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Subdivision by WAVES – Weighted AVEraging Schemes

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Abstract

The Catmull-Clark subdivision algorithm consists of an operator that can be decomposed into a refinement and a smoothing operator, where the refinement operator splits each face with m vertices into m quadrilateral subfaces and the smoothing operator replaces each internal vertex with an affine combination of its neighboring vertices and itself.

In this paper, we generalize the Catmull-Clark scheme. We consider an arbitrarily fixed number r of weighted averaging steps and allow that these r smoothing operators are different. These *w*(eighted) *ave*(raging) *s*(chemes) form an infinite class of stationary subdivision schemes, which we call *wave shemes*. This class includes the Catmull-Clark scheme and the midpoint schemes. For regular meshes, wave schemes generalize the tensor product Lane-Riesenfeld subdivision algorithm.

We analyze the smoothness of stationary wave surfaces at extraordinary points using established methods for analyzing midpoint subdivision. For regular meshes, we analyze the smoothness of non-stationary wave schemes which need not be asymptotically equivalent to stationary schemes. Wave surfaces are smooth at their regular and, in most cases, extraordinary points.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling — Curve, surface, solid, and object representations

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1 Introduction

The limiting surfaces of the Lane-Riesenfeld subdivision algorithm [LR80] are tensor product spline surfaces. Midpoint subdivision generalizes Lane-Riesenfeld subdivision in so far as it can be applied to arbitrary rather than only regular quadrilateral meshes. A midpoint subdivision scheme consists of an operator $M_n = A^{n-1}R$ of degree $n \in \mathbb{N}$, which is used successively to subdivide an input mesh $\mathcal{M}$. The *refinement operator* R maps $\mathcal{M}$ to the quadrilateral mesh $R\mathcal{M}$, where the edges of $R\mathcal{M}$ connect the center with all edge midpoints for each face of $\mathcal{M}$ and any vertex of $\mathcal{M}$ with all adjacent edge midpoints as shown at the top of Figure 1.1. The *averaging operator* A maps $\mathcal{M}$ to the dual mesh $A\mathcal{M}$ that connects the centers of adjacent faces as shown at the bottom of Figure 1.1.

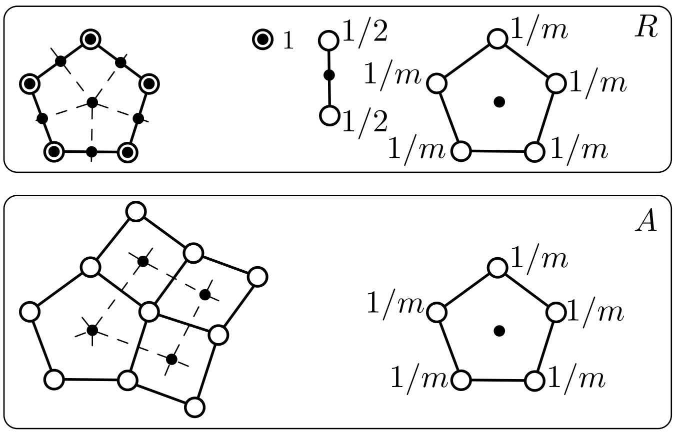

Figure 1.1: Two basic operators for subdividing quadrilateral meshes, where m is the valence of a vertex or face: refinement operator R (top) and averaging operator A (bottom).

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The midpoint schemes of degree 2 and 3 were described first by Doo, Sabin, Catmull and Clark [DS78, CC78]. The limiting surfaces generated by these schemes are smooth everywhere as shown , e. g., in [PR98] by analyzing the spectral properties of the subdivision matrix numerically. Numerically computing or estimating the spectral properties of the subdivision matrix is a bottleneck in the analysis of infinite classes of subdivision schemes. Therefore, the C^1 result for midpoint subdivision could be extended in [ZS01] merely up to degree 9 and only recently, further geometric arguments were developed that helped to prove that midpoint subdivision for any degree $n \geq 2$ generates C^1 subdivision surfaces everywhere [PC11].

The Catmull-Clark subdivision operator can be decomposed into a refinement operator R (see Figure 1.1, top) and a smoothing operator $B_{\alpha,\beta}$ whose mask is shown in Figure 1.2. In [CC78] the weights depend on the valence and are given by $\alpha(m) = 1-3/m$, $\beta(m) = 2/m$ and $\gamma(m) = 1/m$.

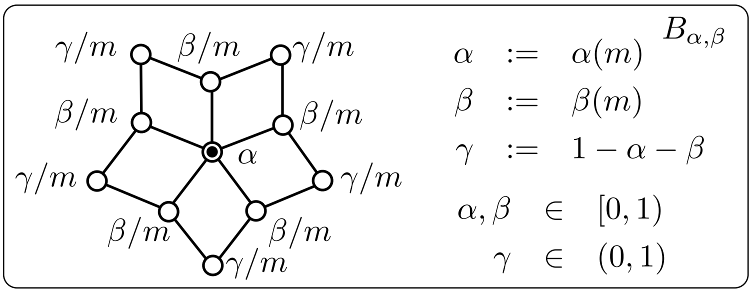

Figure 1.2: Smoothing operator $B_{\alpha,\beta}$ for a vertex of valence m.

In this paper, we generalize the Catmull-Clark scheme to what we call *weighted averaging schemes* or just *"waves"*. "Waves" form an infinite class of stationary subdivision schemes, which includes the Catmull-Clark and the midpoint schemes.

We further develop the techniques used in [PC11] to analyze the smoothness of the resulting wave subdivision surfaces.

2 Wave subdivision

A *wave scheme* W of degree $n \geq 2$ is given by

$$W := \begin{cases} B_r \cdots B_1 R & , \quad \text{if } n = 2r+1 \\ AB_r \cdots B_1 R & , \quad \text{if } n = 2r+2 \end{cases},$$

where R and A are the refinement and averaging operators respectively, shown in Figure 1.1, and where $B_i := B_{\alpha_i,\beta_i}$ is a smoothing operator, shown in Figure 1.2, with non-negative functions α_i and β_i satisfying $0 < \alpha_i + \beta_i < 1$. The weight functions α_i and β_i can be different and every wave scheme defines and denotes a stationary subdivision scheme.

If $\alpha_i \equiv 1/4$ and $\beta_i \equiv 1/2$, then $B_i = A^2$ and the wave scheme of degree n is the midpoint scheme of degree n for arbitrary meshes .

3 Smoothness for regular meshes

In this section, we analyze "waves" for regular meshes. Hence, $m = 4$ and the weight functions α_i and β_i can be viewed as constants.

A regular quadrilateral mesh $\mathcal{C}$ can be represented by the biinfinite matrix $\mathcal{C} = [\mathbf{c_i}]_{\mathbf{i}\in\mathbb{Z}^2}$ of its vertices $\mathbf{c_i}$, which are connected by the edges $\mathbf{c_j}\mathbf{c_{j+e_\mathit{k}}}$, $\mathbf{j} \in \mathbb{Z}^2, k = 1,2$, as shown in Figure 3.1, where

$$[\mathbf{e}_1\ \mathbf{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

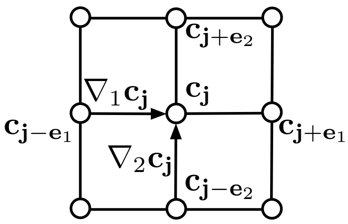

Figure 3.1: A subnet of a regular quadrilateral mesh.

To analyze smoothness, we need the (backward) differences

$$\begin{aligned} \nabla_k \mathbf{c}_\mathbf{i} &= \mathbf{c}_\mathbf{i} - \mathbf{c}_{\mathbf{i}-\mathbf{e}_k}\,, \quad k=1,2,\\ \nabla \mathbf{c}_\mathbf{i} &= [\nabla_1 \mathbf{c}_\mathbf{i}\ \nabla_2 \mathbf{c}_\mathbf{i}]\,,\\ \nabla\nabla \mathbf{c}_\mathbf{i} &= \nabla(\nabla \mathbf{c}_\mathbf{i}) = [\nabla_1\nabla_1 \mathbf{c}_\mathbf{i}\ \nabla_1\nabla_2 \mathbf{c}_\mathbf{i}\ \nabla_2\nabla_1 \mathbf{c}_\mathbf{i}\ \nabla_2\nabla_2 \mathbf{c}_\mathbf{i}]\,, \end{aligned}$$

and the mesh $\mathcal{C}_{\nabla\nabla} = [\nabla\nabla \mathbf{c}_\mathbf{i}]_{\mathbf{i}\in\mathbb{Z}^2}$ of the second order differences $\nabla\nabla \mathbf{c}_\mathbf{i}$.

More generally than in the rest of this paper, in this section we consider weighted averaging operators A_γ defined by non-negative biinfinite sequences $\gamma := (\gamma_\mathbf{i})_{\mathbf{i}\in\mathbb{Z}^2} \geq 0$ such that $\sum_{\mathbf{i}\in\mathbb{Z}^2} \gamma_\mathbf{i} = 1$ and where

$$\left[\mathbf{b}_\mathbf{i}\right]_{\mathbf{i}\in\mathbb{Z}^2} := A_\gamma\, \mathcal{C}$$

is defined by

$$\mathbf{b}_\mathbf{i} := \sum_{\mathbf{j}\in\mathbb{Z}^2} \gamma_{\mathbf{i}-\mathbf{j}} \mathbf{c}_\mathbf{j}\,.$$

Note that the set of all averaging operators A_γ contains all operators $B_r \cdots B_1$ and $AB_r \cdots B_1$.

Furthermore, in this and only in this section, we consider non-stationary "waves" given by a sequence $(W_k)_{k\in\mathbb{N}}$ of operators

$$W_k := A_{\gamma_k} R\,, \quad \gamma_k := (\gamma^k_\mathbf{i})_{\mathbf{i}\in\mathbb{Z}^2}\,,$$

i. e., we study the limits of meshes

$$\mathcal{C}_k := \left[\mathbf{c}^k_\mathbf{i}\right]_{\mathbf{i}\in\mathbb{Z}^2} := W_k \cdots W_1 \mathcal{C}$$

as $k \to \infty$ for any initial control mesh $\mathcal{C}$.

For our analysis, we define the radius of a mask γ_k by

$$r_k := \min \left\{ m \in \mathbb{N} \mid \sum_{\mathbf{i}\in I_m} \gamma^k_\mathbf{i} = 1, \text{ where } I_m := \{-m,\dots,m\}^2 \right\}$$

and observe that $\mathbf{c}^k_\mathbf{j}$ is influenced by $\mathbf{c}^{k-1}_\mathbf{i}$ only if

$$\|2\mathbf{i} - \mathbf{j}\| \leq 1 + r_k\,.$$

Hence, at most $(2+r_k)^2$ many $\mathbf{c}^{k-1}_\mathbf{i}$ influence $\mathbf{c}^k_\mathbf{j}$.

Theorem 3.1. (C^1 condition for regular meshes)
Let $W = (W_k)_{k\in\mathbb{N}}$ be a wave scheme with $\lim_{k\to\infty} r_k/2^k = 0$. Then any sequence of the tensor product splines

$$\mathbf{s}^n_k(\mathbf{x}) := \sum_{\mathbf{i}\in\mathbb{Z}^2} \mathbf{c}^k_\mathbf{i} N^n(2^k x - i) N^n(2^k y - j)$$

converges uniformly to a continuous function $\mathbf{s}(\mathbf{x})$ over any compact domain D, where $N^n(x)$ denotes the cardinal B-spline with knots $0,1,\dots,n$ and $n \geq 2$, and where $\mathbf{s}(\mathbf{x})$ does not depend on n.

Further if there is some $\varepsilon \in (0,1/2]$ such that for all k

$$\sum_{j\in\mathbb{Z}} \sum_{i\in 2\mathbb{Z}} \gamma^k_{ij} \in [\varepsilon, 1-\varepsilon] \ni \sum_{i\in\mathbb{Z}} \sum_{j\in 2\mathbb{Z}} \gamma^k_{ij}\,, \tag{3.1}$$

and

$$\lim_{k\to\infty} (1-\varepsilon)^k r_k = 0\,,$$

then $\mathbf{s}(\mathbf{x})$ is in C^1.

Proof. Let

$$\|\mathcal{C}\|_\infty := \sup_{\mathbf{i}\in\mathbb{Z}^2} \|\mathbf{c}_\mathbf{i}\| \quad \text{and} \quad \|U\| := \sup_{\|\mathcal{C}\|_\infty = 1} \|U\mathcal{C}\|_\infty$$

for any operator U and any regular mesh $\mathcal{C}$.

Since

$$\|\nabla R\mathcal{C}\|_\infty = \frac{1}{2}\|\nabla\mathcal{C}\|_\infty\,, \quad \nabla A_\gamma \mathcal{C} = A_\gamma \nabla \mathcal{C}\,, \quad \text{and} \quad \|A_\gamma \mathcal{C}\|_\infty \leq \|\mathcal{C}\|_\infty\,,$$

we conclude that

$$\|\nabla \mathcal{C}_k\|_\infty \leq 2^{-k} \|\nabla\mathcal{C}\|_\infty\,.$$

Further, every value $\mathbf{s}^n_k(\mathbf{x})$ lies in the convex hull of $(n+1)^2$ points $\mathbf{c}^k_\mathbf{i}$, which together lie in the convex hull K of $(n+3+r_k)^2$ many points $\mathbf{c}^{k-1}_\mathbf{j}$. Since K also contains $\mathbf{s}^m_{k-1}(\mathbf{x})$ for all $m \leq n$ and since

$$\text{diameter}(K) \leq \frac{2(n+2+r_k)}{2^{k-1}} \|\nabla\mathcal{C}\|_\infty\,,$$

we get

$$\sup_{\mathbf{x}\in D} \|\mathbf{s}_k^n(\mathbf{x}) - \mathbf{s}_{k-1}^n(\mathbf{x})\| \le \frac{2(n+2+r_k)}{2^{k-1}} \|\nabla \mathcal{C}\|_\infty .$$

Hence $(\mathbf{s}_k^n)_{k\in\mathbb{N}}$ form Cauchy sequences over D, which converge to the same limit for all n. This proves the first statement of the theorem.

Next we consider the partial derivatives

$$\frac{\partial}{\partial x}\mathbf{s}_k^n(\mathbf{x}) = 2^k \nabla_1 \mathbf{c}_\mathbf{i}^k N^{n-1}(2^k x - i) N^n(2^k y - j)$$

and show that the differences $\nabla\nabla_1 \mathbf{c}_\mathbf{i}^k$ of their control points converge uniformly to zero.

For $\left[\mathbf{b}_{ij}\right]_{(i,j)\in\mathbb{Z}^2} := \nabla_1^2 R\, \mathcal{C}$, we observe that

$$\| \left[\mathbf{b}_{ij}\right]_{(i,j)\in\mathbb{Z}^2} \|_\infty = \frac{1}{2}\|\nabla_1^2\, \mathcal{C}\|_\infty$$

while $\mathbf{b}_{ij} = \mathbf{0}$ for every second i. This implies

$$\|\nabla_1^2 W_k\, \mathcal{C}\|_\infty = \|A_{\gamma_k} \nabla_1^2 R\, \mathcal{C}\|_\infty \le \frac{1-\varepsilon}{2}\|\nabla_1^2\, \mathcal{C}\|_\infty .$$

Since

$$\begin{aligned} \|\nabla_2\nabla_1 A_{\gamma_k} R\, \mathcal{C}\|_\infty = \|A_{\gamma_k}\| \|\nabla_2\nabla_1 R\, \mathcal{C}\|_\infty &= \frac{1}{4}\|\nabla_2\nabla_1\, \mathcal{C}\|_\infty \\ &\le \frac{1-\varepsilon}{2}\|\nabla_2\nabla_1\, \mathcal{C}\|_\infty , \end{aligned}$$

we obtain altogether

$$\|\nabla\, (2^k \nabla_1\, \mathcal{C}_k)\|_\infty \le (1-\varepsilon)^k \|\nabla\nabla_1\, \mathcal{C}\|_\infty .$$

Hence, we can continue as above and obtain that $\mathbf{s}(\mathbf{x})$ is in C^1 and that $(2\nabla_i W_k)_{k\in\mathbb{N}}, i = 1, 2,$ are derivative schemes of $(W_k)_{k\in\mathbb{N}}$. □

From the proof we also get

Corollary 3.2. (Estimates for second order difference schemes)

$$\|\nabla_i \nabla_j W_k\| < \frac{1}{2} \quad \textit{for} \quad i, j \in \{1,\, 2\} .$$

Example 3.3. (Two non-stationary wave schemes)

(1) The condition $\varepsilon \le \sum_{j\in\mathbb{Z}} \sum_{i\in 2\mathbb{Z}} \gamma_{ij} \le 1-\varepsilon$ *in (3.1) means that*

$$\varepsilon \le \|A_\gamma \mathcal{M}\|_\infty \le 1-\varepsilon$$

for a mesh $\mathcal{M}$ *whose columns are alternating zero or* $[\ldots 1\, 1 \ldots]^t$*. These inequalities still hold if we average* $A_\gamma \mathcal{M}$ *with any other scheme* $A_{\overline{\gamma}}$*. Consequently, since* $r_k = O(k)$*, any sequence of "waves"*

$$W_k := B_k \cdots B_1 R$$

defines a non-stationary C^1 *wave scheme* $(W_k)_{k\in\mathbb{N}}$ *for regular meshes.*

(2) The tensor product Rvachev scheme [Rva90]

$$W = (W_k)_{k\in\mathbb{N}} \quad \textit{with} \quad W_k = A^k R$$

is a non-stationary wave scheme with $r_k = O(k)$ *and* $\varepsilon = 1/2$ *for which Theorem (3.1) applies.*

4 Basic observations

In this section, we consider "waves" for arbitrary quadrilateral meshes with extraordinary vertices or faces. Interior vertices or faces of a quadrilateral mesh are called *extraordinary* if their valence does not equal 4.

Subdividing by R, $B_{\alpha,\beta}$, and A does not increase the number of extraordinary elements and isolates these elements. Therefore, it suffices to consider only (sub)meshes with one extraordinary vertex, as illustrated in Figure 4.1. These meshes are called *ringnets*.

Definition 4.1. (Ring and ringnet)
Let $\mathcal{N}_0$ *be the subnet of* $\mathcal{N}$ *consisting of the extraordinary vertex or face of* $\mathcal{N}$*. The* k*-th ring around* $\mathcal{N}_0$ *is denoted by* $\mathcal{N}_k$ *and the mesh consisting of* $\mathcal{N}_0, \ldots, \mathcal{N}_k$ *by* $\mathcal{N}_{0\ldots k}$*. The latter is called a* (regular) k-ringnet *or* (regular) k-net *for short. Furthermore, the submesh* $\mathcal{N}_{i\ldots j}$ *consists of* $\mathcal{N}_i, \ldots, \mathcal{N}_j$ *.*

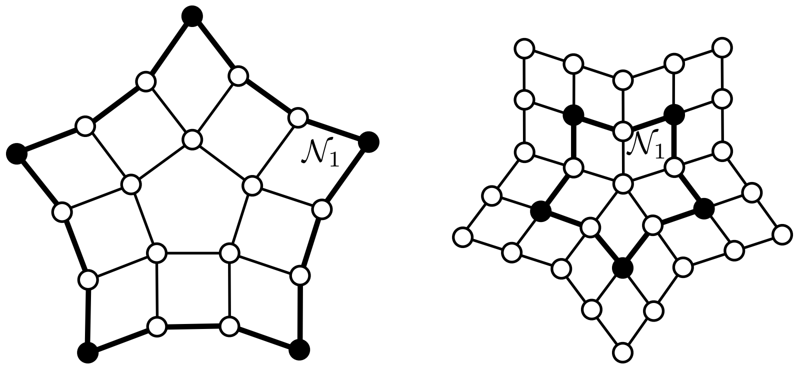

Figure 4.1: Examples of rings and ringnets: a 1-ringnet with an extraordinary face of valence 5 (left) and a 2-ringnet with an extraordinary vertex of valence 5 (right). The first rings $\mathcal{N}_1$ in both meshes are marked by bold lines and the convex corners of $\mathcal{N}_1$ are marked by ● .

Given a ringnet $\mathcal{N}$ and a wave operator W of degree n, we generate the sequence $\mathcal{N}^{(l)} = W^l\mathcal{N}$ and denote $(\mathcal{N}^{(l)})_{i...j}$ and $(\mathcal{N}^{(l)})_i$ by $\mathcal{N}^{(l)}_{i...j}$ and $\mathcal{N}^{(l)}_i$, respectively.

We say that a (sub)mesh $\mathcal{N}$ *influences* another subdivided (sub)mesh $\mathcal{M}$ if, during the subdivision, every vertex in $\mathcal{N}$ has an effect on some vertex in $\mathcal{M}$ and if additionally all vertices in $\mathcal{M}$ depend on $\mathcal{N}$.

Refining any ringnet $\mathcal{N}$ by a wave scheme W, it follows by induction that the vertices of $W\mathcal{N}$ influenced by a vertex in $\mathcal{N}$ form an l ring neighborhood, where l does not depend on α and β even if these are zero. Hence we obtain

Lemma 4.2. (Equivalent masks for wave schemes and midpoint schemes)
The size and topological form of the masks of a wave scheme W of degree n only depend on its degree. Consequently, any vertex in any mesh $\mathcal{N}$ influences equivalent submeshes in $W\mathcal{N}$ and $M_n\mathcal{N}$, where M_n is the midpoint scheme of degree n.

Remark 4.3. (Core mesh)
For

$$r = \left\lfloor \frac{n-1}{2} \right\rfloor ,$$

the r-net $\mathcal{N}_{0...r}$ of $\mathcal{N}$ consists of all vertices influencing $\mathcal{N}^{(l)}_0$ for some $l \geq 1$. It is called the core (mesh) *of $\mathcal{N}$ with respect to the wave scheme W of degree n.*

Depending on the context, we treat any mesh as a matrix whose rows represent the vertices or as the set of all vertices.

It is straightforward to prove

Lemma 4.4. (Dependence after a subdivision step)
$\mathcal{N}_{0...r+k}$ determines $\mathcal{N}^{(1)}_{0...r+2k}$ for $k \geq 0$, i. e.,

$$\mathcal{N}^{(1)}_{0...r+2k} = (W\mathcal{N}_{0...r+k})_{0...r+2k} .$$

If we subdivide just the regular parts of any $\mathcal{N}^{(k)}$, we obtain for every k a limiting surface $\mathbf{s}_k$. Since $\mathbf{s}_{k+1}$ contains $\mathbf{s}_k$, we can consider the difference surface $\mathbf{r}_k = \mathbf{s}_{k+1} \backslash \mathbf{s}_k$ whose control points are contained in a sufficiently large subnet $\mathcal{N}^{(k)}_{0...\rho}$ with $\rho \geq n$ not depending on k. Due to Lemma 4.4, the operator W restricted to ρ-nets can be represented by a stochastic matrix $S = S_\rho$ called the *subdivision matrix*, i. e.,

$$\mathcal{N}^{(k+1)}_{0...\rho} = S\mathcal{N}^{(k)}_{0...\rho} . \tag{4.1}$$

Lemma 4.5. (Dependence property of a core mesh)
For any $\rho > 0$ there is some constant q such that for all $k \geq q$ every core vertex (vertex in $\mathcal{N}_{0...r}$) influences all vertices in $\mathcal{N}^{(k)}_{0...\rho}$, which is denoted by

$$\mathcal{N}_{0...r} \Rrightarrow \mathcal{N}^{(k)}_{0...\rho} .$$

Proof. For sufficiently large l and any $s \geq 0$, every vertex in $\mathcal{N}_{0...r}$ influences all vertices in $\mathcal{N}^{(l+s)}_0$, all vertices in $\mathcal{N}^{(l+s+1)}_{0...1}$, and so on. Hence, we obtain the lemma with $q = l + \rho$. □

Theorem 4.6. (C^0-property of W)
The subdivision surfaces generated by W are C^0 continuous.

Proof. Since the subdivision matrix S is stochastic, i. e., S is a non-negative and real matrix and each row of S sums to 1, the dominant eigenvalue of S is 1. Due to Lemma 4.5, there is an integer $l \geq 1$ such that

$$\mathcal{N}_{0...r} \Rrightarrow \mathcal{N}^{(l)}_{0...\rho} = S^l\mathcal{N}_{0...\rho} .$$

This implies that S^l has a positive column (for every core vertex) and, according to [MP89, Theorem 2.1], any sequence $(S^i\,\mathbf{c})$ converges to a multiple of the vector $[1\ \dots\ 1]^{\mathrm{t}}$ as $i \to \infty$ for all real vectors $\mathbf{c}$. Therefore, the only dominant eigenvalue of S is 1 and it has algebraic multiplicity 1.

Hence, the difference surfaces $\mathbf{s}_i \backslash \mathbf{s}_{i-1}$ converge to a point and the surfaces generated by W are continuous. □

To analyze the spectrum of the subdivision matrix S, we order any ρ-net $\mathcal{N}$ such that

$$\mathcal{N} = \begin{bmatrix} \mathcal{N}_{0\dots r} \\ \mathcal{N}_b \\ \mathcal{N}_a \\ \mathcal{N}_{r+2} \\ \vdots \\ \mathcal{N}_\rho \end{bmatrix},$$

where $\mathcal{N}_a$ consists of the convex corners and $\mathcal{N}_b$ of all other points in $\mathcal{N}_{r+1}$ (see Figure 4.1 for an illustration of the convex corners). With this arrangement, the subdivision matrix S has the lower triangular form

$$S = \begin{bmatrix} C & & & & & \\ * & B & & & & \\ * & * & A & & & \\ * & * & * & 0 & & \\ \vdots & & & \ddots & \ddots & \\ * & \dots & \dots & \dots & * & 0 \end{bmatrix},$$

where

$$\mathcal{N}^{(1)}_{0\dots r} = C\,\mathcal{N}_{0\dots r}\,, \tag{4.2}$$

$$\mathcal{N}^{(1)}_b = \begin{bmatrix} * & B \end{bmatrix} \begin{bmatrix} \mathcal{N}_{0\dots r} \\ \mathcal{N}_b \end{bmatrix}, \quad \text{and} \tag{4.3}$$

$$\mathcal{N}^{(1)}_a = \begin{bmatrix} * & * & A \end{bmatrix} \begin{bmatrix} \mathcal{N}_{0\dots r} \\ \mathcal{N}_b \\ \mathcal{N}_a \end{bmatrix}. \tag{4.4}$$

To verify this, we recall from Remark 4.3 that any point influencing the core mesh influences some $\mathcal{N}^{(l)}_0$ and thus belongs to the core mesh. This implies Equation (4.2) and shows that $\mathcal{N}_{r+1}$ influences only points in $(W\mathcal{N})_{r+1\dots\infty}$ and hence, that $\mathcal{N}_{r+2}$ influences only points in $(W\mathcal{N})_{r+2\dots\infty}$, etc. Since $\mathcal{N}_a$ does not influence any point in $\mathcal{N}^{(1)}_b$, Equations (4.3) and (4.4) follow. Moreover, due to Lemma 4.4, $(W\mathcal{N})_{r+2}$ is determined by $\mathcal{N}_{0\dots r+1}$ and $(W\mathcal{N})_{r+3}$ is determined by $\mathcal{N}_{0\dots r+2}$, etc.

Hence, the eigenvalues of S are zero or are the eigenvalues of the blocks C, B, and A.

Lemma 4.7. (Spectral radii of B and A)
The spectral radii ρ_B and ρ_A of B and A satisfy

$$\rho_B \le \left(\frac{1}{2}\right)^{\lfloor \frac{n}{2} \rfloor + 1} \quad \text{and} \quad \rho_A \le \left(\frac{1}{4}\right)^{\lfloor \frac{n}{2} \rfloor + 1}.$$

In particular, $\rho_B, \rho_A \le 1/4$ for $n \ge 2$.

Proof. Since A is non-negative, we get [HJ85, Corollary 6.1.5]

$$\rho_A \le \|A\|_\infty = \|A\mathbf{1}\|_\infty, \quad \text{where} \quad \mathbf{1} := [1\ \dots\ 1]^{\mathrm{t}}.$$

The vector $A\mathbf{1}$ represents the convex corners of $\mathcal{N}^{(1)}_{r+1}$ if $\mathcal{N}_{0\dots r} = 0$, $\mathcal{N}_b = 0$, $\mathcal{N}_a = 1$, and $\mathcal{N}_{r+2\dots\rho} = 0$. One can easily verify that the (scalar-valued) vertices of these convex corners are

$$\frac{1}{4} \cdot \frac{1-\alpha_1(4)-\beta_1(4)}{4} \cdot \dots \cdot \frac{1-\alpha_r(4)-\beta_r(4)}{4} \quad \text{for } n = 2r+1 \text{ and}$$

$$\frac{1}{4} \cdot \frac{1-\alpha_1(4)-\beta_1(4)}{4} \cdot \dots \cdot \frac{1-\alpha_r(4)-\beta_r(4)}{4} \cdot \frac{1}{4} \quad \text{for } n = 2r+2.$$

Since $1-\alpha_i(4)-\beta_i(4) \in (0,1)$, this concludes the proof of the second statement. The first statement can be proved similarly. □

5 The characteristic map

For the C^1 analysis of wave subdivision, we need to investigate the eigenvectors and eigenvalues of the subdivision matrix S. We do this by subdividing special grid meshes as in [PC11] and recall the basic definitions in this section.

Definition 5.1. (Grid mesh)
A primal grid mesh *of valence m and frequency f is a planar primal ringnet with the vertices*

$$\mathbf{g}_{ij}^l = \begin{bmatrix} \mathrm{Re}(g_{ij}^l) \\ \mathrm{Im}(g_{ij}^l) \end{bmatrix} \in \mathbb{R}^2 ,$$

where $g_{ij}^l = ie^{\hat{\imath}2\pi lf/m} + je^{\hat{\imath}2\pi(l+1)f/m} \in \mathbb{C}$ and $i,j \geq 0$, $l \in \mathbb{Z}_m$, $\hat{\imath} = \sqrt{-1}$.
A dual grid mesh *of valence m and frequency f consists of the vertices*

$$\mathbf{h}_{ij}^l = \frac{1}{4}(\mathbf{g}_{i-1,j-1}^l + \mathbf{g}_{i,j-1}^l + \mathbf{g}_{i-1,j}^l + \mathbf{g}_{i,j}^l), \quad i,j \geq 1,\ l \in \mathbb{Z}_m$$

(see Figure 5.1). For fixed l, the vertices $\mathbf{g}_{ij}^l$ or $\mathbf{h}_{ij}^l$ with $(i,j) \neq (0,0)$ of a grid mesh $\mathcal{N}$ build the l-th segment *of $\mathcal{N}$. The* segment angle *of $\mathcal{N}$ is $\varphi = 2\pi f/m$. The half-line from the center $\mathbf{g}_{00}^l$ through $\mathbf{g}_{10}^l$ is called the l-th* spoke*, denoted by $S_l(\mathcal{N})$ or S_l for short.*

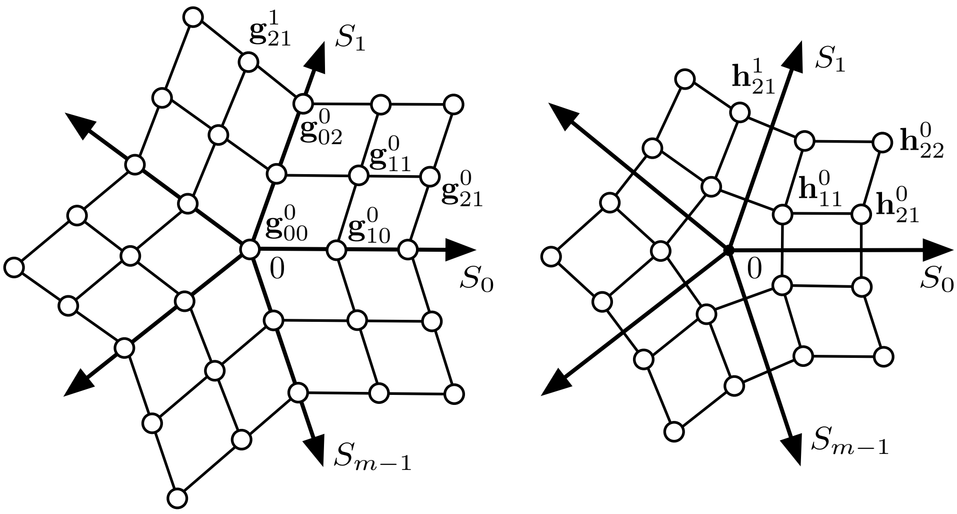

Figure 5.1: A primal grid mesh (left) and a dual grid mesh (right) with valence 5 and frequency 1.

Topologically, any ringnet $\mathcal{M}$ is equivalent to a grid mesh $\mathcal{N}$. Therefore, we use the same indices for equivalent vertices and denote the vertices of $\mathcal{M}$ by $\mathbf{p}_{ij}^l$. For a primal ringnet, $\mathbf{p}_{00}^0, \ldots, \mathbf{p}_{00}^{m-1}$ all denote the same vertex.

Definition 5.2. (Symmetric ringnet)
A planar ringnet of valence m with the vertices $\mathbf{p}_{ij}^l$ in $\mathbb{R}^2$ is called rotationally symmetric *with frequency f, if*

$$\mathbf{p}_{ij}^{l+1} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \mathbf{p}_{ij}^l \quad \textit{with} \quad \varphi = 2\pi f/m .$$

A planar ringnet $\mathcal{N} \in \mathbb{R}^2$ is called reflection symmetric *if its permutation $\widetilde{\mathcal{N}}$ consisting of the points $\widetilde{\mathbf{p}_{ij}^l} := \mathbf{p}_{ji}^{(m-1)-l}$ equals the conjugate ringnet $\overline{\mathcal{N}}$ consisting of the points*

$$\overline{\mathbf{p}_{ij}^l} = \overline{\begin{bmatrix} p_{ij,x}^l \\ p_{ij,y}^l \end{bmatrix}} = \begin{bmatrix} p_{ij,x}^l \\ -p_{ij,y}^l \end{bmatrix} ,$$

i. e.,

$$\widetilde{\mathcal{N}} = \overline{\mathcal{N}} .$$

A rotationally and reflection symmetric ringnet is called symmetric.

Using the technique established in [PC11], we construct and analyze a characteristic map of a wave scheme

$$W = B_r \cdots B_1 R \quad \text{or} \quad W = AB_r \cdots B_1 R$$

with $B_i = B_{\alpha_i,\beta_i}$. We follow [PC11] and use results stated there for midpoint subdivision that are also valid for “waves” since their proofs are only based on

- symmetry preservation,

- convex combinations computed during subdivision, and
- influence relations as considered in Lemma 4.2.

Theorem 5.3. ($\mathcal{M}_\infty$ and λ_φ)
Let $\mathcal{M}$ be the core mesh of a grid mesh with frequency f and segment angle $\varphi := 2f\pi/m \in (0,\pi)$. Let

$$\mathcal{M}_k := \frac{(U^k\mathcal{M})_{0\ldots r}}{\|(U^k\mathcal{M})_{0\ldots r}\|},$$

where $\|\cdot\|$ denotes any matrix norm. Then the following statements hold.

(a) The sequence $(\mathcal{M}_k)_{k\in\mathbb{N}}$ converges to a symmetric eigennet $\mathcal{M}_\infty$ with segment angle φ and a positive eigenvalue λ_φ, which depends only on φ but not on f and m. $(\mathcal{M}_\infty)_{0\ldots 1}$ has at most one zero control point. Additionally, we define $\lambda_\pi := |\gamma_\pi|$, where γ_π is the maximum eigenvalue associated with a rotationally symmetric eigenvector with segment angle π.

(b) Restricting W to core meshes, the eigenvalue λ_φ is the dominant eigenvalue of the eigenspaces of frequencies f and $m-f$ and it has geometric and algebraic multiplicity 2.

(c) $\lambda_\alpha > \lambda_\theta > \lambda_\pi$ for $0 < \alpha < \theta < \pi$.

This can be proved as (5.4), (5.7), (6.3), and (6.4) in [PC11]. For midpoint subdivision schemes for quadrilateral meshes, λ_π is equal to $1/4$ and equal to the subdominant eigenvalue μ_0 of frequency 0. This implies that $\lambda_{2\pi/m}$ is subdominant. However, for "waves", λ_π can be smaller than μ_0. Therefore, we use the following lemma to show that $\lambda_{2\pi/m}$ is subdominant for $m > 4$.

Lemma 5.4. ($\lambda_{\pi/2} = 1/2$)

(a) For $m = 4$, the operator W has the subdominant eigenvalue $1/2$.

(b) $\lambda_{\pi/2} = 1/2$ holds for any m and f such that $\frac{2f\pi}{m} = \frac{\pi}{2}$.

Proof. We consider a scalar-valued eigenmesh $\lambda\mathcal{M} = W\mathcal{M}$ with eigenvalue λ and segment angle $\pi/2$. Since there is a basis of rotationally symmetric eigenmeshes, we may assume that $\mathcal{M}$ is such an eigenmesh. For $m = 4$ and due to Lemma 3.2, we have that

$$\lambda\mathcal{M}_{\nabla\nabla} = W_{\nabla\nabla}\mathcal{M}_{\nabla\nabla}$$

and $\|W_{\nabla\nabla}\| < 1/2$. It follows that $|\lambda| < 1/2$ or that $\mathcal{M}_{\nabla\nabla} = 0$, meaning that $\mathcal{M}$ is an affine image of a regular grid $\mathcal{G}$, i. e., a linear combination of the constant mesh $[1\ \ldots\ 1]^t$ with eigenvalue 1 and the two coordinates of $\mathcal{G}$. Since $W\mathcal{G} = \frac{1}{2}\mathcal{G}$, (a) follows. Due to symmetry, the subdivided mesh $W\mathcal{M}$ does not depend on f, whence (b) follows. □

Next we consider two ringnets of frequency 0 with different valencies m_1 and m_2, but equal (first) segments. If we apply R, A or $B_{\alpha,\beta}$ to these nets, all segments remain equal provided $\alpha(m_1) = \alpha(m_2)$ and $\beta(m_1) = \beta(m_2)$. Hence, we get

Lemma 5.5. ($\lambda_{2\pi/m}$ and μ_0)
For constant functions $\alpha_1, \beta_1, \ldots, \alpha_r, \beta_r$, i. e., $\alpha_i(m) \equiv \alpha_i(4)$ and $\beta_i(m) \equiv \beta_i(4)$, the subdominant eigenvalue μ_0 of frequency 0 does not depend on the valence m and hence

$$|\mu_0| \le 1/2 = \lambda_{\pi/2} < \lambda_{2\pi/m}$$

for $m \ge 5$ due to Lemma 5.4 and Theorem 5.3 (c).

Lemma 5.4 together with Theorem 5.3 and Lemma 4.7 can be used as in the proof of Theorem (7.3) in [PC11] to derive the following theorem.

Theorem 5.6. (Subdominant eigenvalue)
Let ρ be as in Equation (4.1) and let W be a wave operator of degree n mapping the space of ρ-ringnets of valence m into itself with $n \ge 2$ and $1 - \alpha_i - \beta_i \in (0,1)$. Let $\mathcal{M}$ be a ρ-grid mesh of valence m and frequency 1. For $m \ge 3$, the meshes

$$\mathcal{M}_k := \frac{W^k\mathcal{M}}{\|W^k\mathcal{M}\|}$$

converge to a subdominant eigenmesh $\mathcal{C}$ of W called the characteristic mesh *of W and its eigenvalue $\lambda_{2\pi/m}$ has geometric and algebraic multiplicity 2, if*

$$\lambda_{2\pi/m} > \begin{cases} |\mu_0(m)| & , \quad m \ge 4 \\ \max\{|\mu_0(m)|, \rho_B, \rho_A\} & , \quad m = 3 \end{cases}, \tag{5.1}$$

where μ_0 is the subdominant eigenvalue of frequency 0 and ρ_B and ρ_A are the spectral radii defined in Lemma 4.7.

Due to Lemma 5.5, Inequality (5.1) is satisfied if the weight functions $\alpha_1, \beta_1, \ldots, \alpha_r, \beta_r$ are constant and $m \ge 5$. Thus, we get

Corollary 5.7. (Subdominant eigenvalue for $m \ge 5$ with constant functions α_i and β_i)
The meshes $\mathcal{M}_k$ as in Theorem 5.6 converge to the characteristic mesh of W and its eigenvalue $\lambda_{2\pi/m}$ has geometric and algebraic multiplicity 2, if $m \ge 5$ and if the weight functions $\alpha_1, \beta_1, \ldots, \alpha_r, \beta_r$ are constant functions of the valence m.

6 Smoothness for irregular meshes

Let $\mathcal{C}$ be the characteristic mesh of valence m of the wave scheme

$$W = B_r \cdots B_1 R \quad \text{or} \quad W = AB_r \cdots B_1 R$$

with $B_i = B_{\alpha_i,\beta_i}$, $\alpha_i, \beta_i \in [0,1)$, and $1 - \alpha_i - \beta_i \in (0,1)$. It defines the control mesh of a characteristic map, which is a surface ring consisting of m segments.

Theorem 6.1. **(C^1-property of W)**
The wave scheme W of degree $n \geq 2$ is a C^1 subdivision algorithm if Inequality (5.1) is satisfied. Particularly, W generates C^1 surfaces for valencies $m \geq 5$ if the weight functions α_i and β_i are constant functions of the valence m.

Proof. To simplify the notation, we identify the real plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by the bijection $\mathbb{R}^2 \ni [x\ y]^t \mapsto x + \hat{\imath} y \in \mathbb{C}$. Let $\mathbf{c}(x,y) : \Omega \to \mathbb{C}$ be 3 segments of the characteristic map of W, where $\Omega = \Omega_{-1} \cup \Omega_0 \cup \Omega_1$ as shown at the left of Figure 6.1, and $\mathbf{c}_{|\Omega_i}$ is the i-th segment for $i = -1, 0, 1$.

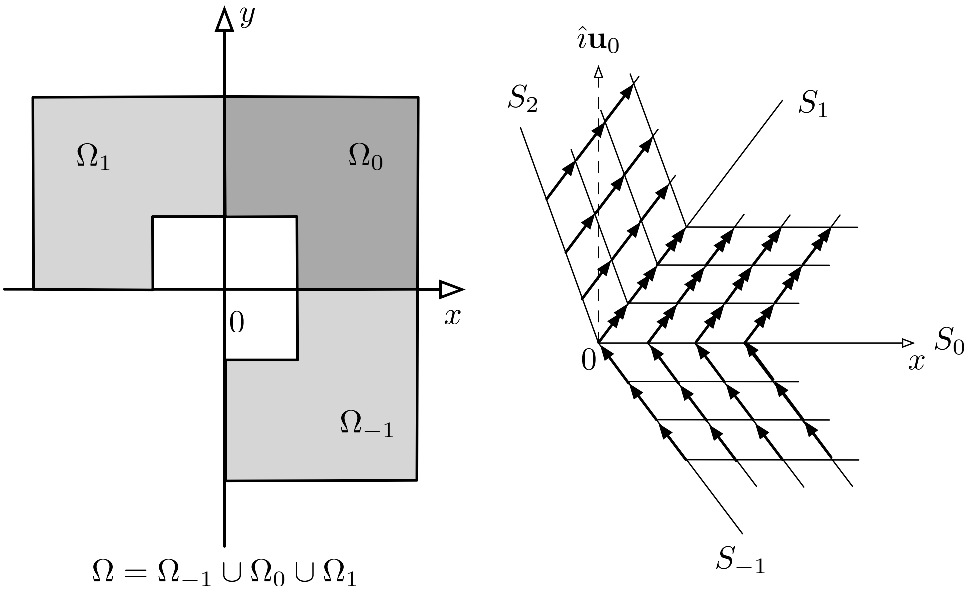

Figure 6.1: The domain Ω of $\mathbf{c}$ (left) and the -1, 0, 1-th segments of a grid mesh (right), where the y-edges in the three segments are marked by arrows and the y-edges in the 0-th segment are especially marked by double arrows.

First, we observe a grid mesh $\mathcal{M}$ as shown at the right of Figure 6.1 such that the subdivided and normalized meshes $\mathcal{M}_k = (W^k \mathcal{M})_{0\ldots\rho} / \|(W^k \mathcal{M})_{0\ldots\rho}\|$ converge to the characteristic mesh $\mathcal{C}$ due to Theorem 5.6 and Corollary 5.7. If n is odd, we require $\mathcal{M}$ to be primal and otherwise to be dual. Let $\mathcal{E}_k = \nabla_2(\mathcal{M}_k)$ and $\mathcal{E} = \nabla_2(\mathcal{C})$, where the edge set of a ringnet $\mathcal{K} = [\mathbf{p}_{ij}^k]$ is defined by

$$\nabla_2(\mathcal{K}) := \{\nabla_2 \mathbf{p}_{i,j}^0 = \mathbf{p}_{i,j}^0 - \mathbf{p}_{i,j-1}^0 \mid i \geq 0, j > 0\},$$

as illustrated at the right of Figure 6.1. These and other edges control the directions of the partial derivatives $\mathbf{c}_y(\Omega_0)$. Furthermore, we add both $\mathbf{u}_1$ and $\hat{\imath}\mathbf{u}_0$ to $\mathcal{E}_k$ and $\mathcal{E}$, where $\mathbf{u}_1$ is the edge direction of the spoke S_1 and $\hat{\imath}\mathbf{u}_0$ is the edge direction of the spoke S_0 rotated by $+\pi/2$. Refining, averaging, and smoothing a mesh also means its edges are averaged by the masks shown in Figure 6.2. In particular, the edges in $\mathcal{E}_k$ are either, due to symmetry, parallel to $\mathbf{u}_1$ and $\hat{\imath}\mathbf{u}_0$ or obtained by iteratively averaging the edges in $\mathcal{E}_{k-1}$ and multiplying these by positive numbers because of the normalization. Thus, we know that $\mathcal{E}_k$ lies in the cone spanned by $\mathcal{E}_{k-1}$, i. e., in the cone

$$\mathcal{D}_0 := \begin{cases} [0,\infty)\, e^{\hat{\imath}[\pi/2,\, 2\pi/3]} & , \quad \text{if } m = 3 \\ [0,\infty)\, e^{\hat{\imath}[2\pi/m,\, \pi/2]} & , \quad \text{if } m \geq 5 \end{cases}.$$

Therefore, by induction, all $\mathcal{E}_k$ and $\mathcal{E}$ lie in $\mathcal{D}_0$.

Moreover, since $\mathcal{C}_{0\ldots1}$ is symmetric and has at most one zero control point, at least one of its edges is non-zero. Subdividing $\mathcal{C}$, we can see that every element of $\mathcal{E}$ is a linear combination of $\mathcal{E}$ with non-negative weights and a positive weight for the non-zero element in the 1-ringnet. Hence, $\mathcal{E}$ has no zero elements.

Second, we observe that for a symmetric ringnet $\mathcal{N}$, each element of $\nabla_2(2R\mathcal{N})$, $\nabla_2(A\mathcal{N})$, and $\nabla_2(B_{\alpha,\beta}\mathcal{N})$ is a convex combination of elements in $\nabla_2(\mathcal{N})$, in $\nabla_2(\mathcal{N})$ reflected at S_1, and in $-\nabla_2(\mathcal{N})$ reflected at S_0, where a reflected element has a weight which is less than or equal to that of the unreflected counterpart. Thus, by induction, we see that $\nabla_2(2^k W^k \mathcal{C}) \subset \mathcal{D}_0$, for $k \geq 0$. Since every partial derivative $\mathbf{c}_y$ over Ω_0 is the limit of a sequence of vectors $\mathbf{v}_k \in \nabla_2(2^k W^k \mathcal{C})$, it follows that $\mathbf{c}_y(\Omega_0) \subset \mathcal{D}_0$.

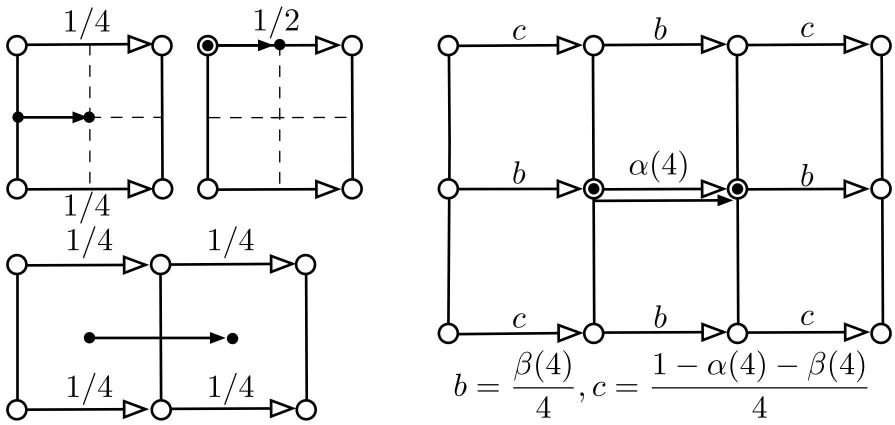

Figure 6.2: Masks for $R_\triangledown$ (top left), $A_\triangledown$ (bottom left), and $(B_{\alpha,\beta})_\triangledown$ (right) on regular meshes.

Next, we show $\mathbf{0} \notin \mathbf{c}_y(\Omega_0)$. Any $\mathbf{c}_y(\mathbf{x})$, $\mathbf{x} \in \Omega_0$, is a convex combination of $\mathcal{F}_{-1}$ or $\mathcal{F}_0$, where $\mathcal{F}_i$ is the set of all y-edges in the segments i and $i+1$ of $2^k U^k \mathcal{C}$ for sufficiently large k. We observe

$$\mathcal{F}_{-1} = e^{\pi/2-2\pi/m}\mathcal{F}_0 \subseteq \begin{cases} (0,\ \infty)\, e^{\hat{\imath}[\pi/3,\, 2\pi/3]} & , \quad \text{if } m=3 \\ (0,\ \infty)\, e^{\hat{\imath}[2\pi/m,\, \pi-2\pi/m]} & , \quad \text{if } m \ge 5 \end{cases} ,$$

which implies $\mathbf{0} \notin \mathbf{c}_y(\Omega_0)$. Hence, $\mathbf{c}_y(\Omega_0) \subset \mathcal{D} := \mathcal{D}_0 \backslash \{\mathbf{0}\}$ and similarly $\mathbf{c}_x(\Omega_0) \subset \mathcal{D} - \pi/2$.

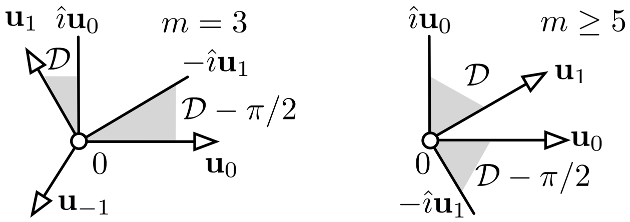

Figure 6.3: The direction cones $\mathcal{D}$ and $\mathcal{D} - \pi/2$.

Since each pair in $(\mathcal{D},\ \mathcal{D} - \pi/2)$ is linearly independent (see Figure 6.3), $\mathbf{c}$ is regular over Ω_0 and hence, the characteristic map of W is regular. Because $\mathbf{c}_y(\Omega_0) \subset \mathcal{D}$, $\mathbf{c}$ does not map any line segment between two points in Ω_0 to a closed curve, meaning that $\mathbf{c}$ is injective over Ω_0. Moreover, since $\mathcal{M}$ is a symmetric grid mesh whose zeroth segment lies in $[0,\ \infty)\, e^{\hat{\imath}[0,\, 2\pi/m]} =: \mathcal{A}$ and W preserves symmetry, it implies $\mathbf{c}(\Omega_0) \subset \mathcal{A}$ and $\mathbf{c}$ maps the interior of Ω_0 into the interior of $\mathcal{A}$. Hence, the total characteristic map of W is injective. Finally, Reif's C^1-criterion [Rei95, Theorem 3.6] is satisfied, which concludes the proof. □

Example 6.2. (Wave scheme of degree 3)
Using the discrete Fourier transform *for the subdivision matrix of $W = B_{\alpha,\beta}R$ with $\alpha, \beta \in [0,1)$, $1-\alpha-\beta \in (0,1)$, we get*

$$\mu_0(m) = \frac{3\alpha(m) + \beta(m) + \sqrt{(3\alpha(m)+\beta(m))^2 - 4t\ \alpha(m)}}{8}$$

and

$$\lambda_{\frac{2\pi}{m}}(m) = \frac{4+t+(2-t)c + \sqrt{(2-t)^2c^2 + 2(4+t)(2-t)c + (4-t)^2}}{16},$$

where $t = 2\alpha(4) + \beta(4)$ and $c = \cos(2\pi/m)$. According to Theorem 6.1, if $\alpha(m) = \alpha(4)$, $\beta(m) = \beta(4)$, and $m \ge 5$, then W generates C^1 surfaces around extraordinary points of valence m. Otherwise, W generates C^1 surfaces around extraordinary points of valence m if $\lambda_{\frac{2\pi}{m}}(m) > \mu_0(m)$ holds, since it can be easily verified that

$$\lambda_{\frac{2\pi}{m}}(m) \ge \frac{5}{16} > \frac{1}{4} \ge \max\{\rho_A,\ \rho_B\}\ .$$

7 Conclusion

In this paper, we have analyzed the smoothness of wave subdivision surfaces. The established C^1 analysis tools for quadrilateral meshes in [PC11] have been generalized to the weighted averaging operator $B_{\alpha,\beta}$. For regular meshes non-stationary wave schemes have been analyzed. Furthermore, a deeper understanding of the spectral properties of the subdivision matrices at extraordinary points is provided.

Tools to analyze wider and infinite classes of subdivision schemes are developed in this paper and hopefully help to advance the state of the art towards general C^1 analysis tools for other subdivision schemes that can be factorized into simple convex combination operators.

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