PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 17 · 2024 · Pages 32–39

On fast converging sequences for constants of Bendersky type: Stirling, Glaisher–Kinkelin, Bendersky–Adamchik and some others

André Pierro de Camargo[a]

Communicated by Stefano De Marchi

Abstract

We exhibit some arbitrarily high order converging sequences for Bendersky constants which improves and generalizes the results of [J. Number Theory 133 (2013) pp. 2465–2469]. Our results rely on the Euler-Maclaurin formula and where already known to Benderski (1933) in some extent, but it apparently remained little known since. We also explain how to accelerate the convergence of the classical converging sequences by means of Richardson extrapolation.

1 Introduction

Let $B_0, B_1, B_2, \ldots$ and $H_1, H_2, H_3, \ldots$, denote the Bernoulli and the harmonic numbers, respectively

$$B_0 = 1 \text{ and } \sum_{\ell=0}^{j} B_\ell \begin{pmatrix} j+1 \\ \ell \end{pmatrix} = 0, \; j \geq 1, \quad H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{k}, \; k \geq 1. \tag{1}$$

In 1933, Bendersky [2] studied the limits

$$\log(A_j) := \lim_{n\to\infty} w_{j,n}, \; j \geq 0, \tag{2}$$

$$\begin{aligned} w_{j,n} &= \sum_{k=1}^{n} k^j \log(k) - \left(\frac{n^{j+1}}{j+1} + \frac{1}{2} n^j + \sum_{q=2}^{j+1} \frac{(j)!}{(j-[q-1])!q!} B_q n^{j-[q-1]} \right) \log(n) \\ &+ \frac{n^{j+1}}{(j+1)^2} - \sum_{q=2}^{j} \frac{(j)!}{(j-[q-1])!q!} B_q \left(H_j - H_{j+1-q} \right) n^{j-[q-1]}. \end{aligned} \tag{3}$$

The number $A_0 = \sqrt{2\pi}$ is the Stirling constant, A_1 is known as the Glaisher–Kinkelin constant and the numbers A_2 and A_3 are usually called the Bendersky–Adamchik constants [15].

It was known at least since Hardy [9, pp. 332-333] that the extension $\zeta(-s)$ of the Riemann zeta function for negative real part, i. e., $\Re e(s) > 0$, can be defined in terms of the constant term in the asymptotic expansion (in n) of

$$\sum_{k=1}^{n} k^s$$

(see also [14]). In the very same vein, one might expect that $\zeta'(-j)$ can be somehow linked to $\log(A_j)$. For instance, Choi and Srivastava [6] obtained

$$\log(A_2) = -\zeta'(-2), \quad \log(A_3) = -\zeta'(-3) - \frac{11}{720}, \tag{4}$$

and concluded that

$$\log(A_2) = \frac{\zeta(3)}{4\pi^2}. \tag{5}$$

[a]Centro de Matemática, Computação e Cognição; Universidade Federal do ABC - UFABC; Rua Santa Adélia, 166, bairro Bangu, cep 09210-170 ; Santo André, SP - Brasil (Email: andrecamargo.math@gmail.com).

Actually (4) was obtained previously by Adamchik [1] in the more general form:

$$\log(A_j) = \frac{B_{j+1}H_j}{j+1} - \zeta'(-j),\ j \geq 0. \tag{6}$$

It is worth noting that these results, and even the following generalized form of (5), were known much earlier by Ramanujan, [3] pp. 273–276:

$$\log(A_{2j}) = \frac{(-1)^{j+1}(2j)!}{2(2\pi)^{2j}}\zeta(2j+1),\ j \geq 1. \tag{7}$$

For instance, using that [7]

$$\zeta'(-2k) = \frac{(-1)^k(2k)!}{2(2\pi)^{2k}}\zeta(2k+1),\ k \geq 1, \tag{8}$$

one obtains (6) by (7) in the even case.

Note that (6) allows for numerical approximation of $\zeta'(j)$ in terms of $w_{j,n}$. In this direction, it may be of interest to access the quality of the approximation $w_{j,n} \approx \log(A_j)$. As a closely related topic, we could not leave to mention that Odlysco and te Riele used numerical approximations to compute the first 2000 zeros of the Riemann zeta function in the critical strip in their disproof of Merten's conjecture [17]. With respect to more theoretical questions, (7) links $\log(A_{2k})$ to the open problem of asserting the irrationality of $\zeta(2k+1)$ for $k > 1$ (see [19] and the references therein). Apéry succeeded in proving that $\zeta(3)$ is irrational by constructing a fast convergent sequence of rational numbers to $\zeta(3)$. In this direction, maybe the convergence structure of the sequences $w_{j,n}$ could be of some help in this regard (even though $w_{j,n}$ is formed mainly by irrational numbers).

Regarding the convergence rate of $w_{j,n}$, Mortici [15] analyzed the convergence rate of the sequences

$$\begin{aligned} w_{1,n} &= \sum_{k=1}^{n} k\log(k) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\log(n) + \frac{n^2}{4}, \\ w_{2,n} &= \sum_{k=1}^{n} k^2\log(k) - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\log(n) + \frac{n^3}{9} - \frac{n}{12}, \\ w_{3,n} &= \sum_{k=1}^{n} k^3\log(k) - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right)\log(n) + \frac{n^4}{16} - \frac{n^2}{12}. \end{aligned} \tag{9}$$

He showed that, for $n \geq 1$,

$$\begin{aligned} w_{1,n} &= \log(A_1) + \frac{1}{720n^2} - \frac{1}{5040n^4} + \theta_{1,n},\ 0 < \theta_{1,n} < \frac{1}{10080n^6}, \\ w_{2,n} &= \log(A_2) - \frac{1}{360n} + \theta_{2,n},\ 0 < \theta_{2,n} < \frac{1}{7560n^3}, \\ w_{3,n} &= \log(A_3) - \frac{1}{5040n^2} + \theta_{3,n},\ 0 < \theta_{3,n} < \frac{1}{33600n^4}. \end{aligned} \tag{10}$$

Different types of approximations for $\log(A_1), \log(A_2)$ and $\log(A_3)$ were subsequently considered in [12, 13], by replacing the n in the term $\log(n)$ in (9) by some series $n + \sum_{k\geq 1} \frac{a_{2k+1}}{n^{2k+1}}$, or by some continued fractions. In addition, applying a correction method, You [22] obtained sequences converging to $\log(A_1), \log(A_2)$ and $\log(A_3)$ with error terms $O(n^{-6}), O(n^{-5})$ and $O(n^{-8})$, respectively.

Those familiar with Ramanujan constants of divergent series (in the sense of Hardy [9], pp. 326-327) may realize that arbitrarily higher order estimates than those in (10) can be obtained via the Euler-Maclaurin formula (Benderski himself acknowledge this in his original paper [2], but apparently this information passed unnoticed so far). For instance, we have

$$w_{1,n} = \log(A_1) + \frac{1}{720n^2} - \frac{1}{5040n^4} + \frac{1}{10080n^6} - \frac{1}{9504n^8} + \alpha_{1,6,n}, \tag{11}$$

with

$$0 \leq \alpha_{1,6,n} \leq \frac{691}{3603600\, n^{10}}.$$

Estimate (11) is the particular case $j = 1$ and $r = 6$ of the following more general statement which holds for every $j \geq 0$.

Theorem 1.1. *For any* $2r \geq j+2,\ r \in \mathbb{N}$, *we have*

$$w_{j,n} = \log(A_j) + \sum_{u=1}^{2r-j-2}\left[(-1)^{u-1}B_{u+j+1}\frac{(j)!(u-1)!}{(u+j+1)!}\right]\frac{1}{n^u} + \alpha_{j,r,n}, \tag{12}$$

with

$$0 \;\le\; \alpha_{j,r,n}(-1)^{r-j-1} \;\le\; \frac{|B_{2r}|}{(2r)!}(j)!(2r-1-j-1)!\frac{1}{n^{2r-1-j}}.$$

By Theorem 1.1, for each $j \ge 0$ and $r \ge \frac{j+2}{2}$, the sequence

$$\widetilde{w_{j,r,n}} \;:=\; w_{j,n} - \sum_{u=1}^{2r-j-2}\left[(-1)^{u-1}B_{u+j+1}\frac{(j)!(u-1)!}{(u+j+1)!}\right]\frac{1}{n^u} \tag{13}$$

converges to $\log(A_j)$ with error $O(n^{2r-1-j})$. Therefore, arbitrarily high order converging sequences for $\log(A_j)$ can be obtained by (13) by choosing appropriate values of r (like the one in (11)).

Remark 1. The statement of Theorem 1.1 could be simplified using that

$$B_{2k+1} = 0 \;\forall\; k \ge 1. \tag{14}$$

However, we prefer to keep it as is because it simplifies the notation.

Remark 2. For $j=0$, (12) and (14) give

$$\begin{aligned} w_{0,n} &= \sum_{k=1}^{n}\log(k) - \left(n+\frac{1}{2}\right)\log(n) + n \\ &= \log(A_0) \;+\; \sum_{q=1}^{r-1}\left[\frac{B_{2q}}{(2q-1)(2q)}\right]\frac{1}{n^{2q-1}} \;+\; O\left(\frac{1}{n^{2r-1}}\right). \end{aligned}$$

This is the Stirling series for the factorial function [16]: $\log(n)! =$

$$\left(n+\frac{1}{2}\right)\log(n) - n + \sqrt{2\pi} \;+\; \sum_{q=1}^{r-1}\left[\frac{B_{2q}}{(2q-1)(2q)}\right]\frac{1}{n^{2q-1}}$$
$$+ \quad O\left(\tfrac{1}{n^{2r-1}}\right).$$

It is worth noting that, once we know the general form of Theorem 1.1, the convergence of $w_{j,n}$ for $\log(A_j)$, for a given $j \ge 0$, can be accelerated by means of Richardson extrapolation [4, 10]. For instance, writing the first relation of (10) for n and $2n$, we get

$$\begin{cases} w_{1,n} &= \log(A_1) + \dfrac{1}{720n^2} - \dfrac{1}{5040n^4} + O(1/n^6), \\ w_{1,2n} &= \log(A_1) + \dfrac{1}{4}\dfrac{1}{720n^2} - \dfrac{1}{16}\dfrac{1}{5040n^4} + O(1/n^6). \end{cases} \tag{15}$$

Hence,

$$w^{(1)}_{1,n} \;:=\; \frac{4w_{j,2n} - w_{1,n}}{3} \;=\; \log(A_1) + \frac{1}{4}\frac{1}{5040n^4} + O(1/n^6), \tag{16}$$

If we repeat the same procedure above but now with $w'_{1,n}$ in the place of $w_{1,n}$, we shall conclude that there are constants α_0, α_1 and α_2 such that

$$w^{(2)}_{j,n} := \alpha_0 w_{j,n} + \alpha_1 w_{j,2n} \;+\; \alpha_2 w_{j,4n} \;=\; \log(A_j) + O\left(1/n^6\right).$$

By Theorem 1.1, there are still high order (in $1/n$) error terms well defined for $w^{(2)}_{j,n}$, that is

$$w^{(2)}_{j,n} \;=\; \log(A_j) + \frac{c_6}{n^6} \;+\; \frac{c_8}{n^8} \;+\; \frac{c_{10}}{n^{10}} \;+\; O(1/n^{12}),$$

for some constants c_8, c_{10}, c_{12}. Hence, one can continue the process of eliminating the lower error terms in the expansion of $w^{(2)}_{j,n}$. Formally, we have

Theorem 1.2. *If j is odd, for any $k \ge 0$ fixed, there are unique constants $\alpha_0, \alpha_1, \ldots, \alpha_k$ such that*

$$w^{(k)}_{j,n} := \sum_{i=0}^{k}\alpha_i w_{j,2^i n} \;=\; \log(A_j) + O\left(\frac{1}{n^{2k+2}}\right).$$

The constants $\alpha_0, \alpha_1, \ldots, \alpha_k$ are given explicitly by

$$\alpha_i = (-1)^k \prod_{\substack{\ell=0 \\ \ell \neq i}}^{k} \frac{1/(2^\ell)^2}{1/(2^i)^2 - 1/(2^\ell)^2}, \quad i = 0, 1, \ldots, k,$$

and the underlying constant in the "big O" notation depends only on k*.*

Similar processes can also be applied for j even.

2 Proofs

2.1 Proof of Theorem 1.1

Our proof is based upon the Euler–Maclaurin formula ([18], chap. 2 and [20], p. 135): if $f : [0, \infty] \to \mathbb{R}$ has $2r+2$ continuous derivatives,

$$\begin{aligned} \sum_{k=\tau}^{n-1} f(k) &= \int_\tau^n f(x)dx + \sum_{q=1}^{r} \frac{B_q}{(q)!}(f^{(q-1)}(n) - f^{(q-1)}(\tau)) \\ &+ R(f, r, \tau, n), \end{aligned} \tag{17}$$

where $f^{(1)}, f^{(2)}, \ldots$ are the derivatives of f, $B_0, B_1, B_2, \ldots$ are the Bernoulli numbers (1) and the remainder term can be expressed in terms of the r-th derivative of f and the Bernoulli polynomial $B_r(x)$:

$$\begin{aligned} R(f, r, \tau, n) &= (-1)^{r+1} \int_\tau^n \frac{B_r(x - \lfloor x \rfloor)}{(r)!} f^{(r)}(x)dx \\ &= -(n-\tau)\frac{B_{2\mu+2}}{(2\mu+2)!} f^{(2\mu+2)}(\xi), \ \tau < \xi < n, \end{aligned} \tag{18}$$

$\mu = \lfloor r/2 \rfloor$.

Let

$$g_j(x) = x^j \log(x), \ \text{and} \ G_j \frac{x^{j+1}}{j+1} \log(x) - \frac{x^{j+1}}{(j+1)^2}. \tag{19}$$

By induction on ℓ, one can easily prove that

$$g_j^{(\ell)}(x) = \begin{cases} \dfrac{(j)!}{(j-\ell)!} x^{j-\ell} \log(x) + \dfrac{(j)!}{(j-\ell)!} \left(\displaystyle\sum_{\nu=0}^{\ell-1} \frac{1}{j-\nu} \right) x^{j-\ell} & , \ell \leq j \\ (j)!(-1)^{\ell-j-1}(\ell-j-1)! \dfrac{1}{x^{\ell-j}} & , \ell > j. \end{cases} \tag{20}$$

Because $\int_1^n g_j^{(s)}(x)dx$ converges absolutely for $s > j$, for $n \to \infty$, and because $B_r(x - \lfloor x \rfloor)$ is $O(1)$, the first equality in (18) tells us that the improper integral

$$\int_1^\infty \frac{B_s(x - \lfloor x \rfloor)}{(s)!} g_j^{(s)}(x)dx, s > j, \tag{21}$$

is well defined. Hence, we can rewrite the Euler–Maclaurin formula (17) for g_j as

$$w_{j,n} = c_{j,s} + d_{j,n,s}, \tag{22}$$

with

$$
\begin{aligned}
w_{j,n} &:= \sum_{k=1}^{n-1} g_j(k) - G(n) - \sum_{q=1}^{j+1} \frac{B_q}{(q)!} g_j^{(q-1)}(n) + \frac{B_{j+1}}{(j+1)!} g_j^{(j)}(1), \\
c_{j,s} &:= -G(1) - \sum_{\substack{q=1 \\ q \neq j+1}}^{s} \frac{B_q}{(q)!} g_j^{(q-1)}(1) + (-1)^{s+1} \int_1^\infty \frac{B_s(x - \lfloor x \rfloor)}{(s)!} g_j^{(s)}(x) dx \\
d_{j,n,s} &:= -\sum_{q=j+2}^{s} \frac{B_q}{(q)!} g_j^{(q-1)}(n) + (-1)^{s+1} \int_n^\infty \frac{B_s(x - \lfloor x \rfloor)}{(s)!} g_j^{(s)}(x) dx.
\end{aligned}
\tag{23}
$$

Because $\lim_{n\to\infty} g_j^{(q-1)}(n) = 0$ for $q \geq j+2$, (22) tell us that

$$
c_{j,s} = \lim_{n\to\infty} w_{j,n} \tag{24}
$$

actually does not depend on s for $s \geq j+2$.

By (20), we obtain $w_{j,n} =$

$$
\begin{aligned}
& \sum_{k=1}^{n} k^j \log(k) - \frac{n^{j+1}}{j+1} \log(n) + \frac{n^{j+1}}{(j+1)^2} - \frac{1}{2} n^j \log(n) \\
- & \sum_{q=2}^{j+1} \frac{B_q}{(q)!} \left[\frac{(j)!}{(j-[q-1])!} n^{j-[q-1]} \log(n) + \frac{(j)!}{(j-[q-1])!} \left(\sum_{v=0}^{[q-2]} \frac{1}{j-v} \right) n^{j-[q-1]} \right] \\
+ & (j)! \left(\sum_{v=0}^{j-1} \frac{1}{j-v} \right) \\
= & \sum_{k=1}^{n} k^j \log(k) - \left(\frac{n^{j+1}}{j+1} + \frac{1}{2} n^j + \sum_{q=2}^{j+1} \frac{(j)!}{(j-[q-1])!q!} B_q n^{j-[q-1]} \right) \log(n) \\
+ & \frac{n^{j+1}}{(j+1)^2} - \sum_{q=2}^{j} \frac{(j)!}{(j-[q-1])!q!} B_q \left(H_j - H_{j-q+1} \right) n^{j-[q-1]}.
\end{aligned}
\tag{25}
$$

In addition, let A_j be defined by

$$
\log(A_j) = \lim_{n\to\infty} w_{j,n}. \tag{26}
$$

By (22), (24) and (26), we get

$$
\begin{aligned}
w_{j,n} &= \log(A_j) + \sum_{q=j+2}^{s} \frac{B_q}{(q)!} g_j^{(q-1)}(n) + (-1)^s \int_n^\infty \frac{B_s(x - \lfloor x \rfloor)}{(s)!} g_j^{(s)}(x) dx \\
&\overset{(20)}{=} \log(A_j) + \sum_{q=j+2}^{s} \frac{B_q}{(q)!} \left[(j)!(-1)^{[q-1]-j-1}([q-1]-j-1)! \frac{1}{n^{[q-1]-j}} \right] \\
&+ (-1)^s \int_n^\infty \frac{B_s(x - \lfloor x \rfloor)}{(s)!} \left[(j)!(-1)^{s-j-1}(s-j-1)! \frac{1}{x^{s-j}} \right] dx \\
&= \log(A_j) + \sum_{u=1}^{s-j-1} \left[(-1)^{u-1} B_{u+j+1} \frac{(j)!(u-1)!}{(u+j+1)!} \right] \frac{1}{n^u} \\
&+ (-1)^s \int_n^\infty \frac{B_s(x - \lfloor x \rfloor)}{(s)!} g_j^{(s)}(x) dx.
\end{aligned}
\tag{27}
$$

For s even, we have $B_s(t) \leq B_s$ for $t \in [0,1]$ (see [11]). Hence, for $s = 2r$, (27) gives

$$\begin{aligned} w_{j,n} &= \log(A_j) + \sum_{u=1}^{2r-j-2} \left[(-1)^{u-1} B_{u+j+1} \frac{(j)!(u-1)!}{(u+j+1)!} \right] \frac{1}{n^u} \\ &+ \alpha_{j,r,n}, \end{aligned} \tag{28}$$

with

$$\alpha_{j,r,n} = \overbrace{(-1)^{2r-j-2} B_{2r} \frac{(j)!(2r-j-2)!}{(2r)!} \frac{1}{n^{2r-j-1}}}^{\beta_{j,r,n}} + \gamma_{j,r,n} \tag{29}$$

and

$$\begin{aligned} |\gamma_{j,r,n}| &\leq \frac{|B_{2r}|}{(2r)!} \int_n^\infty |g_j^{(2r)}(x)|\, dx \\ &\overset{(20)}{=} \frac{|B_{2r}|}{(2r)!} (j)!(2r-j-2)! \frac{1}{n^{2r-1-j}}. \end{aligned}$$

Because

$$\gamma_{j,r,n} = \lim_{m\to\infty} (-1)^{2r+1} \int_n^m \frac{B_{2r}(x - \lfloor x \rfloor)}{(2r)!} g_j^{(2r)}(x) dx,$$

and because the even Bernoulli numbers alternate in sign

$$|B_{2\ell}| = (-1)^{\ell+1} B_{2\ell}, \ell \geq 1, \tag{30}$$

the second equality of (18) gives us the sign of $\gamma_{j,n}$:

$$\gamma_{j,r,n} (-1) \overbrace{(-1)^r}^{\text{sign of} B_{2r+2}} \overbrace{(-1)^{2r+2-j-1}}^{\text{sign of } g_j^{(2r+2)}} = \gamma_{j,r,n} (-1)^{r-j-2} \geq 0.$$

In the same fashion, the sign of $\beta_{j,r,n}$ in (29) is $(-1)^{r-j-1}$. Hence, $\beta_{j,r,n}$ and $\gamma_{j,r,n}$ have opposite sign and, because $|\gamma_{j,r,n}| \leq |\beta_{j,r,n}|$, we can write

$$0 \leq \alpha_{j,r,n}(-1)^{r-j-1} \leq |B_{2r}| \frac{(j)!(2r-j-2)!}{(2r)!} \frac{1}{n^{2r-j-1}}.$$

The proof is complete in view of (25) and (28): $w_{j,n} =$

$$\begin{aligned} &\sum_{k=1}^n k^j \log(k) - \left(\frac{n^{j+1}}{j+1} + \frac{1}{2} n^j + \sum_{q=2}^{j+1} \frac{(j)!}{(j-[q-1])!q!} B_q n^{j-[q-1]} \right) \log(n) \\ + &\frac{n^{j+1}}{(j+1)^2} - \sum_{q=2}^{j} \frac{(j)!}{(j-[q-1])!q!} B_q \left(H_j - H_{j-q+1} \right) n^{j-[q-1]} \end{aligned}$$

and

$$\begin{aligned} w_{j,n} &= \log(A_j) + \sum_{u=1}^{2r-j-2} \left[(-1)^{u-1} B_{u+j+1} \frac{(j)!(u-1)!}{(u+j+1)!} \right] \frac{1}{n^u} \\ &+ \alpha_{j,r,n}. \end{aligned}$$

□

2.2 Proof of Theorem 1.2

The iterative process for the construction of $w_{j,n}^{(k)}$, commonly known as Richardson extrapolation [4, 10], can be easily understood by means of the theory of Lagrange interpolation [5]. Let $p(x,\mathbf{x},\mathbf{y})$ denote the polynomial of degree at most k that interpolates a given real vector $\mathbf{y}=(y_0,y_1,\dots,y_k)$ at the real nodes $\mathbf{x}=(x_0,x_1,\dots,x_k)$, that is

$$p(x_i,\mathbf{x},\mathbf{y}) \;=\; y_i, i=0,1,\dots,k. \tag{31}$$

Explicitly, [8, p. 74],

$$p(x,\mathbf{x},\mathbf{y})=\sum_{i=0}^{k} y_i \prod_{\substack{\ell=0\\ \ell\neq i}}^{k} \frac{x-x_\ell}{x_i-x_\ell}. \tag{32}$$

Using that the odd Bernoulli numbers $B_{2\ell+1}$ are vanishing for $\ell \geq 1$, we can rewrite identity (12), for j odd, as

$$\log(A_j) \;+\; c_2 n^{-2} \;+\; c_4 n^{-4} \;+\; \dots \;+\; c_k n^{-2k} \;=\; w_{j,n} \;+\; O(n^{-2k-2}), \tag{33}$$

where $c_2, c_4, \dots c_{2k}$ do not depend on n.

Given $k \geq 1$, we can rewrite (33) for $n, 2n, 4n, \dots 2^k n$ as the following system of linear equations

$$\left\{\begin{array}{lllll} \log(A_j)+ & c_2 n^{-2}+ & \dots + c_k n^{-2k} & = & w_{j,n}+O_0(n^{-2k-2}) \\ \log(A_j)+ & c_2(2n)^{-2}+ & \dots + c_k(2n)^{-2k} & = & w_{j,2n}+O_1(n^{-2k-2}) \\ \log(A_j)+ & c_2(4n)^{-2}+ & \dots + c_k(4n)^{-2k} & = & w_{j,4n}+O_2(n^{-2k-2}) \\ \vdots & & \vdots & \vdots & \vdots \\ \log(A_j)+ & c_2(2^k n)^{-2}+ & \dots + c_k(2^k n)^{-2k} & = & w_{j,2^k n}+O_k(n^{-2k-2}). \end{array}\right. \tag{34}$$

By (32) and (34), we have

$$\log(A_j) \;+\; c_2 x \;+\; \dots \;+\; c_k x^k \;=\; p(x,\mathbf{x_k},\mathbf{w})+p(x,\mathbf{x_k},\mathbf{E}), \tag{35}$$

where

$$\left\{\begin{array}{lll} \mathbf{x_k} & = & \left(1/n^2, 1/(2n)^2, \dots, 1/(2^k n)^2\right), \\ \mathbf{w} & = & (w_{j,n}, w_{j,2n}, \dots, w_{j,2^k n}), \\ \mathbf{E} & = & \left(O_0(n^{-2k-2}), O_1(n^{-2k-2}), \dots, O_k(n^{-2k-2})\right). \end{array}\right.$$

For $x \;=\; 0$, (32) and (36) give

$$\log(A_j) \;=\; p(0,\mathbf{x_k},\mathbf{w})+p(0,\mathbf{x_k},\mathbf{E}), \tag{36}$$

with

$$\begin{aligned} p(0,\mathbf{x}^*,\mathbf{w}) &= \sum_{i=0}^{k} w_{j,2^i n} \prod_{\substack{\ell=0\\ \ell\neq i}}^{k} \frac{-1/(2^\ell n)^2}{1/(2^i n)^2-1/(2^\ell n)^2} \\ &= \sum_{i=0}^{k} w_{j,2^i n} \prod_{\substack{\ell=0\\ \ell\neq i}}^{k} \frac{-1/(2^\ell)^2}{1/(2^i)^2-1/(2^\ell)^2} \end{aligned}$$

and

$$p(0,\mathbf{x}^*,\mathbf{E}) \;=\; \sum_{i=0}^{k} O_i(n^{-2k-2}) \prod_{\substack{\ell=0\\ \ell\neq i}}^{k} \frac{-1/(2^\ell)^2}{1/(2^i)^2-1/(2^\ell)^2}.$$

□

Conflict of interest

The author declares no competing interests.

Availability of data

Not applicable.

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