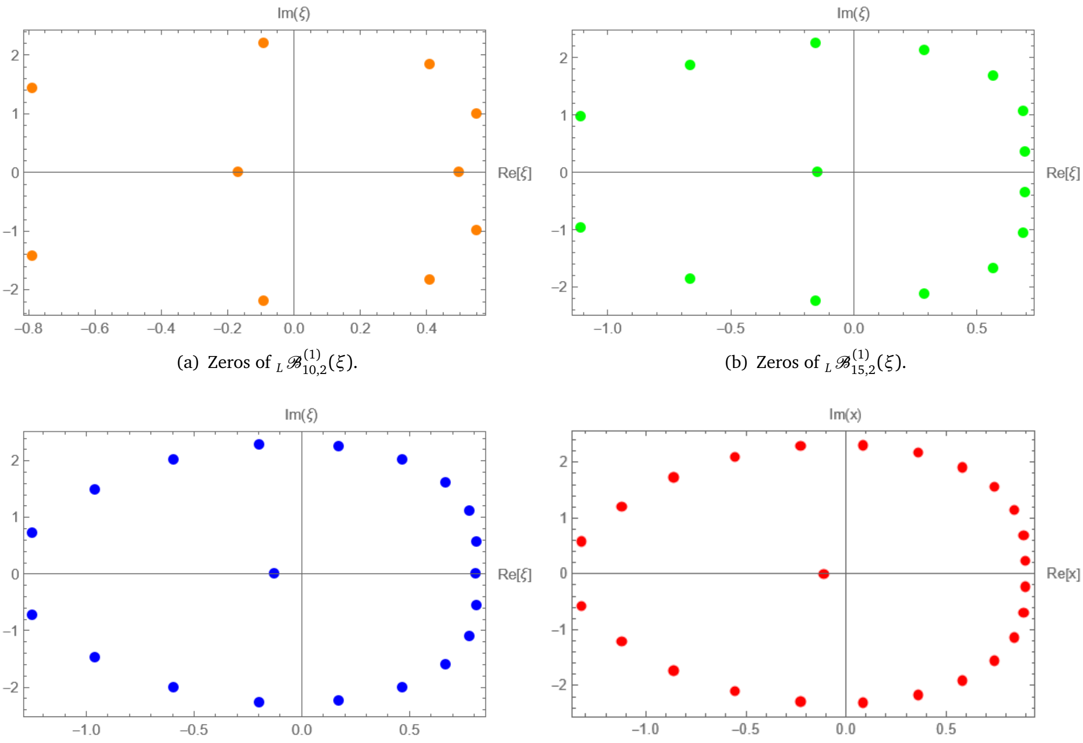
$$\begin{aligned}
{}_L\mathscr{B}^{(1)}_{6,h}(\xi) &= 1958\xi^6 - 9\xi^5(109+h) + 5\xi^4\left(1989+102h+55h^2\right) - 5\xi^3\left(780-169h+99h^2+52h^3\right)\\
&\quad + \xi^2\left(11277+900h+180h^2+390h^3+125h^4\right)\\
&\quad + \frac{1}{2}\xi\left(-5040+3191h-480h^2-50h^3-288h^4+27h^5\right)\\
&\quad + \frac{1}{42}\left(65438-3507h^2-588h^4-863h^6\right).
\end{aligned}$$

In order to illustrate the zero distribution of the mixed-type degenerate Lucas–Bernoulli polynomials we use (3) and Wolfram Mathematica. For instance, for $h = 2$ and $\alpha = 1$ the identity (3) becomes

$$\left[\frac{2-\xi t}{1-\xi t-t^2}\right]\left[\frac{t}{e_2(t)-1}\right]e_2^{\xi}(t),$$

and we can use the command 'Series' for finding the suitable power series of the previous generating function around $t = 0$. Then, the command 'SeriesCoefficient' provides us the corresponding explicit representation for the mixed-type degenerate Lucas–Bernoulli polynomials. Finally, commands such as 'NSolve' allows us to show numerical approximations of such zeros.

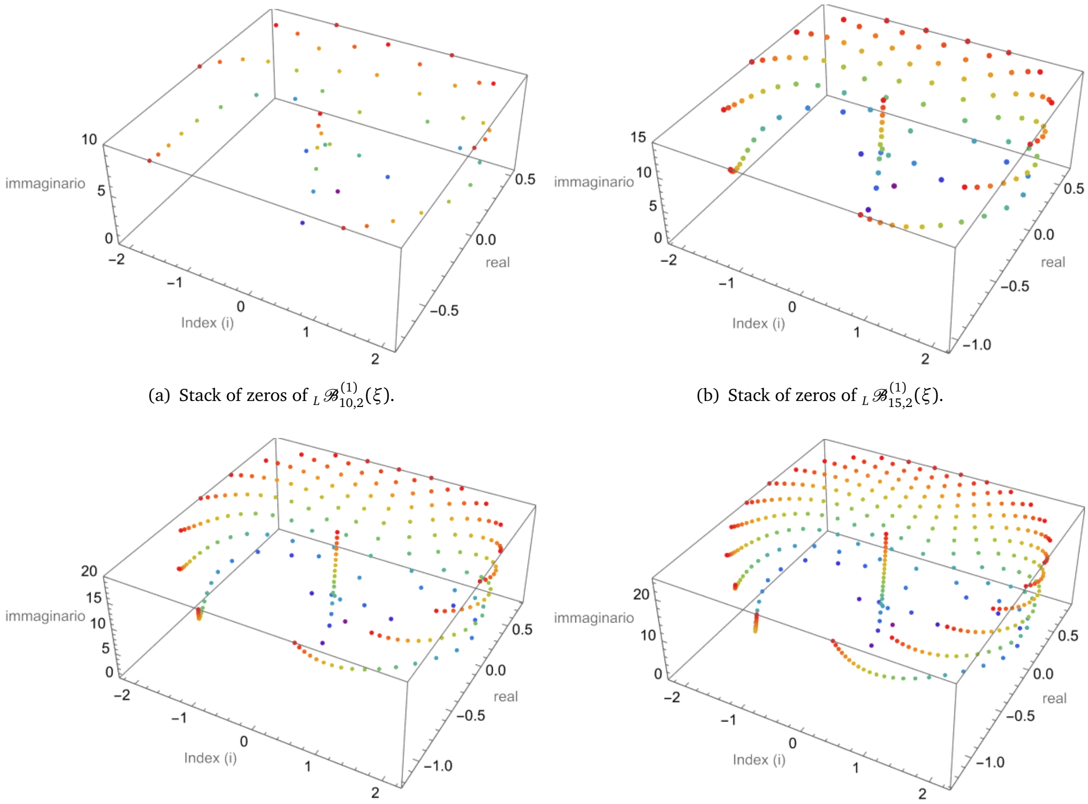
Figure 1 below shows our numerical results for approximate solutions of zeros of ${}_L\mathscr{B}^{(1)}_{n,2}(\xi)$ for $n = 10, 15, 20$ and 25.



(a) Zeros of ${}_L\mathscr{B}^{(1)}_{10,2}(\xi)$.
(b) Zeros of ${}_L\mathscr{B}^{(1)}_{15,2}(\xi)$.
(c) Zeros of ${}_L\mathscr{B}^{(1)}_{20,2}(\xi)$.
(d) Zeros of ${}_L\mathscr{B}^{(1)}_{25,2}(\xi)$.

Figure 1: Zero distribution of ${}_L\mathscr{B}^{(1)}_{n,2}(\xi)$.

Some stacks of zeros of ${}_L\mathscr{B}^{(1)}_{n,2}(\xi)$ from a 3D structure are presented in Figure 2.



(a) Stack of zeros of ${}_L\mathscr{B}^{(1)}_{10,2}(\xi)$.

(b) Stack of zeros of ${}_L\mathscr{B}^{(1)}_{15,2}(\xi)$.

(c) Stack of zeros of ${}_L\mathscr{B}^{(1)}_{20,2}(\xi)$.

(d) Stack of zeros of ${}_L\mathscr{B}^{(1)}_{25,2}(\xi)$.

Figure 2: Stacks of zeros of ${}_L\mathscr{B}^{(1)}_{n,2}(\xi)$, for $n = 10, 15, 20, 25$.

Our numerical results for approximate solutions of zeros of ${}_L\mathscr{B}_{n,2}^{(1)}(\xi)$ for $1 \le n \le 10$ are displayed in Table 1.

Table 1. Approximate solutions of ${}_L\mathscr{B}_{n,2}^{(1)}(\xi) = 0$

degree n	ξ
1	-0.333333
2	$0.0833333 - 0.702179i$, $0.0833333 + 0.702179i$
3	-0.233585, $0.381499 - 1.45634i$, $0.381499 + 1.45634i$
4	$0.0384798 - 0.287308i$, $0.0384798 + 0.287308i$, $0.249399 - 1.92989i$, $0.249399 + 1.92989i$
5	-0.206477, $0.00738165 - 2.08735i$, $0.00738165 + 2.08735i$, $0.363441 - 0.914712i$, $0.363441 + 0.914712i$
6	$-0.205705 - 2.04121i$, $-0.205705 + 2.04121i$, -0.0802425, 0.21218, $0.394844 - 1.44539i$, $0.394844 + 1.44539i$
7	$-0.387527 - 1.89117i$, $-0.387527 + 1.89117i$, -0.194565, $0.325077 - 1.83731i$, $0.325077 + 1.83731i$, $0.410845 - 0.691275i$, $0.410845 + 0.691275i$
8	$-0.546594 - 1.72446i$, $-0.546594 + 1.72446i$, -0.159147, $0.208413 - 2.06561i$, $0.208413 + 2.06561i$, 0.386799, $0.474551 - 1.1748i$, $0.474551 + 1.1748i$
9	$-0.678335 - 1.5671i$, $-0.678335 + 1.5671i$, -0.183184, $0.061912 - 2.16912i$, $0.061912 + 2.16912i$, $0.469087 - 1.5567i$, $0.469087 + 1.5567i$, $0.488957 - 0.561711i$, $0.488957 + 0.561711i$
10	$-0.787114 - 1.42787i$, $-0.787114 + 1.42787i$, -0.168034, $-0.0889383 - 2.19096i$, $-0.0889383 + 2.19096i$, $0.409687 - 1.83684i$, $0.409687 + 1.83684i$, 0.49959, $0.550591 - 0.991377i$, $0.550591 + 0.991377i$

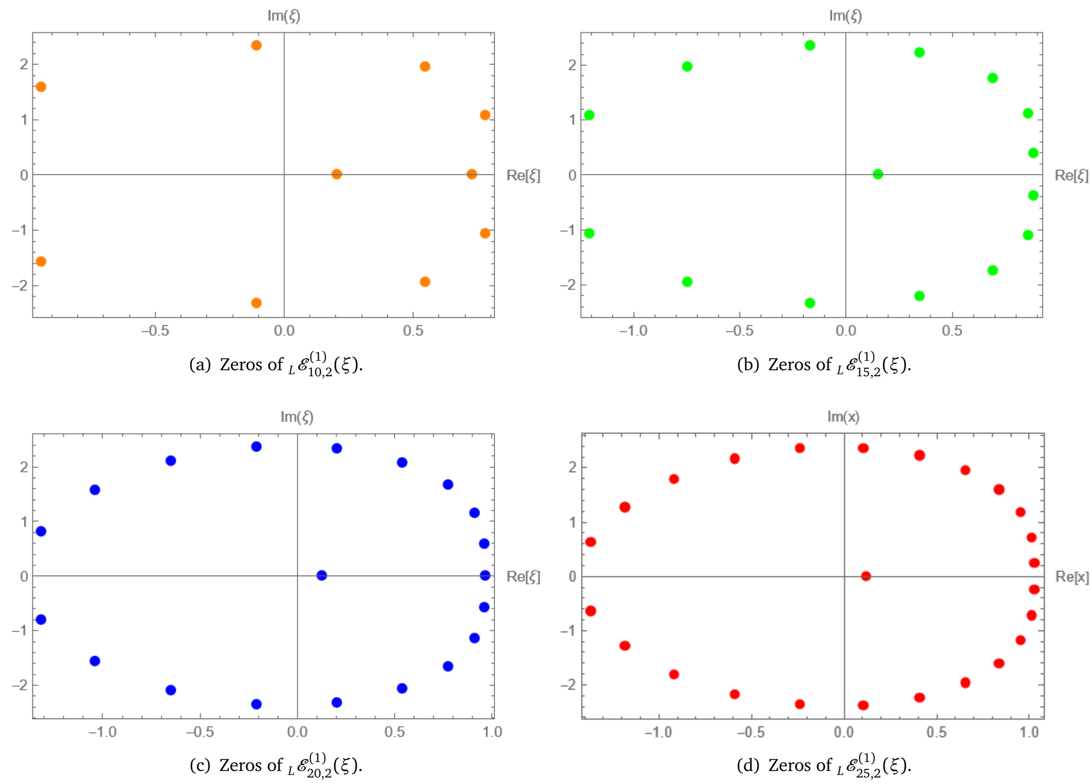
Similarly, for any $n \in \mathbb{N}_0$, the first few mixed-type generalized degenerate Lucas-Euler polynomials for $\alpha = 1$ are given as:

$$\begin{aligned}
{}_L\mathscr{E}^{(\alpha)}_{0,h}(\xi) &= 2,\\
{}_L\mathscr{E}^{(\alpha)}_{1,h}(\xi) &= 3\xi - 1,\\
{}_L\mathscr{E}^{(\alpha)}_{2,h}(\xi) &= 6\xi^2 - 3\xi + h - 2\xi h + 4,\\
{}_L\mathscr{E}^{(\alpha)}_{3,h}(\xi) &= 17\xi^3 - 9(1+h)\xi^2 + \left(30 + \frac{15h}{2} + 4h^2\right)\xi - \left(\frac{11}{2} + 2h^2\right),\\
{}_L\mathscr{E}^{(\alpha)}_{4,h}(\xi) &= 66\xi^4 - 2(17+18h)\xi^3 + 6\left(32+6h+5h^2\right)\xi^2 - \left(57+24h+26h^2+12h^3\right)\xi\\
&\quad +(48+9h+6h^3),\\
{}_L\mathscr{E}^{(\alpha)}_{5,h}(\xi) &= 327\xi^5 - 5(33+34h)\xi^4 + 5\left(260+35h+33h^2\right)\xi^3 - 5\left(93+60h+36h^2+26h^3\right)\xi^2\\
&\quad + \left(840 + \frac{365h}{2} + 80h^2 + 115h^3 + 48h^4\right)\xi - \frac{3}{2}\left(74+15h^2+16h^4\right),\\
{}_L\mathscr{E}^{(\alpha)}_{6,h}(\xi) &= 1958\xi^6 - 9(109+110h)\xi^5 + 5\left(1956+201h+190h^2\right)\xi^4\\
&\quad -5\left(763+576h+221h^2+186h^3\right)\xi^5\\
&\quad +\left(10800+2160h+1380h^2+1065h^3+692h^4\right)\xi^2\\
&\quad -\frac{1}{2}\left(4758+1440h+1595h^2+720h^3+1240h^4+480h^5\right)\xi\\
&\quad +\frac{15}{2}\left(192+38h+9h^3+16h^5\right).
\end{aligned}$$

In order to illustrate the zero distribution of the mixed-type degenerate Lucas–Euler polynomials we use (4) and Wolfram Mathematica. For instance, for $h = 2$ and $\alpha = 1$ the identity (4) becomes

$$\left[\frac{2-\xi t}{1-\xi t - t^2}\right]\left[\frac{2}{e_2(t)+1}\right]e_2^{\xi}(t).$$

Figure 3 shows our numerical results for approximate solutions of zeros of ${}_L\mathscr{E}^{(1)}_{n,2}(\xi)$ for $n = 10, 15, 20$ and 25.



(a) Zeros of ${}_L\mathscr{E}^{(1)}_{10,2}(\xi)$.

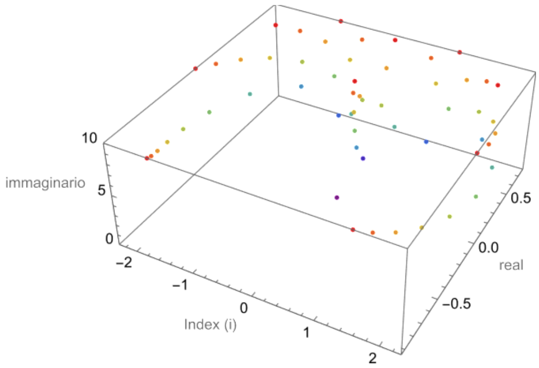
(b) Zeros of ${}_L\mathscr{E}^{(1)}_{15,2}(\xi)$.

(c) Zeros of ${}_L\mathscr{E}^{(1)}_{20,2}(\xi)$.

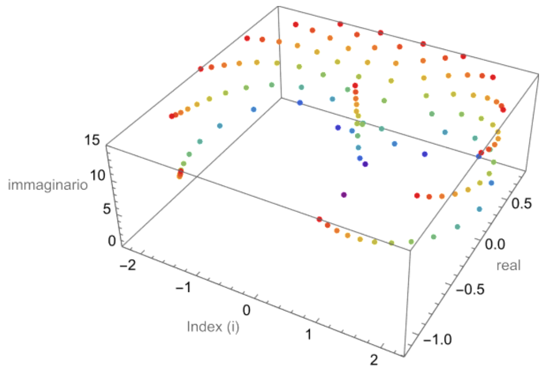
(d) Zeros of ${}_L\mathscr{E}^{(1)}_{25,2}(\xi)$.

Figure 3: Zero distribution of ${}_L\mathscr{E}^{(\alpha)}_{n,h}(\xi)$, for $n = 10, 15, 20, 25$.

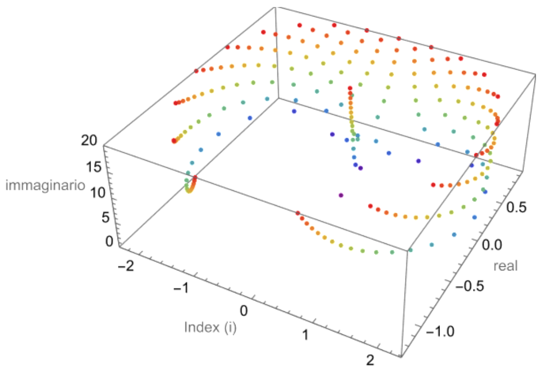
Some stacks of zeros of ${}_L\mathscr{E}^{(1)}_{n,2}(\xi)$ from a 3D structure are presented in Figure 4.



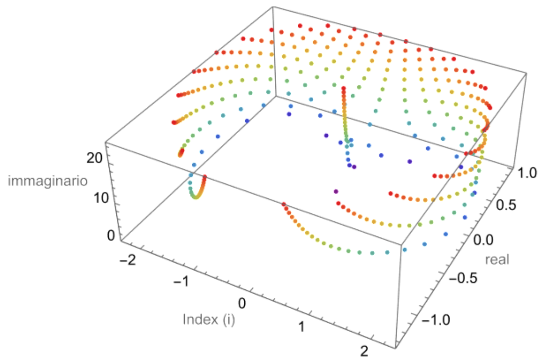
(a) Stack of ${}_L\mathscr{E}^{(1)}_{10,2}(\xi)$.



(b) Stack of ${}_L\mathscr{E}^{(1)}_{15,2}(\xi)$.



(c) Stack of ${}_L\mathscr{E}^{(1)}_{20,2}(\xi)$.



(d) Stack of ${}_L\mathscr{E}^{(1)}_{25,2}(\xi)$.

Figure 4: Stacks of zeros of${}_L\mathscr{E}^{(1)}_{n,2}(\xi)$, for $n = 10, 15, 20, 25$.

Finally, our numerical results for approximate solutions of zeros of ${}_L\mathscr{E}^{(1)}_{n,2}(\xi)$ for $1 \le n \le 10$ were obtained and summarized in Table 2.

Table 2. Approximate solutions of ${}_L\mathscr{E}^{(1)}_{n,2}(\xi) = 0$

degree n	ξ
1	0.333333
2	$0.583333 - 0.812233i, \quad 0.583333 + 0.812233i$
3	$0.243538, \quad 0.672348 - 1.67592i, \quad 0.672348 + 1.67592i$
4	$0.350215 - 2.16871i, \quad 0.350215 + 2.16871i, \quad 0.452816 - 0.390989i,$ $0.452816 + 0.390989i$
5	$-0.0233197 - 2.29585i, \quad -0.0233197 + 2.29585i, \quad 0.216201,$ $0.687391 - 1.04747i, \quad 0.687391 + 1.04747i$
6	$-0.322818 - 2.22057i, \quad -0.322818 + 2.22057i, \quad 0.434323 - 0.169228i,$ $0.434323 + 0.169228i, \quad 0.644624 - 1.60236i, \quad 0.644624 + 1.60236i$
7	$-0.546698 - 2.06374i, \quad -0.546698 + 2.06374i, \quad 0.202239,$ $0.490608 - 1.98515i, \quad 0.490608 + 1.98515i, \quad 0.706266 - 0.778563i,$ $0.706266 + 0.778563i$
8	$-0.714297 - 1.89011i, \quad -0.714297 + 1.89011i, \quad 0.272373, \quad 0.29185 - 2.20762i,$ $0.29185 + 2.20762i, \quad 0.621836, \quad 0.725567 - 1.27447i, \quad 0.725567 + 1.27447i$
9	$-0.841352 - 1.72518i, \quad -0.841352 + 1.72518i, \quad 0.0859129 - 2.3074i,$ $0.0859129 + 2.3074i, \quad 0.18959, \quad 0.661899 - 1.66059i,$ $0.661899 + 1.66059i, \quad 0.748778 - 0.619972i, \quad 0.748778 + 0.619972i$
10	$-0.939519 - 1.57713i, \quad -0.939519 + 1.57713i, \quad -0.106246 - 2.32413i,$ $-0.106246 + 2.32413i, \quad 0.20594, \quad 0.547759 - 1.94449i,$ $0.547759 + 1.94449i, \quad 0.729981, \quad 0.780049 - 1.06245i, \quad 0.780049 + 1.06245i$

4.1 Approximation-theoretic remarks and a numerical application

Beyond the identity-based development, the mixed-type families $\{{}_L\mathscr{B}^{(\alpha)}_{n,h}(\xi)\}$ and $\{{}_L\mathscr{E}^{(\alpha)}_{n,h}(\xi)\}$ may be used as *polynomial bases* for approximating sufficiently smooth (or analytic) functions on a bounded interval. For a fixed choice of parameters (h, α), one may expand a target function f in the truncated form

$$f(\xi) \approx \sum_{n=0}^{N} c_n\, {}_L\mathscr{B}^{(\alpha)}_{n,h}(\xi) \qquad \text{or} \qquad f(\xi) \approx \sum_{n=0}^{N} d_n\, {}_L\mathscr{E}^{(\alpha)}_{n,h}(\xi), \tag{20}$$

where the coefficients can be determined by collocation, least-squares fitting, or interpolation. A natural collocation choice is to take the nodes as the (real/complex) zeros of the next basis polynomial, e.g., the zeros of ${}_L\mathscr{B}^{(\alpha)}_{N+1,h}$ or ${}_L\mathscr{E}^{(\alpha)}_{N+1,h}$. Our zero plots in Figures 1–3 therefore provide a practical guideline for selecting numerically well-distributed nodes (especially when complex collocation is considered).

As a simple numerical illustration, consider approximating an analytic function such as $f(\xi) = e^{\xi}$ on a real interval by the Lucas–Bernoulli basis with fixed (h, α). Choosing N and collocation points $\{\xi_j\}_{j=0}^{N}$ (for example, the real zeros when available, or the real parts of complex-conjugate pairs), one solves the linear system $f(\xi_j) = \sum_{n=0}^{N} c_n\, {}_L\mathscr{B}^{(\alpha)}_{n,h}(\xi_j)$ for the coefficients c_n. The stability of this procedure is closely related to the analyticity restrictions in Remark 2 (via the generating-function radius of convergence and the additional requirement $|ht| < 1$ when $h \neq 0$), which control the growth of the basis coefficients and, in turn, the conditioning of the collocation matrix.

5 Conclusions

In this work, we have studied the properties of mixed-type generalized degenerate Lucas–Bernoulli polynomials and mixed-type generalized degenerate Lucas–Euler polynomials establishing various identities and structural relationships. Through theoretical development and computational analysis, we have derived explicit formulas and recurrence relations, providing a solid foundation for their study in both analytical and combinatorial contexts.

An interesting feature of our research is the analysis of the zeros of these polynomials, which reveal intriguing distribution patterns. We have identified recurring structures that provide numerical evidence for the geometric form of the critical set (or zero attractor) where zeros of ${}_L\mathscr{B}_{n,h}^{(\alpha)}(\xi)$ and ${}_L\mathscr{E}_{n,h}^{(\alpha)}(\xi)$ accumulate.

Several promising directions remain open for future research:

- A deeper investigation into the asymptotic distribution of zeros could offer further insight into their behavior as the degree of the polynomial tends to infinity.
- Exploring potential applications in numerical analysis, differential equations, and physical models where Bernoulli and Euler polynomials have proven useful.
- Extending these results to multivariable versions or non-commutative settings could open new perspectives in algebra and analysis.
- Developing fast algorithms for computing expansion coefficients and quadrature/collocation rules based on these mixed-type bases (including stability analysis with respect to the degenerate parameter h).

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