

## Positivity-preserving and elementary stable nonstandard method for a COVID-19 SIR model

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### Abstract

The main purpose of this work is to build a numerical method for solving an epidemiological model that describes the spread of COVID-19 in some countries. The method is constructed using a NonStandard Finite Difference (NSFD) discretization for the analyzed model, in order to preserve its positivity and equilibrium points properties. Numerical simulations testify the best performance of the proposed scheme with respect to the related Standard Finite Difference (SFD) method, the famous explicit four-stage order-four Runge-Kutta known as RK4, and another positivity-preserving nonstandard method.

### 1 Introduction

Physical situations that evolve in space and time are often described with appropriate differential equations. Solving them exactly is not always possible, and therefore it is necessary to use numerical methods [34]. Despite this, it may happen that some qualitative characteristics of the model are a-priori known. For example, the model can represent an epidemiological phenomenon, and therefore its exact solution is non-negative over the entire integration interval. Furthermore, if the model refers to a periodic chemical reaction [8, 9], the exact solution is known to oscillate with some frequency [29]. Also, the equilibrium points of the differential system to be solved can be a-priori known, as well as conservation laws [11, 20, 21] and behavior of the solution subjected to perturbations [15]. Therefore, when selecting the numerical method, it is necessary to pay attention to its stability properties. Usually, one only looks at the linear stability [34] properties of a numerical method. However, more generally, given a property  $P$  of the considered model, we say that the numerical method used for its solution is  $P$ -stable if it is able to preserve  $P$  for each choice of the discretization steps [2]. In summary, when it is not possible to calculate the exact solution of differential equations, it is important to select a numerical method that provides a solution with the same properties as the exact one. In this context, very useful approaches consist in the use of NonStandard Finite Difference (NSFD) numerical methods and geometric numerical integrators. Regarding the geometric numerical integration, there are several papers in the scientific literature that lead to the construction of methods that preserve energy, mass, positivity, and other known characteristics of the considered problem [7, 12, 23]. Concerning the preservation of positivity, interesting articles have been written both in the field of geometric numerical integration and NSFD discretizations [31, 32, 42]. Usually, when a method is required to preserve positivity, it is necessary to pay a price in terms of order of accuracy, which cannot be greater than one. However, recently, some work has been done in which, for some types of problems, this order barrier can be overcome [6, 36].

NSFD methods [2] are a generalization of Standard Finite Difference (SFD) discretizations and have been introduced to avoid numerical instabilities [13]. In fact, they can be constructed to preserve the known properties of the analyzed model. Therefore, the major advantage of these methods is to be *dynamically consistent* with the problem, with respect to one or more properties  $P$ , for all the choices of the discretization steps [40]. By means of classical discretizations, well-known numerical methods are obtained, such as Linear Multistep and Runge-Kutta schemes [34]. They are obtained by approximating the involved derivatives using Taylor series expansions and their combinations. However, in practice, the qualitative properties of the exact solution are not necessarily preserved for each step-size value. Starting from a standard numerical method, it is possible to obtain a relative NSFD scheme, by modifying the first one according to two rules, which will be explained in the next section. These two rules must be applied to improve the stability properties of the initial SFD method. Usually, the starting SFD method is explicit and its modification is performed to preserve this structure. In this way, the computational cost of the NSFD method is the same as that of the SFD one. In the scientific literature about NSFD numerical methods, the preservation of positivity [41] and equilibrium points [2] of the model of interest are mainly investigated. Recently, it has been observed that such methods can also preserve the possible oscillating character of the model [14]. The use of NSFD methods for differential equation models has increased over the years, also because it has been observed that the classic routines available on software (such as the *ode45* and *ode15s* functions of Matlab) can fail in the numerical solution of such problems, not being able to capture their fundamental properties [35].

Currently, given the COVID-19 pandemic that has characterized the entire planet in the last two and a half years, the most fashionable scientific models are the epidemiological ones. Furthermore, other serious respiratory diseases have characterized the world in the past years, such as SARS (Severe Acute Respiratory Syndrome), MERS (Middle East Respiratory Syndrome), Ebola, and others, representing a real challenge for the health system. In this context, modeling [1, 3, 4, 5, 22, 43, 45, 46] proved to be very useful, as thanks to it, it has been possible to predict the progress of epidemics, making sure to take the necessary countermeasures in time, preventing the health systems of the involved nations from collapsing. To derive a model for epidemics,

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the most followed route is that of compartmentalization. This means that the overall population is divided into classes, and the obtained model can tell us how these classes of people vary over space and time. Epidemiological models have also been used in other fields, for example to study the spread of fake news [16, 17, 18]. One of the first and most famous models, introduced by Kermack and McKendrick, is the SIR [30], which consists of a system of three Ordinary Differential Equations (ODEs), where the population is divided into the following categories:  $S(t)$  are healthy individuals at time  $t$ , Susceptible to infection;  $I(t)$  are Ill individuals at time  $t$ , and therefore vehicles of infection;  $R(t)$  are individuals who recovered or died at time  $t$ , therefore they could be considered as Removed. The SIR can then be generalized by adding the dependence on space, or by adding other categories of population, to be more refined. For example, a famous generalization of the SIR model is the SEIR [24, 33], in which the class  $E(t)$  of individuals Exposed to the virus at time  $t$ , but not yet infectious, is added. In this work, we consider a particular version of SIR model, consisting of three ODEs, which describes the spread of the COVID-19 epidemic in Pakistan [19].

The main aim of this paper is to derive a new NSFD scheme for the considered epidemiological model, that has been analyzed in [19], where the authors propose a standard numerical method for its solution. In this work, we demonstrate that our NSFD method is able to preserve the positivity of the exact solution and also the characteristics of the equilibrium points of the model, both for the values of the parameters used in [19] and for others introduced here, thanks to which it is possible to describe the qualitative spread of the COVID-19 epidemic in Bangladesh.

This paper is organized as follows. In Section 2 we recall how to construct NSFD numerical methods. In Section 3 we describe a mathematical model for the spread of COVID-19 in Pakistan proposed in [19]. In Section 4 we derive an NSFD numerical scheme for the model solution, capable of preserving the positivity and equilibrium points properties, also by changing its parameters. In Section 5 we perform numerical simulations to compare the derived NSFD scheme with the related SFD method, the RK4, and another positivity-preserving nonstandard method. In Section 6 we discuss the results obtained in this work.

## 2 Preliminaries

In this section, we briefly recall the properties of NSFD numerical methods and some concepts and definitions that we will use in the paper.

Consider an initial value problem, in hypothesis of existence and unicity of the solution, described by the first-order system of Ordinary Differential Equations (ODEs)

$$\begin{cases} y'(t) = f(t, y), \\ y(t_0) = y_0, \end{cases} \quad t \in I := [t_0, T], \tag{1}$$

where  $y: I \rightarrow \mathbb{R}^m$  and  $f: I \times \mathbb{R} \rightarrow \mathbb{R}^m$ . Then, consider the time discretization  $\{t_k = t_0 + kh, k = 0, \dots, N, t_N = T\}$  of the continuous interval  $[t_0, T]$  related to (1). Let us indicate with  $y_k$  the approximation of the exact solution at the grid point  $t_k$ , i.e.  $y_k \approx y(t_k)$ . A very simple example of SFD discretization for the first-order derivative of the function  $y$  of (1) is

$$y'(t_k) \approx \frac{y_{k+1} - y_k}{h}. \tag{2}$$

Starting from the SFD discretization (2), the NSFD rules [38, 39] allow to modify it as

$$y'(t_k) \approx \frac{y_{k+1} - \psi(h)y_k}{\phi(h)}, \tag{3}$$

where the two functions  $\psi(h)$  and  $\phi(h)$  are non-negative and must satisfy, for  $h \rightarrow 0$ ,

$$\psi(h) = 1 + O(h), \quad \phi(h) = h + O(h^2). \tag{4}$$

They are called *numerator function* and *denominator function*, respectively. The property (4) allows to obtain an NSFD numerical method that is consistent with the continuous problem (1). Furthermore, this property can be generalized to different starting standard discretizations associated with the first-order derivative, and also if higher order derivatives are present (see, e.g., [2, 39]). The functions  $\psi(h)$  and  $\phi(h)$  are usually chosen to improve the stability properties of the starting SFD method, and even to preserve some a-priori known features of the exact solution of the reference problem (1) [37, 41].

The NSFD rules also allow to approximate the function  $f$  of (1) in a different way than the classical case. In fact, non-local representations are granted for the non-linear terms involved in it, such as for example [38, 39]

$$y^2(t_k) \approx y_{k+1}y_k, \quad y^3(t_k) \approx \frac{2y_{k+1}^2y_k^2}{y_{k+1} + y_k}. \tag{5}$$

As before, the non-local representations that are introduced cannot be generic, as attention must be paid to consistency of the NSFD method with the continuous problem (1). As for the first rule, also in this case an accurate choice of the non-local representations can improve the stability properties of the starting SFD method, allowing the nonstandard numerical scheme to preserve a-priori known qualitative features of the model.

In this work, we will exploit both NSFD properties (3) and (5) to construct an explicit nonstandard method for a SIR model for COVID-19. We will prove that this method, in addition to having very good stability properties, is elementary stable and positivity-preserving. We recall below the definitions of positivity of a system of ODEs, and of positivity and elementary stability of a numerical method.

**Definition 2.1.** *Positivity of an initial value problem* [39]. The initial value problem (1) is said to be positive if the exact solution satisfies  $y(t) \geq 0, \forall t \in I$ , starting from non-negative initial conditions.

**Definition 2.2.** *Positivity of a numerical method* [39]. Consider a numerical method for the solution of the initial value problem (1) on the discrete grid  $\{t_k = t_0 + kh, k = 0, \dots, N, t_N = T\}$ . The numerical method is said to be positivity-preserving if the numerical solution satisfies  $y_k \geq 0, \forall k = 1, \dots, N$ , given  $y_0 \geq 0$ .

**Definition 2.3.** *Elementary stability of a numerical method* [2]. A finite difference numerical method is called elementary stable if, for any value of the step-size  $h$ , its only fixed-points  $\tilde{y}_k$  are those of the differential system (1), and if the linear stability properties of each fixed point are the same for both the differential system and the discrete method.

Recall that the equilibrium points, or fixed-points, of autonomous differential equations of the type  $y' = f(y)$  are constant vectors  $\tilde{y}$  satisfying

$$f(\tilde{y}) = 0.$$

A criterion to study the linear stability of the equilibrium points lies in the analysis of the eigenvalues  $\lambda_j$  of the Jacobian matrix  $J$  of  $f$  at  $\tilde{y}$ . A fixed-point is

- linearly stable iff  $|\operatorname{Re}\lambda_j| \leq 1$  for all  $j$ ;
- linearly unstable iff  $|\operatorname{Re}\lambda_j| > 1$  for at least one  $j$ .

A physical interpretation of the linear stability of equilibrium points is given by Lyapunov theory. In fact, a fixed-point  $\tilde{y}$  is said to be

- stable if it holds

$$\forall \varepsilon > 0, \exists \delta > 0: \text{ if } \|y_0 - \tilde{y}\| < \delta \implies \|y(t) - \tilde{y}\| < \varepsilon, \quad \forall t \geq 0;$$

- asymptotically stable if it is stable and it holds

$$\lim_{t \rightarrow \infty} \|y(t) - \tilde{y}\| = 0.$$

In other words, a linearly stable fixed-point attracts, in some neighborhood, other solutions of the differential system as  $t \rightarrow \infty$ . A linearly stable fixed point is stable when setting initial conditions that are close to it leads to a solution that is close to it, and asymptotically stable when setting initial conditions that are close to it leads to a solution that converges to it. Instead, an unstable fixed-point repulses the solutions.

Regarding epidemic models, an alternative way to study the stability of the equilibrium points is given by the reproductive number  $R_0$ , which is a threshold parameter governing the time evolution of a disease. When  $R_0 < 1$ , the epidemic is under control, otherwise it spreads immeasurably.

### 3 Epidemiological model

Over the years, several mathematical models have been constructed to predict the spread of infectious diseases (see, e.g., [1, 5, 25, 44, 46]).

In this section, we consider a modified version of the famous compartmental Kermack-McKendrick SIR model [30], which is based on the subdivision of a population into three classes: Susceptible (to the disease), Infectious, and Recovered people. Specifically, in this paper we analyze the following system of three coupled non-linear differential equations [19]:

$$\frac{dS(t)}{dt} = b - k(1 - \alpha S(t)I(t)) - \alpha k \beta S(t)I(t) - \mu S(t), \tag{6a}$$

$$\frac{dI(t)}{dt} = k(1 - \alpha S(t)I(t)) + \alpha k \beta S(t)I(t) - (d_0 + \gamma + \mu)I(t), \tag{6b}$$

$$\frac{dR(t)}{dt} = \gamma I(t) - \mu R(t). \tag{6c}$$

Here,  $S(t), I(t)$  and  $R(t)$  represent the number in million of Susceptible, Infectious, and Removed people of the population on day  $t$ , respectively. All the parameters involved are positive and their meaning is described in Table 1. There, we also report the values of the parameters used in [19] on the left, and those we propose in this paper on the right, thanks to which the qualitative trend of the COVID-19 epidemic in other countries, such as Bangladesh in the period May-December 2021 [26], can be reproduced. The total population, denoted by  $N(t)$ , is given by the sum  $S(t) + I(t) + R(t)$ . Note that adding the equations (6a), (6b) and (6c) gives the conservation law [42]

$$\frac{dN(t)}{dt} = b - \mu N(t) - d_0 I(t), \tag{7}$$

which describes how the total population evolves over time. It does not always happen that the sum of the components of a system of ODEs gives a conservation law, but in this case it is necessary to take this into account in the construction of the method. In fact,  $N(t)$  has a precise physical meaning, representing the total population at time  $t$ . Generally, compartmental models such as the one analyzed here give rise to conservation laws of this type.

In [19], the authors prove the positivity of the model (6) and study its equilibrium points, setting parameters values for which it is possible to reproduce the trend of the COVID-19 epidemic in Pakistan. Furthermore, they numerically solve the model using explicit Euler method, which is the simplest and most well-known SFD-based one-step numerical scheme for initial value problems.

**Table 1:** Physical description of the parameters of the SIR model (6), and relative values chosen in [19] on the left, and those used in this paper on the right.

Parameters	Physical descriptions	Pakistan	Bangladesh
$d_0$	Death due to COVID-19	0.02	0.015
$\mu$	Natural death	0.0062	0.0054
$b$	Birth rate	10.7	0.018
$\beta$	Protection rate	0.009	0.009
$k$	Constant rate	0.00761	0.00761
$\alpha$	Isolation rate	0.009	0.009
$\gamma$	Recovery rate	0.0003	0.9

In this work we show that by choosing other parameters values for the SIR model (6), i.e. the ones on the right of Table 1, some explicit classical SFD methods, such as Euler method, but also the famous RK4 [28], are inefficient (they are not stable and do not retain positivity and equilibrium points properties for some values of  $h$ ). Then, starting from the explicit Euler method, we construct an NSFD numerical scheme which is still explicit and is also endowed with very good stability properties.

The ODEs system (6) is characterized by the following properties (see [19]).

- **Positivity condition.** The exact solution  $y(t) = (S(t), E(t), I(t))$  is positive over the interval  $[0, \infty)$ , starting from non-negative initial conditions;
- **Equilibrium points.** For some values of the system parameters, such as those proposed in [19] and also the ones used there, good approximations of the equilibrium points are the following:
  - Disease-Free Equilibrium (DFE), denoted by  $E_0$ ,

$$E_0 = (S^0, 0, 0) = \left(\frac{b}{\mu}, 0, 0\right); \tag{8}$$

- Endemic Equilibrium (EE), denoted with  $E^* = (S^*, I^*, R^*)$ ,

$$S^* = \frac{b - (\mu + d_0 + \gamma)I^*}{\mu}, \tag{9a}$$

$$I^* = \frac{k\mu}{k\alpha(1 - \beta)(\mu + d_0 + \gamma - b)I^* + \mu(\mu + d_0 + \gamma)}, \tag{9b}$$

$$R^* = \frac{\gamma}{\mu}I^*. \tag{9c}$$

The DFE and EE points are linearly stable, and in particular they result asymptotically stable [19] for both the pairs of parameters in Table 1. Furthermore, the reproductive number of (6) is given by (see [19])

$$R_0 = \frac{k\alpha(1 - \beta)b}{\mu^2}.$$

For both the pairs of parameters reported in Table 1, it easy to note that it holds  $R_0 < 1$ .

### 4 Construction of the nonstandard scheme and related properties

In this section, we derive an NSFD method to solve the ODEs system (6), proving its positivity and elementary stability.

Note that the presence of the conservation law (7) imposes constraints in the construction of an NSFD scheme. First of all, the denominator function has to be the same for all the involved equations. Secondly, if some terms appear in more than one equation, the discretizations used for them should be the same. Considering also the positivity and equilibrium points properties of the SIR model (6), we propose the following nonstandard method for its solution:

$$\frac{S_{k+1} - S_k}{\phi(h)} = b - k + k\alpha S_k I_k - \alpha k \beta S_{k+1} I_k - \mu S_{k+1}, \tag{10a}$$

$$\frac{I_{k+1} - I_k}{\phi(h)} = k - k\alpha S_k I_k + \alpha k \beta S_{k+1} I_k - (d_0 + \gamma + \mu)I_{k+1}, \tag{10b}$$

$$\frac{R_{k+1} - R_k}{\phi(h)} = \gamma I_{k+1} - \mu R_{k+1}. \tag{10c}$$

We take the following denominator function  $\phi(h)$ :

$$\phi(h) = \frac{1 - e^{-akh}}{\alpha k}. \tag{11}$$

Note that the function  $\phi(h)$  thus defined respects the NSFD rules recalled in Section 2, since it is non-negative and satisfies

$$\phi(h) = h - \frac{h^2}{2}(\alpha k) + O(h^3), \quad h \rightarrow 0.$$

Below, we prove that the NSFD method (10) respects the conservation law (7), also preserving the positivity condition and the equilibrium points properties. Firstly, note that the method (10) can be formulated as

$$S_{k+1} = \frac{\phi(b-k) + (1 + \phi k \alpha I_k) S_k}{1 + \phi \mu + \phi \alpha k \beta I_k}, \tag{12a}$$

$$I_{k+1} = \frac{\phi k + (1 - \phi k \alpha S_k) I_k + \phi \alpha k \beta S_{k+1} I_k}{1 + \phi(d_0 + \gamma + \mu)}, \tag{12b}$$

$$R_{k+1} = \frac{R_k + \phi \gamma I_{k+1}}{1 + \phi \mu}. \tag{12c}$$

Hence, the proposed NSFD scheme is explicit and has the same cost as the related standard scheme, i.e. explicit Euler method.

**Proposition 4.1.** *The NSFD scheme (10) preserves the conservation law (7).*

*Proof.* Adding the right and left side of the equations (10) gives

$$\frac{N_{k+1} - N_k}{\phi(h)} = b - \mu N_{k+1} - d_0 I_{k+1},$$

which is a nonstandard discretization for (7). □

**Theorem 4.2.** *The NSFD scheme (10) preserves the positivity of the model (6) for each  $h > 0$ , if  $b \geq k$  and  $\beta \geq 1 + \mu/\alpha k$ .*

*Proof.* In order to prove the positivity of the NSFD method (10), we look at the rewriting (12). Since  $S_0, I_0$  and  $R_0$ , representing numbers of individuals in the population, are non-negative values, we proceed by induction assuming  $S_k \geq 0, I_k \geq 0$  and  $R_k \geq 0$ .

Remembering that  $\phi(h)$  is non-negative for each choice of  $h > 0$ , note that to have  $S_{k+1} \geq 0$ , it is sufficient that  $b - k \geq 0$ . This condition is valid for both the pairs of parameters reported in Table 1.

To prove that  $I_{k+1} \geq 0$ , we define the function  $g(I_k) := I_{k+1}$ . Note that if  $g(I_k)$  has only negative critical points, and if it is positive at  $I_k = 0$  and such that  $g(0) \leq g(I_k)$  for at least one value of  $I_k \geq 0$ , it automatically follows that  $I_{k+1} \geq 0$ . Indeed, in this case  $g(I_k)$  would be a non-negative increasing function for  $I_k \geq 0$ . The first-order derivative of  $g(I_k)$  is given by

$$g'(I_k) = \frac{\alpha^2 \beta^2 I_k^2 k^2 \phi^2 + (1 + \mu \phi)^2 - \alpha k \phi (1 + \mu \phi) (S_k + \mu \phi S_k - \beta (2I_k + b \phi - k \phi + S_k))}{(1 + \alpha \beta I_k k \phi + \mu \phi)^2 (1 + (d_0 + \gamma + \mu) \phi)}.$$

Then, the critical points of  $g(I_k)$  are given by

$$I_{k1,2} = \frac{-1 - \mu \phi \pm \sqrt{\alpha k \phi (1 + \mu \phi) (-b \beta \phi + \beta k \phi + S_k - \beta S_k + \mu \phi S_k)}}{\alpha \beta k \phi}. \tag{13}$$

Note that one of the two critical points is always negative (the one with the minus sign in front of the square root). The other critical point is negative iff

$$(1 + \mu \phi) (-1 - \mu \phi + \alpha k \phi (\beta \phi (-b + k) + S_k (1 - \beta + \mu \phi))) \leq 0.$$

This happens for each  $S_k \geq 0$  if  $b \geq k$  and  $\beta \geq 1 + \mu \phi$ . Obviously, since  $\phi$  assumes maximum value at  $1/\alpha k$ , to express the condition  $\beta \geq 1 + \mu \phi$  in such a way that it is independent from  $h$ , we finally assume  $\beta \geq 1 + \mu/\alpha k$ . To conclude this part of the proof, we observe that  $g(0) = (k \phi)/(1 + (d_0 + \gamma + \mu) \phi) \geq 0$  and that  $\lim_{I_k \rightarrow \infty} g(I_k) = \infty \geq g(0)$ .

Finally, since  $I_{k+1}$  is non-negative, it also holds that  $R_{k+1} \geq 0$ . □

Note that for both pairs of parameters of Table 1, the hypothesis  $b \geq k$  is true, while the hypothesis  $\beta \geq 1 + \mu/\alpha k$  is not satisfied. In the next remark we therefore explain why, in any case, our scheme preserves positivity even for very large values of  $h$ , as observed numerically.

*Remark 1.* Let us observe that one of the two critical points (13) of the function  $g$  defined in Theorem 4.2 can be positive, if we do not assume  $\beta \geq 1 + \mu/\alpha k$ . Let us indicate it with  $I_k^*$ . As seen during the proof of the theorem,  $I_k^*$  is a minimum for  $g$ . Therefore, if we prove that  $g(I_k^*) \geq 0$ , automatically it follows that  $g(I_k) \geq 0$  for each  $I_k \geq 0$ . This leads to the positivity of  $I_{k+1}$ , which is what we want to prove. Note that

$$g(I_k^*) = \frac{-1 + 2\sqrt{\alpha k \phi (1 + \mu \phi) (-b \beta \phi + \beta k \phi + S_k - \beta S_k + \mu \phi S_k)} + \phi(-\mu + \alpha \beta \beta k \phi + \alpha k(-1 + \beta - \mu \phi) S_k)}{\alpha \beta k \phi (1 + (d_0 + \gamma + \mu) \phi)}.$$

This quantity is always non-negative if the polynomial  $p(S_k) = c_0 + c_1 S_k + c_2 S_k^2$  satisfies  $p(S_k) \geq 0$ , where

$$\begin{aligned} c_0 &= 1 + \phi(\mu^2 \phi + 2\mu(1 + \alpha \beta k(-3b + 2k)\phi^2) + \alpha \beta k \phi(4k + b(-6 + \alpha \beta \beta k \phi^2))), \\ c_1 &= 2\alpha k \phi(-1 + \beta - \mu \phi)(-3 - 3\mu \phi + \alpha \beta \beta k \phi^2), \\ c_2 &= \alpha^2 k^2 \phi^2 (1 - \beta + \mu \phi)^2. \end{aligned}$$

Then, if the coefficients  $c_i$ ,  $i = 0, 1, 2$ , are non-negative, we get a sufficient condition for the positivity of the method (10), even when the hypothesis  $\beta \geq 1 + \mu/ak$  is not satisfied. By requesting, for example,  $c_i \geq 0$ ,  $i = 0, 1, 2$ , for the parameters reported on the left of Table 1, we get  $0 \leq \phi \leq 261.06$ . By doing the same with the parameters reported on the right of Table 1, we get  $0 \leq \phi \leq 119977.90$ . Let us observe that  $\phi(h) = h\varphi(z)$ , where  $\varphi(z) = (1 - e^{-z})/z$  and  $z = akh$ . Since  $0 \leq \varphi(z) \leq 1$ , we can choose very large values of  $h$  being sure that the nonstandard method (10) still preserves the positivity, for both pairs of parameters in Table 1.

Now, we also prove that the constructed method preserves the properties of the equilibrium points of the model.

**Theorem 4.3.** *The NSFD scheme (10) is elementary stable.*

*Proof.* We have to prove that the equilibrium points  $E_0$  (8) and  $E^*$  (9) are linearly stable for the NSFD method (10), as in the case of the continuous model (6).

The Jacobian  $J = \partial f / \partial y := (j_{n,m})_{n,m=1}^3$  of the NSFD scheme (see the formulation (12)) has the following entrances:

$$\begin{aligned} j_{1,1} &= \frac{1 + \phi k \alpha I_k}{1 + \phi \mu + \phi \alpha k \beta I_k}, & j_{2,1} &= \frac{-\phi k \alpha I_k + \phi k \alpha \beta I_k j_{1,1}}{1 + \phi(d_0 + \gamma + \mu)}, & j_{3,1} &= \frac{\phi \gamma}{1 + \phi \mu} j_{2,1}, \\ j_{1,2} &= \frac{\phi k \alpha (S_k(1 + \phi \mu - \beta) + \phi \beta (k - b))}{(1 + \phi \mu + \phi \alpha k \beta I_k)^2}, & j_{2,2} &= \frac{1 + \phi k \alpha (-S_k + \beta(j_{1,2} I_k + S_{k+1}))}{1 + \phi(d_0 + \gamma + \mu)}, & j_{3,2} &= \frac{\phi \gamma}{1 + \phi \mu} j_{2,2}, \\ j_{1,3} &= 0, & j_{2,3} &= 0, & j_{3,3} &= \frac{1}{1 + \phi \mu}. \end{aligned}$$

Evaluating the Jacobian  $J$  at the disease-free equilibrium point  $E_0 = (\frac{b}{\mu}, 0, 0)$  leads to

$$J(E_0) = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix},$$

where

$$\begin{aligned} a_1 &= \frac{1}{1 + \phi \mu}, & a_2 &= \frac{\phi k \alpha (b(1 + \phi \mu - \beta) + \phi \beta \mu (k - b))}{\mu(1 + \phi \mu)^2}, & a_3 &= 0, \\ a_4 &= 0, & a_5 &= \frac{\alpha b(-1 + \beta)k\phi + \mu(1 + \phi(\mu - \alpha k(b - b\beta + \beta k)\phi))}{\mu(1 + \phi \mu)(1 + (d_0 + \gamma + \mu)\phi)}, & a_6 &= 0, \\ a_7 &= 0, & a_8 &= \frac{\phi \gamma}{1 + \phi \mu} a_5, & a_9 &= \frac{1}{1 + \phi \mu}. \end{aligned}$$

Solving the characteristic equation  $\det(J(E_0) - \lambda I) = 0$ , where obviously

$$J(E_0) - \lambda I = \begin{bmatrix} a_1 - \lambda & a_2 & 0 \\ 0 & a_5 - \lambda & 0 \\ 0 & a_8 & a_9 - \lambda \end{bmatrix},$$

leads to the eigenvalues of  $J(E_0)$ :

$$\det(J(E_0) - \lambda I) = 0 \implies (a_1 - \lambda)(a_5 - \lambda)(a_9 - \lambda) = 0 \implies \lambda_1 = a_1, \lambda_2 = a_5, \lambda_3 = a_9.$$

Note that all the eigenvalues are reals and trivially  $|\lambda_1| = |\lambda_3| < 1$ . Furthermore,  $\lambda_2$  can be rewritten as

$$\lambda_2 = \frac{(1 - \phi \mu R_0)(1 + \phi \mu) - \alpha \beta k^2 \phi^2}{(1 + \phi(d_0 + \gamma + \mu))(1 + \phi \mu)}.$$

Since for the two pairs of parameters in Table 1 it holds  $R_0 < 1$ , then trivially  $\lambda_2 < 1$ . Moreover, by doing some calculations, it is easily shown that it is also true that  $\lambda_2 > -1$ . This proves that the equilibrium point  $E_0$  is linearly stable for the NSFD method.

To prove that  $E^* = (S^*, I^*, R^*)$  is also a linearly stable equilibrium point for the NSFD method, we need to evaluate the Jacobian in correspondence of it. This leads to

$$J(E^*) = \begin{bmatrix} a_1^* & a_2^* & a_3^* \\ a_4^* & a_5^* & a_6^* \\ a_7^* & a_8^* & a_9^* \end{bmatrix},$$

where

$$\begin{aligned} a_1^* &= \frac{\mu(d_0 + \gamma + \mu + 2\alpha k^2 \phi) + c_0}{c_1}, & a_2^* &= \frac{\phi k \alpha (S^*(1 + \phi \mu - \beta) + \phi \beta (k - b))}{(1 + \phi \mu + \phi \alpha k \beta I^*)^2}, & a_3^* &= 0, \\ a_4^* &= \frac{2\alpha k^2 \mu \phi (-1 + \beta - \mu \phi)}{c_1(1 + (d_0 + \gamma + \mu)\phi)}, & a_5^* &= \frac{\alpha^2 \beta^2 (I^*)^2 k^2 \phi^2 + (1 + \phi \mu)^2 - \alpha k \phi (1 + \phi \mu) c_2}{(1 + \alpha \beta I^* k \phi + \mu \phi)^2 (1 + (d_0 + \gamma + \mu)\phi)}, & a_6^* &= 0, \\ a_7^* &= \frac{2\alpha \gamma k^2 \mu \phi^2 (-1 + \beta - \mu \phi)}{c_1(1 + \phi \mu)(1 + (d_0 + \gamma + \mu)\phi)}, & a_8^* &= \frac{\phi \gamma}{1 + \phi \mu} a_5^*, & a_9^* &= \frac{1}{1 + \phi \mu}, \end{aligned}$$

$$c_0 = \sqrt{\mu(4\alpha(-1 + \beta)k^2(b - d_0 - \gamma - \mu) + \mu(d_0 + \gamma + \mu)^2)}, \quad c_1 = \mu(d_0 + \gamma + \mu + 2\alpha\beta k^2\phi + \mu(d_0 + \gamma + \mu)\phi) + c_0 + \mu\phi c_0,$$

$$c_2 = S^* + \mu\phi S^* - \beta(2I^* + b\phi - k\phi + S^*).$$

Subsequently, as before, it is necessary to compute the eigenvalues of  $J(E^*)$ , checking that they are all in modulus less than one. In this case, the characteristic polynomial is given by

$$\det(J(E^*) - \lambda I) = 0 \implies ((a_1^* - \lambda)(a_5^* - \lambda) - a_2^* a_4^*)(a_5^* - \lambda) = 0.$$

As for  $E_0$ , also for  $E^*$  one eigenvalue is  $\lambda_3 = a_5^*$ , which satisfies  $|\lambda_3| < 1$ . The other two eigenvalues are easily derivable from the characteristic polynomial, and by carrying out some algebraic calculations it is possible to see that they are in modulus less than one, using that it holds  $R_0 < 1$  for the two pairs of parameters in Table 1. This proves that  $E^*$  is a linearly stable equilibrium point and concludes the theorem.  $\square$

### 5 Numerical simulations

In this section, we perform numerical simulations to highlight the advantages of the proposed NSFD scheme and confirm the theoretical properties discussed so far. Specifically, we solve the model (6) using the NSFD method (10), the related standard one, i.e. explicit Euler, the famous and well known explicit Runge-Kutta RK4, and a nonstandard method derived by Mickens in [42]. The comparison between the mentioned schemes shows the advantages of the nonstandard method derived in this paper, which, unlike the classical ones, avoids numerical instabilities. Furthermore, compared to the nonstandard method by Mickens, which is stable and preserves the positivity, our NSFD scheme is able to compute the solution with a slightly lower error.

The scheme proposed in [19] for the solution of the model (6) is simply explicit Euler method, which in this case takes the following form:

$$S_{k+1} = S_k + h(b - k(1 - \alpha S_k I_k) - \alpha k \beta S_k I_k - \mu S_k), \tag{15a}$$

$$I_{k+1} = I_k + h(k(1 - \alpha S_k I_k) + \alpha k \beta S_k I_k - (d_0 + \gamma + \mu) I_k), \tag{15b}$$

$$R_{k+1} = R_k + h(\gamma I_k - \mu R_k). \tag{15c}$$

The RK4 is instead an explicit Runge-Kutta method with four stages and order four having the following Butcher tableau [10]:

0				
$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{2}$	0	$\frac{1}{2}$		
$\frac{2}{2}$	0	0	$\frac{1}{2}$	
1	0	0	1	
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Finally, the nonstandard method obtained by Mickens in [42] has been built on a slightly different version of the SIR compared to the one we have considered here. Then, we need to modify the scheme by Mickens in order to apply it on our SIR model (6), as follows:

$$S_{k+1} = \frac{S_k + \phi(b - k)}{1 + \phi(\mu + \alpha k \beta I_k - k \alpha I_k)}, \tag{16a}$$

$$I_{k+1} = \frac{I_k + \phi(k \alpha \beta - k \alpha) I_k S_{k+1} + \phi k}{1 + \phi(\mu + \gamma + d_0)}, \tag{16b}$$

$$R_{k+1} = \frac{R_k + \phi \gamma I_{k+1}}{1 + \phi \mu}. \tag{16c}$$

Mickens has demonstrated that the nonstandard method derived in [42] preserves the positivity for each step-size value on the version of the SIR model analyzed there. Adapting his method to the SIR model (6) considered here, then getting the scheme (16), we have numerically observed that the positivity is still preserved for very large step-sizes, for both pairs of parameters in Table 1.

The initial values of the functions  $S(t), I(t), R(t)$  are taken from [26, 27]. In particular, the initial conditions are  $S(0) = 164.7, I(0) = 0.0078, R(0) = 0.0003$ . The parameters values we set are those of Table 1 on the right.

In order to compute the errors of the used methods, we calculate a reference solution using the Matlab function `ode15s`, requiring maximum accuracy. For the computation of the reference solution, we opted for `ode15s` after verifying that it is able to satisfy the most important properties of the model. The behavior of the reference solution is shown in Figure 1.

Figure 2 shows that explicit Euler is not stable, since for  $h = 4$  the numerical solution and the global error explode. Even reducing the value of the step-size  $h$ , the numerical solution computed by this SFD method continues to be totally different from the exact one, as shown in Figure 3 on the left. Instead, our NSFD method provides in the same case an accurate solution, as shown in Figure 3 on the right. Figure 4 shows that the RK4 method provides an inaccurate and divergent solution setting  $h = 4$ . Differently from the SFD Euler scheme and the RK4 method, the NSFD scheme derived by us reproduces very well the qualitative behavior of the exact solution for the same step-size, as shown in Figure 5.

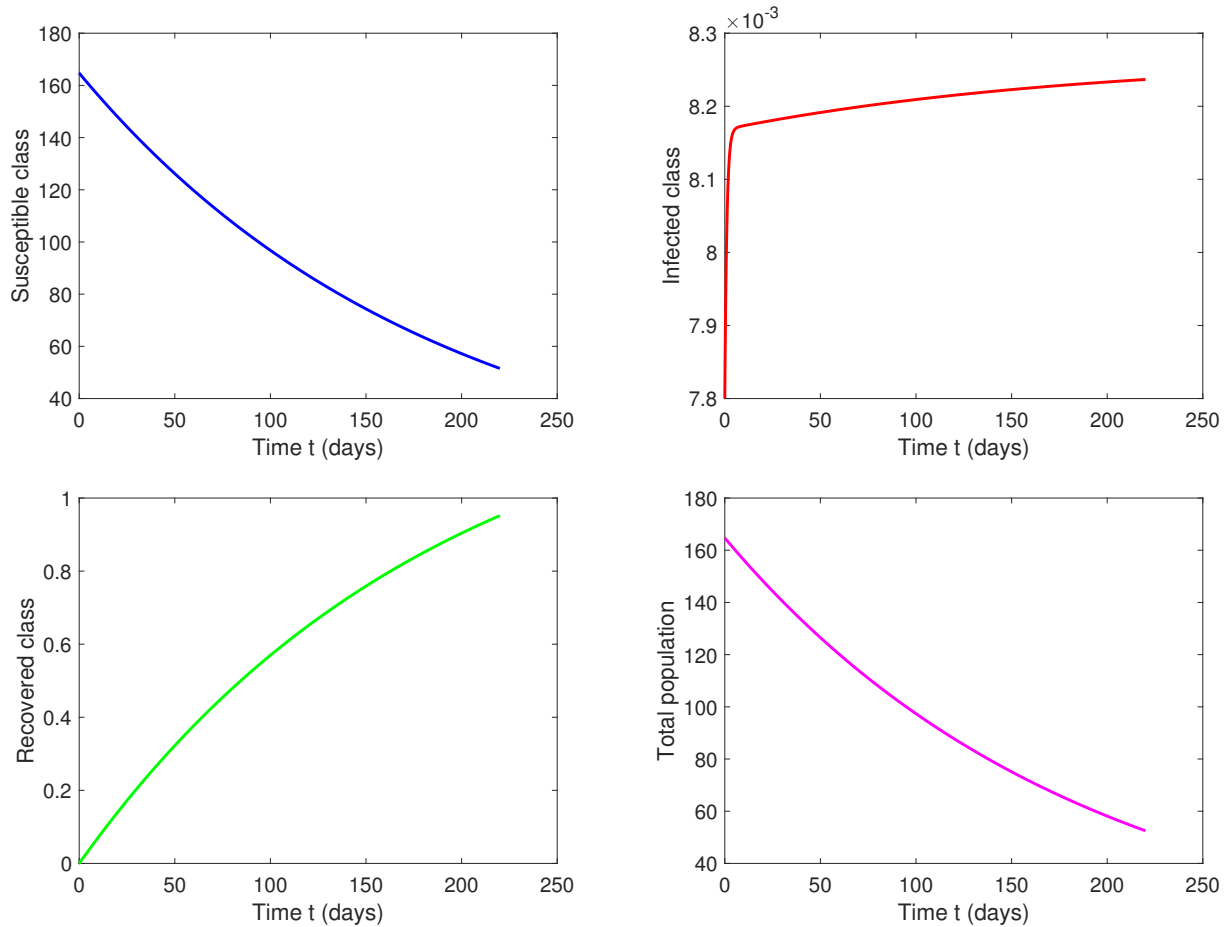


Figure 1: Reference solution of (6), provided by the Matlab function `ode15s`.

In order to numerically confirm that our NSFD method is elementary stable, as proven in the previous section, we report in Table 2 on the right the spectral radius of the Jacobian matrix associated with it relative to the equilibrium point  $E^*$ , setting different values of the step-size  $h$ . Note that the spectral radius of the Jacobian matrix associated with our NSFD method is less than one for all the values of  $h$ , thus confirming the scheme elementary stability. This does not happen for the standard Euler method, as shown in Table 2 on the left. Hence, the explicit Euler scheme is not elementary stable. To show that also the RK4 is not elementary stable, we have solved with it the model (6) by setting initial conditions close to the equilibrium point  $E^*$ . The corresponding results, reported in Figure 6, testify to what has been said, given that the numerical solution explodes rather than tends to the equilibrium point. Note instead, from Figure 7, that by doing the same thing with our NSFD method, the numerical solution tends exactly to  $E^*$ . We underline that it is important to have a stable method that performs well even for large step values, as when an epidemic occurs in a population, it is crucial to predict and estimate the long-term temporal evolution of the disease. Therefore, for long periods, it is desirable to use large step-size values, in order to have a lower computational cost, while maintaining good performance in describing how the disease evolves over time. From this point of view, a nonstandard scheme allows to avoid numerical instabilities, producing reliable solutions that preserve the important properties of the exact one.

Regarding the comparison between our NSFD scheme and the nonstandard method (16), we report the global error by them at the grid point  $t = 2000$  in Table 3. The initial conditions and the parameters used are the same ones already employed in this section. We have numerically verified that the method (16) preserves the positivity and equilibrium point properties for very large values of  $h$ , like our scheme. However, note that our NSFD method gives a slightly lower error than the other nonstandard scheme. Finally, according to the NSFD rules, we show that our nonstandard method has consistency order equal to one, like the corresponding SFD scheme. In fact, the NSFD rules (4) and (5) trivially allow to preserve the consistency of the starting standard method. To do this, just look at Table 3, where it can be seen that halving the step-size, the global error becomes two times smaller, thus confirming the thesis.

## 6 Conclusions

In this paper, we have constructed an NSFD scheme for an epidemiological model, initially used to describe the spread of the COVID-19 epidemic in Pakistan and solved with the explicit Euler method in [19]. Furthermore, by making a different choice for the model parameters, we have obtained qualitative results similar to the trend of the same epidemic in Bangladesh. We have



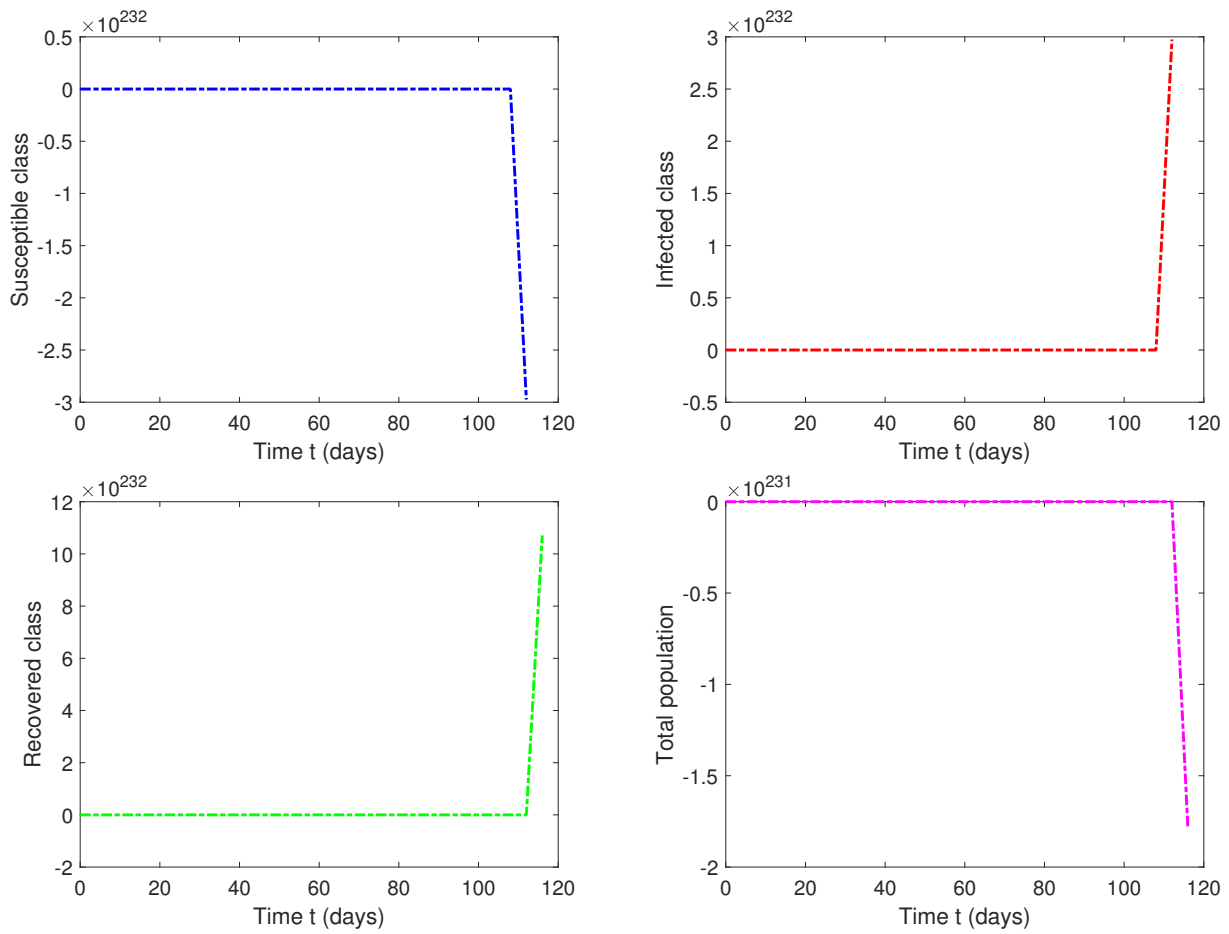


Figure 2: Numerical solutions of (6) provided by the SFD scheme (15), setting  $h = 4$ .

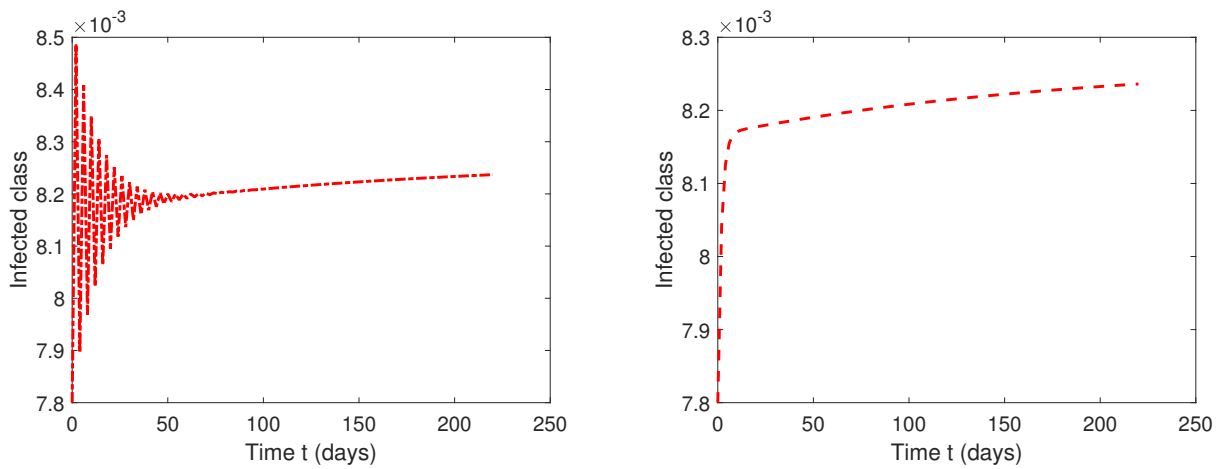


Figure 3: Numerical solution of (6b) provided by the SFD scheme (15) on the left and our NSFD method (10) on the right, setting  $h = 2$ .

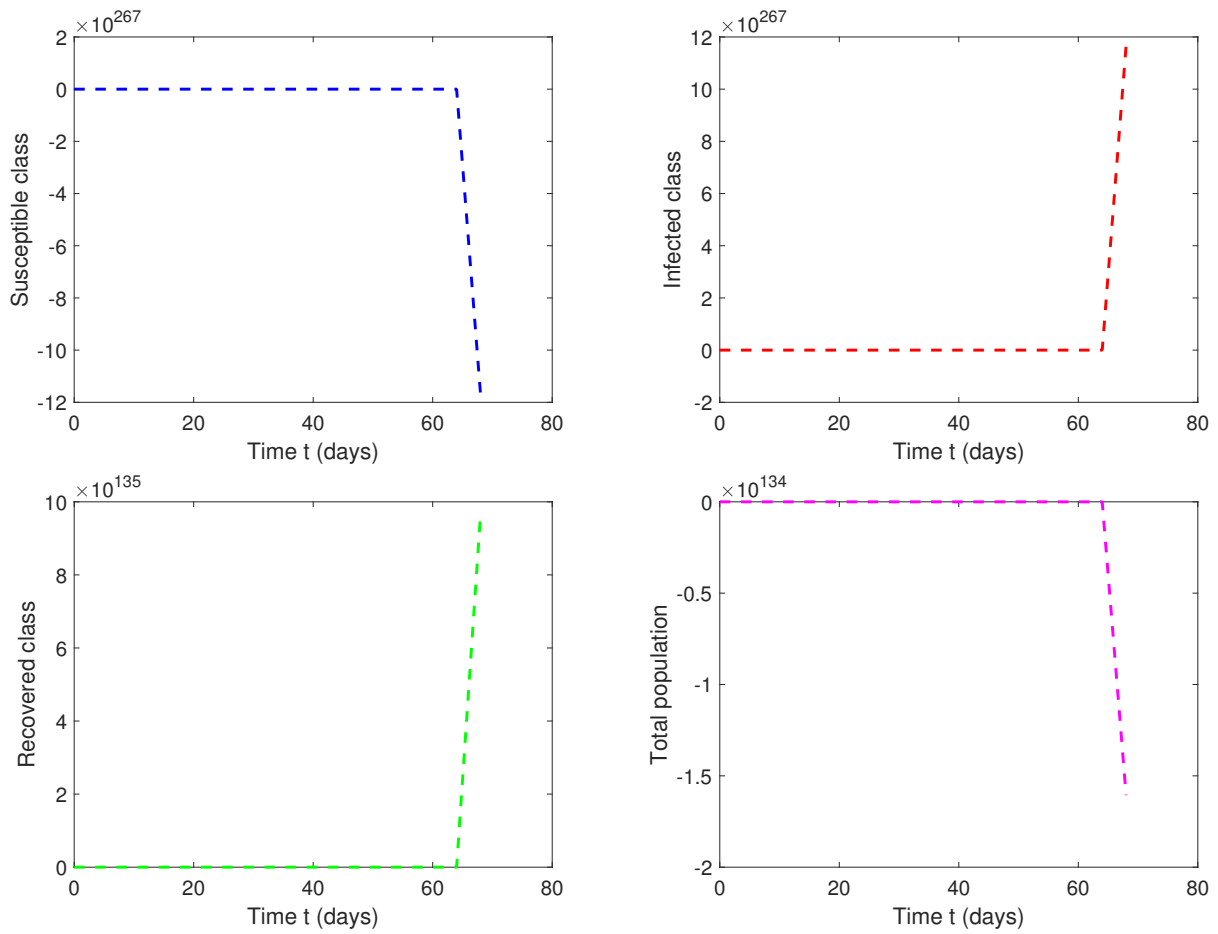


Figure 4: Numerical solutions of (6) provided by the RK4 method, setting  $h = 4$ .

Table 2: Spectral radius  $\rho$  of the Jacobian matrix with respect to the equilibrium point  $E^* = (1.92427, 0.00827, 1.37783)$ , setting different time step-sizes, using the parameters values reported in Table 1 on the right.

$h$	$\rho$ of SFD (15)	$\rho$ of our NSFD (10)
0.0625	0.9997	0.9997
0.125	0.9993	0.9993
0.25	0.9987	0.9987
0.5	0.9973	0.9973
1	0.9946	0.9946
2	0.9892	0.9893
4	2.6821 – Instability	0.9789
8	6.3642 – Instability	0.9586

Table 3: Global error at  $t = 2000$  by the nonstandard method (16) and our NSFD method (10), setting different time step-sizes, using the parameters values reported in Table 1 on the right.

$h$	NSFD (16)	our NSFD (10)
2	$1.0412e-4$	$6.0582e-5$
1	$5.1485e-5$	$2.9956e-5$
0.5	$2.5600e-5$	$1.4895e-5$
0.25	$1.2764e-5$	$7.4267e-6$

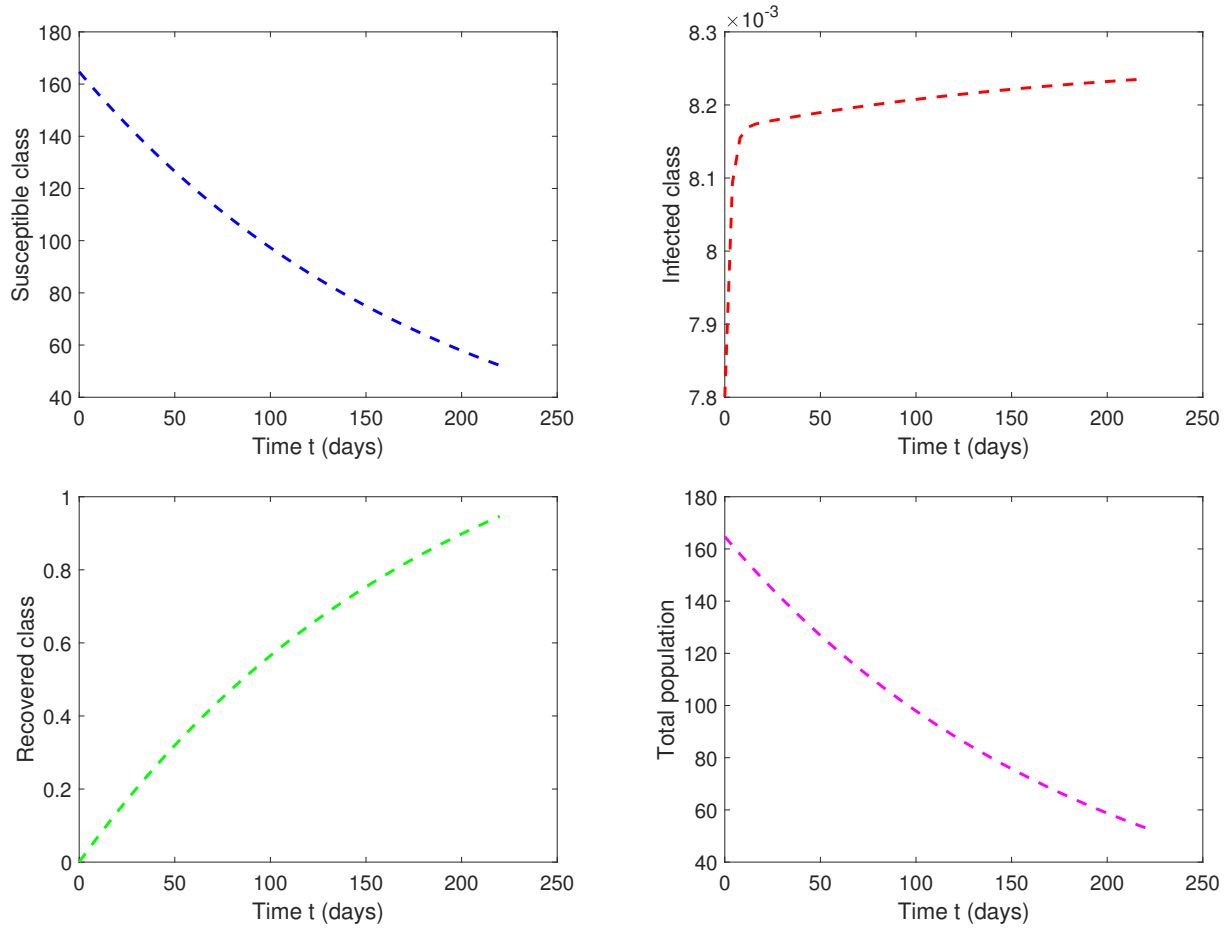


Figure 5: Numerical solutions of (6) provided by our NSFD scheme (10), setting  $h = 4$ .

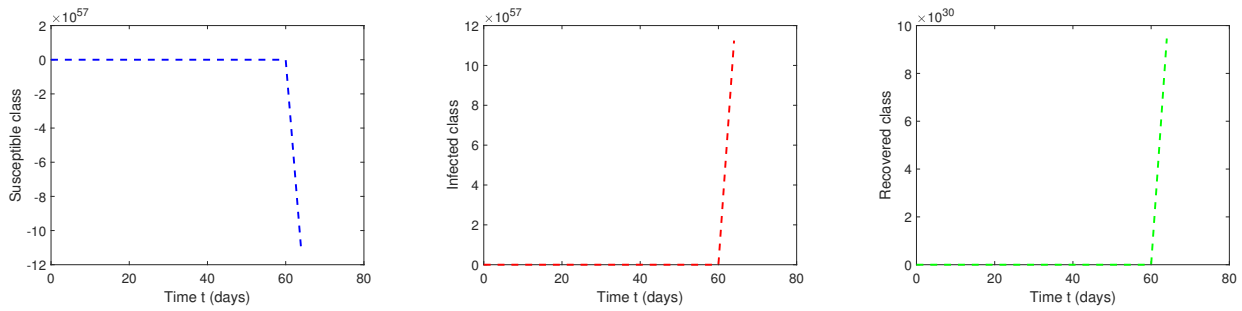


Figure 6: Numerical solution of (6) provided by the RK4 method, setting initial values  $S(0), I(0), R(0)$  close to the equilibrium point  $E^* = (1.92427, 0.00827, 1.37783)$  and  $h = 4$ .

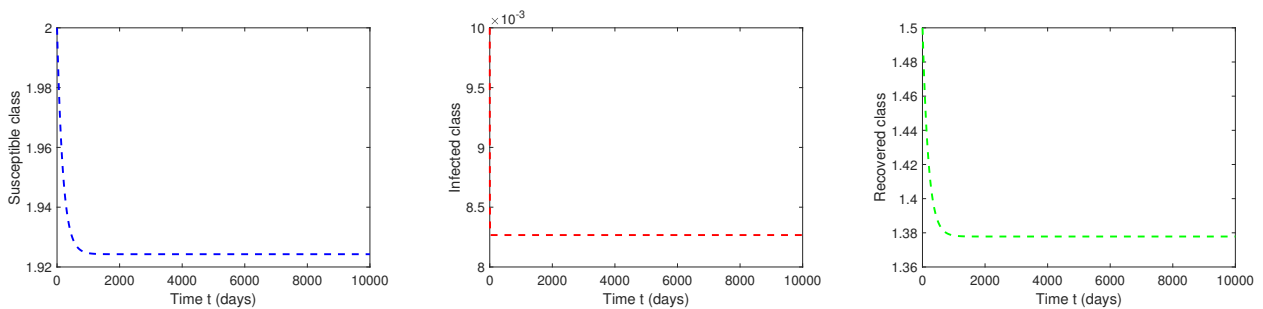


Figure 7: Numerical solution of (6) provided by the proposed NSFD scheme (10), setting initial values  $S(0), I(0), R(0)$  close to the equilibrium point  $E^* = (1.92427, 0.00827, 1.37783)$  and  $h = 4$ .

shown that, by carefully setting the function  $\phi(h)$  and the non-local representations in accordance with the NSFD rules, it has been possible to construct a method with very good stability properties, capable of preserving the positivity and equilibrium points features of the exact solution of the analyzed model. Numerical tests have confirmed that the constructed nonstandard method performs better than both the related standard scheme, and also than the famous Runge-Kutta RK4. Furthermore, the numerical tests have showed the convenience of the method we have obtained also compared to another positivity-preserving nonstandard scheme.

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