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A new class of degenerate Apostol-type Hermite polynomials and applications

Clemente Cesarano[a] · William Ramírez[b] · Subuhi Khan[c]

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Abstract

In this article, a new class of the degenerate Apostol–type Hermite polynomials is introduced. Certain algebraic and differential properties of there polynomials are derived. Most of the results are proved by using generating function methods.

1 Introduction

Throughout this paper, we use the standard notions: $\mathbb{N} := \{1, 2, \ldots\}$; $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$; $\mathbb{Z}$ denotes the set of integers; $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Further let $k \in \mathbb{Z}$ and $\lambda, \mu \in \mathbb{C}$.

On the subject of the Apostol-Bernoulli, Apostol–Euler and Apostol-Genocchi polynomials and their various extensions, a remarkably large number of investigations have appeared in the literature, see for example [3, 4, 7, 8, 9, 12, 14, 15, 17]. Certain results including various relatives of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials are established. Recently, many researchers studied degenerate versions of the familiar polynomials like Bernoulli, Euler, falling factorial and Bell polynomials by using generating functions, umbral calculus, and p-adic integrals, see for example [5, 10, 11, 13, 19].

The 2-variable Kampé de Fériet generalization of the Hermite polynomials are given by (see [2]):

$$H_n(x,y) = n! \sum_{n=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1}$$

It is to be noted that

$$H_n(2x,-1) = H_n(x),$$

where $H_n(x)$ are the ordinary Hermite polynomials [1].

These polynomials satisfy the following generating equation:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{2}$$

The 2–variable degenerate Hermite polynomials $H_n(x,y,\alpha)$ [20, p. 65] are defined by means of the generating function

$$(1+\alpha t)^{\frac{x}{\alpha}}(1+\alpha t^2)^{\frac{y}{\alpha}} = \sum_{n=0}^{\infty} H_n(x,y,\alpha)\frac{t^n}{n!}. \tag{3}$$

Since, $(1+\alpha t)^{\frac{x}{\alpha}} \to e^{xt}$ and $(1+\alpha t^2)^{\frac{y}{\alpha}} \to e^{yt^2}$ as $\alpha \to 0$, it is evident that Equation (3) reduces to Equation (2). That is $H_n(x,y)$ is the limiting case of $H_n(x,y,\alpha)$, when $\alpha \to 0$.

[a]Universitá Telematica Internazionale Uninettuno. Rome, Italy.
[b]Departamento de Ciencias Naturales y Exactas, Universidad de la Costa. Barranquilla, Colombia.
[c]Department of Mathematics, Aligarh Muslim University, Aligarh, India.

The first–kind Stirling number $s(n,k)$ is the number of ways in which n objects can be divided among k non–empty cycles and the second-kind Stirling numbers $S(n,k)$ count the number of ways to partition a set of n elements into exactly k nonempty subsets. The generating functions are given, respectively, by (see [18]):

$$\frac{1}{k!}[\ln(1+t)]^k = \sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!} \tag{4}$$

and

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!}. \tag{5}$$

The generalized falling factorial $(x|\alpha)_n$ with increment α is defined by (see [19, Definition 2.3]):

$$(x|\alpha)_n = \prod_{k=0}^{n-1}(x-\alpha k), \tag{6}$$

for positive integer n, with the convention $(x|\alpha)_0 = 1$, it follows that

$$(x|\alpha)_n = \sum_{k=0}^{n} s(n,k)\alpha^{n-k}x^k. \tag{7}$$

From Binomial theorem, we have

$$(1+\alpha t)^{\frac{x}{\alpha}} = \sum_{n=0}^{\infty}(x|\alpha)_n\frac{t^n}{n!}. \tag{8}$$

Next, we recall the definitions of the degenerate Bernoulli polynomials $\mathcal{B}_n(x;a)$, the degenerate Euler polynomials $\mathcal{E}_n(x;a)$ and the degenerate Genocchi polynomials $\mathcal{G}_n(x;a)$ [6], with parameter $a\in\mathbb{R}$ in the variable x and in a suitable neighborhood of $t=t_0$, by means of the corresponding generating functions.

$$\frac{t}{(1+at)^{\frac{1}{a}}-1}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty}\mathcal{B}_n(x;a)\frac{t^n}{n!}. \tag{9}$$

When $x=0$, $\mathcal{B}_n(a) := \mathcal{B}_n(0;a)$ are the corresponding degenerate Bernoulli numbers. It is to be noted from Equation (9) that

$$\lim_{a\to 0}\mathcal{B}_n(x;a) = B_n(x), \quad n\geq 0, \tag{10}$$

where $B_n(x)$ are the n—th order Bernoulli polynomials [16].

$$\frac{2}{(1+at)^{\frac{1}{a}}+1}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty}\mathcal{E}_n(x;a)\frac{t^n}{n!}. \tag{11}$$

For $x=0$, $\mathcal{E}_n(a) := \mathcal{E}_n(0;a)$ are the corresponding degenerate Euler numbers. It follows from Equation (11) that

$$\lim_{a\to 0}\mathcal{E}_n(x;a)(x;a) = E_n(x), \quad n\geq 0, \tag{12}$$

where $E_n(x)$ are the n—th order ordinary Euler polynomials [16].

$$\frac{2t}{(1+at)^{\frac{1}{a}}+1}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty}\mathcal{G}_n(x;a)\frac{t^n}{n!}. \tag{13}$$

When $x=0$, $\mathcal{G}_n(a):=\mathcal{G}_n(0;a)$ are the corresponding degenerate Genocchi numbers. Consequently from Equation. (13), we have

$$\lim_{a\to 0}\mathcal{G}_n(x;a)=G_n(x), \quad n\geq 0, \tag{14}$$

where $G_n(x)$ are the n−th order ordinary Genocchi polynomials [17].

Waseem A. Khan [21] introduced the Degenerate Hermite–Bernoulli Numbers and Polynomials of the Second Kind by means of the following generating function:

$$\frac{\log(1+at)^{\frac{1}{a}}}{(1+at)^{\frac{1}{a}}-1}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}}=\sum_{n=0}^{\infty}{}_H\mathcal{B}_n(x;y;a)\frac{t^n}{n!}. \tag{15}$$

For $\lambda,u\in\mathbb{C}$ and $\alpha\in\mathbb{N}$ with $u\neq 1$ the generalized degenerate Apostol–type Frobenius Euler–Hermite polynomials of order α which are given by generating function (see [22, P. 569]):

$$\left(\frac{1-u}{\lambda(1+at)^{\frac{1}{a}}-u}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}}=\sum_{n=0}^{\infty}{}_Hh_n(x;y;a;\lambda;u)\frac{t^n}{n!}. \tag{16}$$

Taking $u=-1$ and $\alpha=1$ in (16), we obtain the degenerate Hermite–Euler Polynomials

$$\frac{2}{\lambda(1+at)^{\frac{1}{a}}+1}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}}=\sum_{n=0}^{\infty}{}_H\mathcal{E}_n(x;y;a;\lambda)\frac{t^n}{n!}. \tag{17}$$

Subuhi Khan et. al. [19] introduced and studied the degenerate Apostol-type polynomials order α, denoted by $\mathcal{P}^{(\alpha)}(x;,a;\lambda;\mu;\nu)$ by means of the following generating function:

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}=\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha)}(x;a;\lambda;\mu;\nu)\frac{t^n}{n!}, \tag{18}$$

where $x\in\mathbb{R}$, $\lambda,\mu,\nu\in\mathbb{C}$; $n\in\mathbb{N}_0$.

For $x=0$, $\mathcal{P}_n^{(\alpha)}(a;\lambda;\mu;\nu):=\mathcal{P}_n^{(\alpha)}(0;a;\lambda;\mu;\nu)$ denotes the corresponding the degenerate Apostol-type numbers of order α and are defined as:

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}=\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha)}(a;\lambda;\mu;\nu)\frac{t^n}{n!}. \tag{19}$$

In view of Equation (18), it follows that

$$\lim_{a\to 0}\mathcal{P}^{(\alpha)}(x;,a;\lambda;\mu;\nu)=\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu), \quad n\geq 0, \tag{20}$$

where $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ are the Apostol-type polynomials of order α (see [18]).

This article aims to introduce a new class of degenerate Apostol-type Hermite polynomials. Some algebraic properties and relations for these polynomials ara derived. These results extend certain relations and identies of the related polynomials.

2 Degenerate Apostol-type Hermite polynomials

In this section, the degenerate Apostol–type Hermite polynomials are introduced and certain result for these polynomials are derived.

Definition 2.1. For arbitrary real or complex parameter α and for $a \in \mathbb{Z}^+$, the degenerate Apostol-type Hermite polynomials ${}_H\mathcal{P}^{(\alpha)}(x,y;,a;\lambda;\mu;\nu)$, are defined, in a suitable neighborhood of $t=0$, by means of the generating function:

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}} = \sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}, \tag{21}$$
$$|t| < \left|\log\left(\frac{-1}{\lambda}\right)\right|, \quad 1^{\alpha} := 1.$$

Taking $x=y=0$ in Equation (21) we obtain the corresponding degenerate Apostol-type Hermite numbers defined as:

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha} = \sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(a;\lambda;\mu;\nu)\frac{t^n}{n!}. \tag{22}$$

We note the following limit case:

$$\lim_{a\to 0}\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}} = \left(\frac{2^{\mu}t^{\nu}}{\lambda e^t+1}\right)^{\alpha} e^{xt+yt^2},$$

that is

$$\lim_{a\to 0}\sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu;\nu)\frac{t^n}{n!},$$

where ${}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu;\nu)\frac{t^n}{n!}$ denotes the Apostol type Hermite polynomials. Consequently, we have

$$\lim_{a\to 0} {}_H\mathcal{F}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} = {}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;\mu;\nu)\frac{t^n}{n!}.$$

Also, we note the following limits:

$$\begin{aligned}(-1)^{\alpha}\lim_{a\to 0} {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;1;0;\nu) &= {}_HB_n^{(\alpha)}(x,y),\\ \lim_{a\to 0} {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;1;1;0) &= {}_HE_n^{(\alpha)}(x,y),\\ \lim_{a\to 0} {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;1;1;1) &= {}_HG_n^{(\alpha)}(x,y).\end{aligned}$$

Example 2.1. For any $\lambda \in \mathbb{C}\setminus\{-1,1\}$, $a=\alpha=\mu=\nu=1$, the first few degenerate Apostol-type Hermite polynomials are given as:

$$\begin{aligned}{}_H\mathcal{P}_0^{(\alpha)}(x,y;a;\lambda;\mu;\nu) &= \frac{2}{\lambda+1},\\ {}_H\mathcal{P}_1^{(\alpha)}(x,y;a;\lambda;\mu;\nu) &= \frac{2}{\lambda+1}x - \frac{2\lambda}{(\lambda+1)^2},\\ {}_H\mathcal{P}_2^{(\alpha)}(x,y;a;\lambda;\mu;\nu) &= -\frac{2}{\lambda+1}x^2 + \left[\frac{2}{\lambda+1} - \frac{2\lambda}{(\lambda+1)^2}\right]x + \left[\frac{4\lambda}{\lambda+1} + \frac{4\lambda^2}{(\lambda+1)^3}\right],\\ {}_H\mathcal{P}_3^{(\alpha)}(x,y;a;\lambda;\mu;\nu) &= \frac{2}{\lambda+1}x^3 - \left[\frac{6}{\lambda+1} + \frac{6\lambda}{(\lambda+1)^2}\right]x^2 - \frac{12\lambda}{(\lambda+1)^2}y - \frac{12\lambda^3}{(\lambda+1)^4}\\ &\quad + \left[\frac{12y}{\lambda+1} + \frac{12\lambda^2}{(\lambda+1)^3} + \frac{6\lambda}{(\lambda+1)^2} + \frac{4}{\lambda+1}\right]x.\end{aligned}$$

Similarly, for any $\lambda \in \mathbb{C} \setminus \{-1, 1\}$, $a = \alpha = \mu = \nu = 1$, the first few degenerate Apostol-type Hermite numbers are given as:

$$\begin{aligned}
{}_H\mathcal{P}_0^{(\alpha)}(a;\lambda;\mu;\nu) &= \frac{2}{\lambda+1},\\
{}_H\mathcal{P}_1^{(\alpha)}(a;\lambda;\mu;\nu) &= \frac{2\lambda}{(\lambda+1)^2},\\
{}_H\mathcal{P}_2^{(\alpha)}(a;\lambda;\mu;\nu) &= \left[\frac{4\lambda}{\lambda+1}+\frac{4\lambda^2}{(\lambda+1)^3}\right],\\
{}_H\mathcal{P}_3^{(\alpha)}(a;\lambda;\mu;\nu) &= -\frac{12\lambda^3}{(\lambda+1)^4}.
\end{aligned}$$

3 Properties of the degenerate Apostol-type Hermite polynomials ${}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)$

In this section, we establish some basic properties for the degenerate Apostol-type Hermite polynomials, considered in the previous section.

Theorem 3.1. *The degenerate Apostol-type Hermite polynomials ${}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)$ of order α and variable x, satisfy the following summation formula:*

$$ {}_H\mathcal{P}_n^{(\alpha+\beta)}(x+y,z+w;a;\lambda;\mu;\nu) = \sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{P}_k^{(\beta)}(y,w;a;\lambda;\mu;\nu)\,{}_H\mathcal{P}_{n-k}^{(\alpha)}(x,z;a;\lambda;\mu;\nu). \tag{23}$$

Proof. Consider

$$\begin{aligned}
\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha+\beta)}(x+y,z+w;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha+\beta}(1+at)^{\frac{x+y}{a}}(1+at^2)^{\frac{z+w}{a}}\\
&= \left(\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,z;a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\beta)}(y,w;a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{P}_{n-k}^{(\alpha)}(x,z;a;\lambda;\mu;\nu)\,{}_H\mathcal{P}_k^{(\beta)}(y,w;a;\lambda;\mu;\nu)\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we get result (23).

Theorem 3.2. *For $n \in \mathbb{N}$, let $\{{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\}_{n\geq 0}$ be the sequence of degenerate Apostol-type Hermite polynomials in the variable x. They satisfy the following relations:*

$$(i)\ {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu) = {}_H\mathcal{P}_n^{(\alpha)}(x+a,y;a;\lambda;\mu;\nu) - an\,{}_H\mathcal{P}_{n-1}^{(\alpha)}(x,y;a;\lambda;\mu;\nu).$$

$$(ii)\ {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu) = {}_H\mathcal{P}_n^{(\alpha)}(x,y+a;a;\lambda;\mu;\nu) - an(n-1)\,{}_H\mathcal{P}_{n-2}^{(\alpha)}(x,y;a;\lambda;\mu;\nu).$$

Proof (i). From generating function (21), we have

$$\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}} = \sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}. \tag{24}$$

$$\begin{aligned}
\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x+a}{a}}(1+at^2)^{\frac{y}{a}} &= (1+at)\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\\
\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x+a,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\\
&\quad +at\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x+a,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} \\ &+\sum_{n=0}^{\infty} n\,{}_H\mathcal{P}_{n-1}^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{at^n}{n!}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x+a,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\big[{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu) \\ &+an{}_H\mathcal{P}_{n-1}^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\big]\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of t^n in both sides of the equation, the result is

$${}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu) = {}_H\mathcal{P}_n^{(\alpha)}(x+a,y;a;\lambda;\mu;\nu)-an{}_H\mathcal{P}_{n-1}^{(\alpha)}(x,y;a;\lambda;\mu;\nu).$$

(ii). From generating function (21), we have

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}} = \sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}. \tag{25}$$

$$\begin{aligned}\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y+a}{a}} &= (1+at^2)\sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} \\ \sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x,y+a;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} \\ &+at^2\sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x,y+a;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} \\ &+\sum_{n=0}^{\infty} n\,{}_H\mathcal{P}_{n-2}^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{an(n-1)t^n}{n!}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x+a,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\big[{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu) \\ &+an(n-1){}_H\mathcal{P}_{n-2}^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\big]\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of t^n in both sides of the equation, the result is

$${}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu) = {}_H\mathcal{P}_n^{(\alpha)}(x+a,y;a;\lambda;\mu;\nu)-an(n-1){}_H\mathcal{P}_{n-2}^{(\alpha)}(x,y;a;\lambda;\mu;\nu).$$

Theorem 3.3. *For $n \in \mathbb{N}$, let $\{_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\}_{n\geq 0}$ be the sequence of degenerate Apostol–type Hermite polynomials in the variable x. They satisfy the following relations:*

$$(i)\ \frac{\partial\, {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)}{\partial x} = \sum_{k=0}^{n-1} n(-1)^k a^k \frac{k!}{k+1}\binom{n-1}{k}{}_H\mathcal{P}_{n-1-k}^{(\alpha)}(x,y;a;\lambda;\mu;\nu).$$

$$(ii)\ \frac{\partial\, {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)}{\partial y} = \sum_{k=0}^{n-k} n(n-1)(-1)^k a^k \frac{2k!}{k+1}\binom{n-2}{2k}{}_H\mathcal{P}_{n-2k-2}^{(\alpha)}(x,y;a;\lambda;\mu;\nu).$$

Proof (i). Partially differentiating (2.1) with respect to x, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\partial}{\partial x}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}\frac{\partial}{\partial x}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}}\\
&= \left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}}\ln(1+at)\frac{1}{a}\\
&= \left(\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}a^{n+1}t^{n+1}\frac{1}{a}\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}{}_H\mathcal{P}_{n-k}^{(\alpha)}(x,y;a;\lambda;\mu;\nu)(-1)^k a^k\binom{n}{k}\frac{k!}{k+1}\frac{t^{n+1}}{n!}.
\end{aligned}$$

Thus, we have

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n-1}{}_H\mathcal{P}_{n-1-k}^{(\alpha)}(x,y;a;\lambda;\mu;\nu)(-1)^k a^k n\binom{n-1}{k}\frac{k!}{k+1}\frac{t^n}{n!}.$$

Comparing the coefficients of t^n in both sides of the equation, the result is

$$\frac{\partial\, {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)}{\partial x} = \sum_{k=0}^{n-1} n(-1)^k a^k \frac{k!}{k+1}\binom{n-1}{k}{}_H\mathcal{P}_{n-1-k}^{(\alpha)}(x,y;a;\lambda;\mu;\nu).$$

(ii). Partially differentiating (2.1) with respect to y, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\partial}{\partial y}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}\frac{\partial}{\partial y}(1+at^2)^{\frac{y}{a}}\\
&= \left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}}\ln(1+at^2)\frac{1}{a}\\
&= \left(\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}a^{n+1}t^{2n+2}\frac{1}{a}\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}{}_H\mathcal{P}_{n-k}^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{(-1)^k}{k+1}a^k\frac{t^{n+k+2}}{(n-k)!}.
\end{aligned}$$

Thus, we have

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n-k}{}_H\mathcal{P}_{n-2-2k}^{(\alpha)}(x,y;a;\lambda;\mu;\nu)(-1)^k a^k n(n-1)\binom{n-2}{2k}\frac{2k!}{k+1}\frac{t^n}{n!}.$$

Comparing the coefficients of t^n in both sides of the equation, the result is

$$\frac{\partial\, {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)}{\partial y} = \sum_{k=0}^{n-k} n(n-1)(-1)^k a^k \frac{2k!}{k+1}\binom{n-2}{2k}{}_H\mathcal{P}_{n-2k-2}^{(\alpha)}(x,y;a;\lambda;\mu;\nu).$$

Theorem 3.4. *For $n \in \mathbb{N}$, let $\{{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\}_{n\geq 0}$ be the sequence of degenerate Apostol-type Hermite polynomials in the variable x. They satisfy the following relation:*

$$\begin{aligned}\sum_{k=0}^{n} {}_H\mathcal{P}_{n-k}^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\,{}_H\mathcal{P}_k^{(\alpha)}(x,y;a;\lambda;\mu;\nu) &= \sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{P}_k^{(\alpha)}(2x,2y;a;\lambda;\mu;\nu) \\ &\quad\times {}_H\mathcal{P}_{n-k}^{(\alpha)}(a;\lambda;\mu;\nu).\end{aligned} \tag{26}$$

Proof. Consider the following expressions:

$$\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}} = \sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}, \tag{27}$$

$$\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}} = \sum_{n=0}^{\infty}{}_H\mathcal{P}_m^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}. \tag{28}$$

From (27) and (28), we have

$$\begin{aligned}\left(\frac{2^\mu t^\nu}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{2\alpha}(1+at)^{\frac{2x}{a}}(1+at^2)^{\frac{2y}{a}} &= \sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} \\ &\quad\times\sum_{n=0}^{\infty}{}_H\mathcal{P}_m^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} \\ \sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(a;\lambda;\mu;\nu)\frac{t^n}{n!}\sum_{n=0}^{\infty}{}_H\mathcal{P}_m^{(\alpha)}(2x,2y;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} \\ &\quad\times\sum_{n=0}^{\infty}{}_H\mathcal{P}_m^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!} \\ \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{P}_{n-k}^{(\alpha)}(a;\lambda;\mu;\nu)\,{}_H\mathcal{P}_k^{(\alpha)}(2x,2ya;\lambda;\mu;\nu)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{P}_{n-k}^{(\alpha)}(x,y;a;\lambda;\mu;\nu) \\ &\quad\times{}_H\mathcal{P}_k^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}.\end{aligned}$$

Hence, we get assertion (26).

Theorem 3.5. *The degenerate Apostol-type Hermite polynomials ${}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)$ of order α and variable x, are defined by the following series expansion:*

$${}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu) = \sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{P}_{n-k}^{(\alpha)}(y;a;\lambda;\mu;\nu)(x|a)_n. \tag{29}$$

Proof. Using equations, (8) and (21) and applying the Cauchy-product rule in the resultant equation, it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{{}_H\mathcal{P}_{n-k}^{(\alpha)}(y;a;\lambda;\mu;\nu)(x|a)_n t^n}{n!} = \sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}. \tag{30}$$

Equating the coefficients of same powers of t in Equation. (30), assertion (29) is proved.

Theorem 3.6. *For $n \in \mathbb{N}_0$, the degenerate Apostol–type Hermite polynomials ${}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)$, are related with the degenerate Apostol–type Hermite–Bernoulli polynomials of the second kind by means of the following identity.*

$$\begin{aligned}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu) &= \frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}\lambda\,{}_H\mathcal{B}_k(x+1;y;a;\lambda)\,{}_H\mathcal{P}_{n+1-k}^{(\alpha)}(a;\lambda;\mu;\nu) \\ &\quad-\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}{}_H\mathcal{B}_k(x;y;a;\lambda)\,{}_H\mathcal{P}_{n+1-k}^{(\alpha)}(a;\lambda;\mu;\nu).\end{aligned} \tag{31}$$

Proof. From generating function (21), we have

$$\begin{aligned}
&\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}}\\
&=\left[\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right]^{\alpha}\frac{1}{\log(1+at)^{\frac{1}{a}}}\left[\frac{\lambda\log(1+at)^{\frac{1}{a}}}{\lambda(1+at)^{\frac{1}{a}}-1}\right](1+at)^{\frac{x+1}{a}}(1+at^2)^{\frac{y}{a}}\\
&\quad-\left[\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right]^{\alpha}\frac{1}{\log(1+at)^{\frac{1}{a}}}\left[\frac{\log(1+at)^{\frac{1}{a}}}{\lambda(1+at)^{\frac{1}{a}}-1}\right](1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}}.
\end{aligned} \tag{32}$$

Taking into account the generating functions (15), (21) and (22) in Equation. (32), it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{z^n}{n!} &= \frac{\lambda}{z}\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(a;\lambda;\mu;\nu)\frac{z^n}{n!}\sum_{n=0}^{\infty}{}_H\mathcal{B}_k(x+1;y;a;\lambda)\frac{z^n}{n!}\\
&\quad-\frac{1}{z}\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(a;\lambda;\mu;\nu)\frac{z^n}{n!}\sum_{n=0}^{\infty}{}_H\mathcal{B}_k(x;y;a;\lambda)\frac{z^n}{n!}.
\end{aligned}$$

Using the Cauchy product rule in the right hand side of the above equation and, then, equating the coefficients of identical powers of t in both sides of resultant equation, assertion (31) is proved.

Theorem 3.7. *For $n\in\mathbb{N}_0$, the degenerate Apostol-type Hermite polynomials ${}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)$, are related with the degenerate Apostol–type Hermite–Euler Polynomials by means of the following identity:*

$$\begin{aligned}
{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu) &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\lambda{}_H\mathcal{E}_k(x+1;y;a;\lambda){}_H\mathcal{P}_{n-k}^{(\alpha)}(a;\lambda;\mu;\nu)\\
&\quad-\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{E}_k(x;y;a;\lambda){}_H\mathcal{P}_{n-k}^{(\alpha)}(a;\lambda;\mu;\nu).
\end{aligned} \tag{33}$$

Proof. Following the same line of proof as in Theorem 3.7, and using of equations (17), (21) and (22) assertion (33) can be proved.

Theorem 3.8. *The following implicit summation formula involving degenerate Apostol-type Hermite polynomials in the variable x, holds true:*

$${}_H\mathcal{P}_n^{(\alpha)}(x+z,y+w;a;\lambda;\mu;\nu)=\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{P}_{n-k}^{(\alpha)}(x,y;a;\lambda;\mu;\nu)H_k(z,w;a). \tag{34}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x+z,y+w;a;\lambda;\mu;\nu)\frac{t^n}{n!} &= \left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x+z}{a}}(1+at^2)^{\frac{y+w}{a}}\\
&= \left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}}(1+at)^{\frac{z}{a}}(1+at^2)^{\frac{w}{a}}\\
&= \left(\sum_{n=0}^{\infty}{}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}H_n(z,w;a)\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{P}_{n-k}^{(\alpha)}(x,y;a;\lambda;\mu;\nu)H_k(z,w;a)\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of t^n, we get result (34).

4 Conclusions

The paper aims at presenting the study of degenerate Apostol-type Hermite polynomials which plays an important role in several diverse field of physics, applied mathematics and engineering. Certain expressions, representations and summations of these polynomials are derived in terms of well-known classical special functions. The results we have considered in this paper indicate the usefulness of the series rearrangement technique used to deal with the theory of special functions. We have derived several implicit summation formulas for the degenerate Apostol-type Hermite polynomials by using different analytical means on their respective generating functions. This process can be extended to derive new relations for conventional and generalized polynomials.

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