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Determining the first radii of meromorphy via orthogonal polynomials on the unit circle

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Abstract

Applying a result concerning a convergence of modified orthogonal Padé approximants constructed from orthogonal polynomials on the unit circle, we prove an analogue of Hadamard's theorem for determining the radius of 1-meromorphy of a function holomorphic on the closed unit disk. Furthermore, we apply our result to study analytic properties of the reciprocal of Szegő functions when their corresponding sequence of Verblunsky coefficients has exponential decay.

1 Introduction

Let μ be a finite positive Borel measure with infinite support $\mathrm{supp}(\mu)$ contained in the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. We write $\mu \in \mathcal{M}$ and define the associated inner product,

$$\langle g, h\rangle := \int g(\zeta)\overline{h(\zeta)}d\mu(\zeta), \quad g, h \in L_2(\mu).$$

Let

$$\varphi_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \ldots$$

be the orthonormal polynomial of degree n with respect to μ having positive leading coefficient; that is, $\langle \varphi_n, \varphi_m\rangle = \delta_{n,m}$. Let

$$\mathbb{B}_R := \{z \in \mathbb{C} : |z| < R\} \qquad \text{and} \qquad \mathbb{B} := \mathbb{B}_1 = \{z \in \mathbb{C} : |z| < 1\}.$$

Denote by $\mathcal{H}(\overline{\mathbb{B}})$ the space of all functions holomorphic on some neighborhood of $\overline{\mathbb{B}}$ and by $R_m(F)$ the radius of the largest disk centered at the origin to which $F \in \mathcal{H}(\overline{\mathbb{B}})$ can be extended meromorphically with at most m poles counting their multiplicity. The constant $R_m(F)$ is commonly known as the radius of m-meromorphy of F. We write R_m when it is clear to which function the notation refers.

Now, let us define subclasses of $\mathcal{M}$. We say that $\mu \in \mathbf{S}$ if and only if μ satisfies the Szegő condition, namely

$$\int_{\mathbb{T}} \log \mu'(\zeta)|d\zeta| > -\infty,$$

where μ' denotes the Radon-Nikodym derivative of μ with respect to the arc length on $\mathbb{T}$. We denote by $\hat{\mathbf{S}}$ the class of all $\mu \in \mathcal{M}$ such that

$$\rho(\mu) := \left(\limsup_{n\to\infty} |\varphi_n(0)|^{1/n}\right)^{-1} > 1. \tag{1}$$

It is well-known that $\hat{\mathbf{S}}$ is the class of all measures meeting the Szegő condition such that the corresponding constant $\rho(\mu) = R_0(D^{-1}) > 1$, where D is the interior Szegő function given by

$$D(z) := \exp\left\{\frac{1}{4\pi}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \mu'(\zeta)|d\zeta|\right\}, \qquad |z| < 1$$

(see (2.1), (2.5), and Theorems 6.2 and 7.4 in [8] for more details).

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For a given function $F \in \mathcal{H}(\overline{\mathbb{B}})$, we define

$$\Delta_{n,m} := \begin{vmatrix} \langle z^{m-1}F, \varphi_n\rangle & \langle z^{m-2}F, \varphi_n\rangle & \cdots & \langle F, \varphi_n\rangle \\ \langle z^{m}F, \varphi_n\rangle & \langle z^{m-1}F, \varphi_n\rangle & \cdots & \langle zF, \varphi_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle z^{2m-2}F, \varphi_n\rangle & \langle z^{2m-3}F, \varphi_n\rangle & \cdots & \langle z^{m-1}F, \varphi_n\rangle \end{vmatrix}. \tag{2}$$

Set

$$l_0 := 1 \qquad \text{and} \qquad l_m := \limsup_{n\to\infty} |\Delta_{n,m}|^{1/n}, \qquad \text{for all } m \geq 1.$$

Our main result is

Theorem 1.1. *Let $\mu \in \hat{\mathbf{S}}$ and $F \in \mathcal{H}(\overline{\mathbb{B}})$. Then, we have*

$$R_1 = \frac{l_1}{l_2}, \tag{3}$$

where by convention $0/0 = \infty$.

Definitely, a natural conjecture can be posed as the following:

Conjecture 1.2. *Let $\mu \in \hat{\mathbf{S}}$ and $F \in \mathcal{H}(\overline{\mathbb{B}})$. Then, for all $m \in \mathbb{N}_0$, we have*

$$R_m = \frac{l_m}{l_{m+1}}, \tag{4}$$

where by convention $0/0 = \infty$.

Although we are only able to prove Conjecture 1 when $m = 1$, Theorem 1.1 itself has an application in localizing the first pole of the reciprocal of the interior Szegő function corresponding $\mu \in \hat{\mathbf{S}}$ using its Verblunsky coefficients (see Corollary 1.5 below).

Conjecture 1 is an analogue of the classical Hadamard theorem for determining the radius of m-meromorphy of an analytic function from its Taylor coefficients stated as the following:

Theorem 1.3 (Hadamard [9]). *Let $F = \sum_{k=0}^{\infty} f_k z^k$ be an analytic function on some neighborhood of $z = 0$. Then, for each $m \geq 0$, we have*

$$R_m = \frac{\hat{l}_m}{\hat{l}_{m+1}}, \tag{5}$$

where $\hat{l}_0 := 1$ and $\hat{l}_m := \limsup_{n\to\infty} |H_{n,m}|^{1/n}$,

$$H_{n,m} := \begin{vmatrix} f_{n-m+1} & f_{n-m+2} & \cdots & f_n \\ f_{n-m+2} & f_{n-m+3} & \cdots & f_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ f_n & f_{n+1} & \cdots & f_{n+m-1} \end{vmatrix}, \quad m \in \mathbb{N}, \quad n \geq m-1,$$

(here, as in (4), by convention $0/0 = \infty$).

The reader can also find the proof of Theorem 1.2 in English in [7].

To support that Conjecture 1 should be true, we consider a function $F = \sum_{k=0}^{\infty} f_k z^k \in \mathcal{H}(\overline{\mathbb{B}})$ and the normalized arc length measure $d\mu = d\theta/2\pi$ on the unit circle. In this case, $\varphi_n(z) = z^n$ for all $n \in \mathbb{N}_0$, the determinant in (2) is

$$\Delta_{n,m} = \begin{vmatrix} f_{n-m+1} & f_{n-m+2} & \cdots & f_n \\ f_{n-m} & f_{n-m+1} & \cdots & f_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n-2m+2} & f_{n-2m+3} & \cdots & f_{n-m+1} \end{vmatrix},$$

(this determinant is the same as $(-1)^{m(m-1)/2} H_{n-m+1,m}$), and the formulas (4) and (5) coincide. Therefore, Theorem 1.2 is a special case of Conjecture 1.

The validity of Theorem 1.1 is dependent on the convergence of modified orthogonal Padé approximants defined as follows.

Definition 1.1. Let $F \in \mathcal{H}(\overline{\mathbb{B}})$ and $\mu \in \mathcal{M}$. Fix $(n, m) \in \mathbb{N}_0 \times \mathbb{N}$. Then, there exists a polynomial $Q^{\mu}_{n,m}$ such that $\deg(Q^{\mu}_{n,m}) \leq m, Q^{\mu}_{n,m} \not\equiv 0$, and

$$\langle z^k Q^{\mu}_{n,m} F, \varphi_{n+1}\rangle = 0, \qquad k = 0, 1, \ldots, m-1. \tag{6}$$

Define the corresponding polynomial

$$P^{\mu}_{n,m}(z) := \sum_{j=0}^{n} \langle Q^{\mu}_{n,m} F, \varphi_j\rangle \varphi_j(z).$$

The rational function

$$R^{\mu}_{n,m} := \frac{P^{\mu}_{n,m}}{Q^{\mu}_{n,m}}$$

is called an (n,m) *(modified) orthogonal Padé approximant of F with respect to μ.*

Finding $Q^{\mu}_{n,m}$ in (6) is equivalent to solving a homogeneous system of m linear equations on $m+1$ unknowns. Therefore, for any pair $(n,m)\in\mathbb{N}_0\times\mathbb{N}$, a polynomial $Q^{\mu}_{n,m}$ always exists but it may not be unique. Since $Q^{\mu}_{n,m}\not\equiv 0$, we normalize it to be a monic polynomial. By a proof similar to a proof of [6, Lemma 3.4], the condition $\Delta_{n+1,m}\neq 0$ and the condition that every solution of (6) has $\deg Q^{\mu}_{n,m}=m$ are equivalent. In turn, they imply the uniqueness of $Q^{\mu}_{n,m}$. Note that the concept of Definition 1.1 was first introduced in [3] in the vector format.

We would like to point out that Barrios Rolanía, López Lagomasino, and Saff [1] used another related construction of orthogonal Padé approximants [12] (which were introduced long before the development of Definition 1.1 and later called standard orthogonal Padé approixmants in [4, Definition 5]) to show a formula similar to (4). To be precise, they proved that under the condition that $\mu\in\hat{\mathbf{S}}$ and $F\in\mathcal{H}(\overline{\mathbb{B}})$,

$$R_m = \frac{\tilde{l}_m}{\tilde{l}_{m+1}},$$

where $\tilde{l}_0 := 1$ and $\tilde{l}_m := \limsup_{n\to\infty}|\tilde{\Delta}_{n,m}|^{1/n}$, and

$$\tilde{\Delta}_{n,m} := \begin{vmatrix} \langle z^{m-1}F,\varphi_n\rangle & \langle z^{m-2}F,\varphi_n\rangle & \cdots & \langle F,\varphi_n\rangle \\ \langle z^{m-1}F,\varphi_{n+1}\rangle & \langle z^{m-2}F,\varphi_{n+1}\rangle & \cdots & \langle F,\varphi_{n+1}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle z^{m-1}F,\varphi_{n+m-1}\rangle & \langle z^{m-2}F,\varphi_{n+m-1}\rangle & \cdots & \langle F,\varphi_{n+m-1}\rangle \end{vmatrix}, \quad m\in\mathbb{N}.$$

Our method used to prove Theorem 1.1 is strongly influenced by the method in [1] (see also [5] a result similar to (3) but the measure μ is supported on the interval $[-1,1]$). Therefore, we need a result concerning the convergence of $\{Q^{\mu}_{n,m}\}_{n\in\mathbb{N}_0}$ with m being fixed in Theorem 1.3 below (see [3, Theorem 1.2] for its proof). To state such convergence result, we need to define another class of measures which is a subclass of $\mathcal{M}$. We say that $\mu\in\mathbf{Reg}$ when $\operatorname{supp}(\mu)=\mathbb{T}$ and

$$\lim_{n\to\infty}|\varphi_n(z)|^{1/n} = |z|, \tag{7}$$

uniformly on compact subsets of $\mathbb{C}\setminus\overline{\mathbb{B}}$. When $\operatorname{supp}(\mu)=\mathbb{T}$, it was shown in [11, Theorem 3.1.1] that the condition (7) is equivalent to the condition

$$\lim_{n\to\infty}\kappa_n^{1/n} = 1. \tag{8}$$

Moreover, since $\overline{\mathbb{B}}$ is convex, if $\mu\in\mathbf{Reg}$, then

$$\lim_{n\to\infty}|s_n(z)|^{1/n} = |z|^{-1},$$

uniformly on compact subsets of $\mathbb{C}\setminus\overline{\mathbb{B}}$, where

$$s_n(z) := \int\frac{\overline{\varphi_n(\zeta)}}{z-\zeta}d\mu(\zeta), \qquad z\in\overline{\mathbb{C}}\setminus\operatorname{supp}(\mu)$$

is the second kind function corresponding to the polynomial φ_n (see the discussion about this result on pages 20-21 in [3]). It is well-known that $\hat{\mathbf{S}}\subset\mathbf{Reg}$.

Theorem 1.4. *Let $F\in\mathcal{H}(\overline{\mathbb{B}})$, $m\in\mathbb{N}$ be fixed, and $\mu\in\mathbf{Reg}$. Denote by $\mathcal{P}_m(F)$ the set of all poles of F in $\mathbb{B}_{R_m}$ and Q^F_m the monic polynomial whose zeros are these poles counting multiplicities. Then, the following assertions are equivalent:*

(a) F has exactly m poles in $\mathbb{B}_{R_m}$.

(b) The polynomials $Q^{\mu}_{n,m}$ for F are uniquely determined for all sufficiently large n and there exists a polynomial Q_m of degree m such that

$$\limsup_{n\to\infty}\|Q^{\mu}_{n,m}-Q_m\|^{1/n} = \theta < 1, \tag{9}$$

where the norm $\|\cdot\|$ denotes the norm induced in the space of polynomials of degree at most m by the maximum of the absolute value of the coefficients. Moreover, if one of the assertions (a) or (b) takes place, then $Q_m = Q^F_m$ and

$$\theta = \frac{\max_{\lambda\in\mathcal{P}_m(F)}|\lambda|}{R_m}. \tag{10}$$

Theorem 1.1 can be used to study analytic properties of interior Szegő functions when certain asymptotic properties of Verblunsky coefficients $\alpha_n := -\overline{\varphi_{n+1}(0)}/\kappa_{n+1}$ are known. In [2, Corollary 2.4], it was proved that

Theorem 1.5. *Assume that μ satisfies the Szegő condition and $D^{-1} \in \mathcal{H}(\overline{\mathbb{B}})$. If*

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = b,$$

for some $b \in \mathbb{C}$, then b^{-1} is a singularity of D^{-1} and D^{-1} is holomorphic in $\mathbb{B}_{|b|^{-1}}$.

The natural question is "What can we say concerning the analytic properties of D^{-1} if α_{n+1}/α_n converges to b at a geometrical rate?" We can make use of Theorem 1.1 to give a proof of the following result:

Corollary 1.6. *Let $b \in \mathbb{B}$. If*

$$\frac{\alpha_{n+1}}{\alpha_n} = b + O(\delta^n), \tag{11}$$

for some $\delta < 1$, then for some $\delta' > 0$, $D^{-1}(z)$ is meromorphic in $\mathbb{B}_{|b|^{-1}+\delta'}$ with a single pole in that disk and the pole is at $z = b^{-1}$. Additionally, if we assume further that $\delta < |b|^2$, then

$$R_1(D^{-1}) = \frac{1}{|b|^3}. \tag{12}$$

Note that Corollary 1.5 is a known result (see [10, Corollary 7.2.2] and [1, Corollary 2]). However, the formula (12) appeared in [1, Corollary 2] is under the condition that $\delta < |b|^4$. Our result is an refinement of Corollary 2 in [1]. The validity of Conjecture 1 would potentially be of great importance in understanding locations of poles of D^{-1} further in $\mathbb{B}_{R_m}$ when the asymptotic properties of α_n are given.

The paper is organized as follows. In Section 2, we state and prove some lemmas and auxiliary results. Finally, the proofs of Theorem 1.1 and Corollary 1.5 are in Section 3.

2 Lemmas and auxiliary results

The following lemma is equivalent to Conjecture 1 when $m = 0$. This serves as an analogue of the Cauchy-Hadamard formula. The reader can find the proof of this result in [8, Theorems 6.2 and 7.4] or [11, Theorem 6.6.1].

Lemma 2.1. *Let $F \in \mathcal{H}(\overline{\mathbb{B}})$ and $\mu \in \mathbf{Reg}$. Then,*

$$l_1 = \limsup_{n\to\infty} |\langle F, \varphi_n\rangle|^{1/n} = \frac{1}{R_0}. \tag{13}$$

Moroever, the series $\sum_{n=0}^{\infty} \langle F, \varphi_n\rangle \varphi_n(z)$ converges to $F(z)$ uniformly on compact subsets of $\mathbb{B}_{R_0}$ and diverges pointwise for all $z \in \mathbb{C}\setminus\overline{\mathbb{B}_{R_0}}$.

It can also be proved that the partial sum of the series in Lemma 2.1 converges to F in the $L_2(\mu)$ space with the following rate of convergence (see Theorem 6.6.1 in [11]).

Lemma 2.2. *Let $F \in \mathcal{H}(\overline{\mathbb{B}})$ and $\mu \in \mathbf{Reg}$. Then,*

$$\limsup_{n\to\infty} \|F - S_n\|_2^{1/n} \leq \frac{1}{R_0}, \tag{14}$$

where $\|\cdot\|_2$ denotes the $L_2(\mu)$ norm and

$$S_n(z) := \sum_{k=0}^{n} \langle F, \varphi_k\rangle \varphi_k(z)$$

denotes the n-th partial sum of the Fourier expansion of F.

Using Lemma 2.1, we obtain the following estimate of l_m.

Lemma 2.3. *Let $F \in \mathcal{H}(\overline{\mathbb{B}})$ and $\mu \in \mathbf{Reg}$. Then,*

$$l_m \leq \frac{1}{R_0 \cdots R_{m-1}} < 1, \qquad m \in \mathbb{N}.$$

Proof of Lemma 2.3. For each $j \in \mathbb{N}_0$, we denote by j_0 the number of poles of F in $\mathbb{B}_{R_j}$ (counting their order). Let Q_j be the monic polynomial of degree j which has a zero at each pole of F in $\mathbb{B}_{R_j}$ and zeros at 0 of order $j - j_0$. Notice that

$$R_j(F) = R_0(z^k Q_j F), \qquad \text{for all } k, j \in \mathbb{N}_0.$$

Then, by Lemma 2.1, we have

$$\limsup_{n\to\infty} |\langle z^k Q_j F, \varphi_n\rangle|^{1/n} = \frac{1}{R_0(z^k Q_j F)} = \frac{1}{R_j(F)}. \tag{15}$$

Fix $m \in \mathbb{N}$. Since $Q_{m-1} \in \operatorname{span}\{1, z, \dots, z^{m-1}\}$, by the properties of the determinants, we obtain

$$\Delta_{n,m} = \begin{vmatrix} \langle Q_{m-1}F, \varphi_n \rangle & \langle z^{m-2}F, \varphi_n \rangle & \cdots & \langle F, \varphi_n \rangle \\ \langle zQ_{m-1}F, \varphi_n \rangle & \langle z^{m-1}F, \varphi_n \rangle & \cdots & \langle zF, \varphi_n \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle z^{m-1}Q_{m-1}F, \varphi_n \rangle & \langle z^{2m-3}F, \varphi_n \rangle & \cdots & \langle z^{m-1}F, \varphi_n \rangle \end{vmatrix}. \tag{16}$$

Analogously, applying the fact that $Q_j \in \operatorname{span}\{1, z, \dots, z^j\}$ for all $1 \le j \le m-2$ and the properties of the determinants, we obtain

$$\Delta_{n,m} = \begin{vmatrix} \langle Q_{m-1}F, \varphi_n \rangle & \langle Q_{m-2}F, \varphi_n \rangle & \cdots & \langle Q_1F, \varphi_n \rangle & \langle F, \varphi_n \rangle \\ \langle zQ_{m-1}F, \varphi_n \rangle & \langle zQ_{m-2}F, \varphi_n \rangle & \cdots & \langle zQ_1F, \varphi_n \rangle & \langle zF, \varphi_n \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle z^{m-1}Q_{m-1}F, \varphi_n \rangle & \langle z^{m-1}Q_{m-2}F, \varphi_n \rangle & \cdots & \langle z^{m-1}Q_1F, \varphi_n \rangle & \langle z^{m-1}F, \varphi_n \rangle \end{vmatrix}.$$

Expanding this determinant, we obtain a sum of $m!$ terms each one of which has exactly one factor representing each column. According to (15), it follows that the n-th root of each one of these terms has lim sup not greater than $(R_0(F)R_1(F)\dots R_{m-1}(F))^{-1}$. Since the number of terms in the expansion of the determinants remains fixed with n, the statement of the lemma follows. □

Define the monic orthogonal polynomial of degree n,

$$\Phi_n(z) := \frac{\varphi_n(z)}{\kappa_n}.$$

It is well-known that the polynomials Φ_n satisfy the following three term recurrence formula:

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \tag{17}$$

where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\overline{z})}$ is the so-called n-th reversed polynomial, and the following relation

$$1 - \left(\frac{\kappa_n}{\kappa_{n+1}}\right)^2 = |\Phi_{n+1}(0)|^2. \tag{18}$$

Using (8) and (18), it is not difficult to check that if $\mu \in \hat{\mathbf{S}}$, then

$$\lim_{n\to\infty} \frac{\kappa_{n+1}}{\kappa_n} = 1. \tag{19}$$

For most parts of the proof of the main theorem, it is required that $\mu \in \mathbf{Reg}$. However, at some points, we need the following relations which are true when $\mu \in \hat{\mathbf{S}}$.

Lemma 2.4. *Let $\mu \in \hat{\mathbf{S}}$ and $F \in \mathcal{H}(\overline{\mathbb{B}})$. Then, we have*

$$\langle zQ_1F, \varphi_{n+1} \rangle = \langle Q_1F, \varphi_n \rangle + \delta_{n,1} \qquad \text{and} \qquad \langle zF, \varphi_{n+1} \rangle = \langle F, \varphi_n \rangle + \delta_{n,2},$$

where the polynomial Q_1 is defined as in the proof of Lemma 2.3, and

$$\limsup_{n\to\infty} |\delta_{n,1}|^{1/n} < \frac{1}{R_1} \qquad \text{and} \qquad \limsup_{n\to\infty} |\delta_{n,2}|^{1/n} < \frac{1}{R_0}.$$

Proof of Lemma 2.4. It follows from the recurrence formula (17) that

$$\frac{1}{\kappa_{n+1}}\langle zQ_1F, \varphi_{n+1} \rangle = \langle zQ_1F, z\Phi_n + \Phi_{n+1}(0)\Phi_n^* \rangle = \langle zQ_1F, z\Phi_n \rangle + \langle zQ_1F, \Phi_{n+1}(0)\Phi_n^* \rangle$$
$$= \frac{1}{\kappa_n}\langle Q_1F, \varphi_n \rangle + \langle zQ_1F, \Phi_{n+1}(0)\Phi_n^* \rangle.$$

Then, from the above equality,

$$|\langle zQ_1F, \varphi_{n+1} \rangle - \langle Q_1F, \varphi_n \rangle|$$
$$\le \left| \langle zQ_1F, \varphi_{n+1} \rangle - \frac{\kappa_n}{\kappa_{n+1}}\langle zQ_1F, \varphi_{n+1} \rangle \right| + \left| \frac{\kappa_n}{\kappa_{n+1}}\langle zQ_1F, \varphi_{n+1} \rangle - \langle Q_1F, \varphi_n \rangle \right|$$
$$\le \left| 1 - \frac{\kappa_n}{\kappa_{n+1}} \right| |\langle zQ_1F, \varphi_{n+1} \rangle| + |\kappa_n \langle zQ_1F, \Phi_{n+1}(0)\Phi_n^* \rangle|. \tag{20}$$

By (1), (8), (13), (18), and (19), we have

$$\limsup_{n\to\infty} \left(\left| 1 - \frac{\kappa_n}{\kappa_{n+1}} \right| |\langle zQ_1F, \varphi_{n+1} \rangle| \right)^{1/n}$$

$$\leq \limsup_{n\to\infty}\left(\frac{|\Phi_{n+1}(0)|^2}{1+\kappa_n\kappa_{n+1}^{-1}}\right)^{1/n}\limsup_{n\to\infty}|\langle zQ_1F,\varphi_{n+1}\rangle|^{1/n} < \frac{1}{R_0(zQ_1F)} = \frac{1}{R_1(F)}. \tag{21}$$

The second term of (20) can be rewritten as

$$|\kappa_n\overline{\Phi_{n+1}(0)}\langle zQ_1F,\Phi_n^*\rangle| = |\kappa_n\overline{\Phi_{n+1}(0)}\langle z(Q_1F-S_{n-1}),\Phi_n^*\rangle|, \tag{22}$$

where S_{n-1} denotes the $(n-1)$-th Fourier sum of Q_1F. Notice that

$$\langle zS_{n-1},\Phi_n^*\rangle = 0$$

because zS_{n-1} is a polynomial of degree at most $n\geq 1$ with a zero of multiplicity ≥ 1 at $z=0$ and Φ_n^* is orthogonal to all such polynomials. Therefore, using (1), (8), (14), and the Holder inequality, it follows from (22) that

$$\limsup_{n\to\infty}|\kappa_n\overline{\Phi_{n+1}(0)}\langle zQ_1F,\Phi_n^*\rangle|^{1/n} < \limsup_{n\to\infty}\|Q_1F-S_{n-1}\|_2^{1/n} \leq \frac{1}{R_0(Q_1F)} = \frac{1}{R_1(F)}. \tag{23}$$

By (21) and (23), it follows from (20) that

$$\limsup_{n\to\infty}|\langle zQ_1F,\varphi_{n+1}\rangle - \langle Q_1F,\varphi_n\rangle|^{1/n} < \frac{1}{R_1(F)},$$

which proves the first part of the lemma.

Similarly, the second part of Lemma 2.4 can be proved by considering

$$\frac{1}{\kappa_{n+1}}\langle zF,\varphi_{n+1}\rangle = \frac{1}{\kappa_n}\langle F,\varphi_n\rangle + \langle zF,\Phi_{n+1}(0)\Phi_n^*\rangle.$$

We leave the details for the reader. □

In order to prove Corollary 1.5, we will need the following three lemmas.

Lemma 2.5. *Let $\mu\in\hat{\mathbf{S}}$. Then,*

$$\langle z^kD^{-1},\varphi_n\rangle = \frac{1}{k!}\sum_{s=0}^{k}\binom{k}{s}\overline{\varphi_n^{(s)}(0)D^{(k-s)}(0)},$$

where $f^{(s)}(z)$ denotes the s-th derivative of $f(z)$.

Lemma 2.6. *Let $n_0\in\mathbb{N}$. Assume that $\Phi_n(0)\neq 0$ for all $n\geq n_0$. Then,*

$$\frac{\Phi_{n+1}^{(1)}(0)}{\Phi_{n+1}(0)} - \frac{\Phi_n^{(1)}(0)}{\Phi_n(0)} = \left(\frac{\Phi_n(0)}{\Phi_{n+1}(0)} - \frac{\Phi_{n-1}(0)}{\Phi_n(0)}\right) + \frac{\Phi_{n-1}(0)}{\Phi_n(0)}|\Phi_n(0)|^2.$$

The reader can find the proofs of Lemmas 2.5 and 2.6 in [1, page 275] and [1, page 276], respectively.

Lemma 2.7. *Let $b\in\mathbb{B}$. If*

$$\frac{\alpha_{n+1}}{\alpha_n} = b + O(\delta^n), \tag{24}$$

for some $\delta<1$, then

$$\frac{\Phi_{n+1}^{(1)}(0)}{\Phi_{n+1}(0)} - \frac{\Phi_n^{(1)}(0)}{\Phi_n(0)} = \frac{|c|^2|b|^{2n}}{\overline{b}} + O(\delta^n).$$

Proof of Lemma 2.7. Recall that $\alpha_n := -\overline{\Phi_{n+1}(0)}$. It is easy to check that (24) implies that

$$\frac{\Phi_n(0)}{\Phi_{n+1}(0)} = \frac{1}{\overline{b}} + O(\delta^n). \tag{25}$$

Since $\alpha_n = \alpha_0\prod_{j=0}^{n-1}(\alpha_{j+1}/\alpha_j)$, (24), and $\prod_{j=n}^{\infty}(1+O(\delta^j)) = 1+O(\delta^n)$, we obtain

$$\alpha_n = ab^n + O(|b|^n\delta^n),$$

for some $a\neq 0$. This implies that

$$\Phi_n(0) = c\overline{b}^n + O(|b|^n\delta^n),$$

where $c = -\overline{(a/b)}$. Since

$$|\Phi_n(0)|^2 = \Phi_n(0)\overline{\Phi_n(0)} = |c|^2|b|^{2n} + O(|b|^{2n}\delta^n),$$

by Lemma 2.6 and (25),

$$\begin{aligned}
\frac{\Phi_{n+1}^{(1)}(0)}{\Phi_{n+1}(0)} - \frac{\Phi_n^{(1)}(0)}{\Phi_n(0)} &= \left(\frac{\Phi_n(0)}{\Phi_{n+1}(0)} - \frac{\Phi_{n-1}(0)}{\Phi_n(0)}\right) + \frac{\Phi_{n-1}(0)}{\Phi_n(0)}|\Phi_n(0)|^2 \\
&= O(\delta^n) + \left(\frac{1}{\overline{b}} + O(\delta^n)\right)\left(|c|^2|b|^{2n} + O(|b|^{2n}\delta^n)\right) \\
&= \frac{|c|^2|b|^{2n}}{\overline{b}} + O(\delta^n).
\end{aligned}$$

□

3 Proofs of Main Results

Proof of Theorem 1.1. By definition and Lemma 2.1, we recall that

$$l_0 = 1, \qquad l_1 = (R_0)^{-1}. \tag{26}$$

From Lemma 2.3, we have that

$$l_2 \le \frac{1}{R_0 R_1} \le 1. \tag{27}$$

If $R_1 = \infty$, then we have that $l_2 = 0$. Hence, $R_1 = l_1/l_2$ as needed (recall that by convention $0/0 = \infty$). Therefore, we can assume that $R_1 < \infty$ which implies that $R_0 < \infty$. Using (26) and (27), we get

$$\frac{l_1}{l_2} \ge \frac{R_0 R_1}{R_0} = R_1. \tag{28}$$

Now, it rests to show that $R_1 \ge l_1/l_2$.

Notice that from (26) and (28),

$$\frac{l_0}{l_1} = R_0 \le R_1 \le \frac{l_1}{l_2}.$$

If $l_0/l_1 = l_1/l_2$, we would have equality throughout and, in particular, $R_1 = l_1/l_2$ as needed. Hence, it is sufficient to consider the case when $R_1 < \infty$ and $l_0/l_1 < l_1/l_2$, or what is the same

$$l_0 l_2 / l_1^2 < 1. \tag{29}$$

Our next objective is to show that given these conditions, the polynomials $Q^{\mu}_{n,1}$ for F are uniquely determined for all sufficiently large n and there exists a polynomial Q_1 of degree 1 such that

$$\limsup_{n\to\infty} \|Q^{\mu}_{n,1} - Q_1\|^{1/n} \le \frac{l_0 l_2}{l_1^2} < 1. \tag{30}$$

Suppose that this has been proved. Then, according to (9) and (10), we have that F has exactly one pole in $\mathbb{B}_{R_1}$ at λ, where λ is the zero of Q_1, and

$$\frac{|\lambda|}{R_1} \le \frac{l_0 l_2}{l_1^2}.$$

This further implies that $R_0 = |\lambda|$. As a result, we obtain

$$\frac{R_0}{R_1} \le \frac{l_0 l_2}{l_1^2} = \frac{R_0 l_2}{l_1}.$$

Cancelling out R_0 on both sides of this inequality, we get

$$R_1 \ge \frac{l_1}{l_2},$$

and we are done. Hence, in order to establish the validity of Theorem 1.1, it is necessary to show that the claim holds when (29) and $R_1 < \infty$ take place.

First, let us prove that $Q^{\mu}_{n,1}$ are unique and have degree 1 for all sufficiently large n. Recall from (13) that

$$\limsup_{n\to\infty} |\Delta_{n,1}|^{1/n} = \limsup_{n\to\infty} |\langle F, \varphi_n\rangle|^{1/n} = \frac{1}{R_0} = l_1. \tag{31}$$

By Lemma 2.4,

$$\begin{aligned}\Delta_{n+1,2} &= \begin{vmatrix} \langle zF, \varphi_{n+1}\rangle & \langle F, \varphi_{n+1}\rangle \\ \langle z^2F, \varphi_{n+1}\rangle & \langle zF, \varphi_{n+1}\rangle \end{vmatrix} = \begin{vmatrix} \langle F, \varphi_n\rangle + \delta_{n,2} & \langle F, \varphi_{n+1}\rangle \\ \langle F, \varphi_{n-1}\rangle + \delta_{n,3} & \langle F, \varphi_n\rangle + \delta_{n,2} \end{vmatrix} \\ &= \langle F, \varphi_n\rangle\langle F, \varphi_n\rangle - \langle F, \varphi_{n-1}\rangle\langle F, \varphi_{n+1}\rangle + 2\delta_{n,2}\langle F, \varphi_n\rangle + \delta_{n,2}^2 - \delta_{n,3}\langle F, \varphi_{n+1}\rangle, \end{aligned} \tag{32}$$

where

$$\limsup_{n\to\infty} |\delta_{n,2}|^{1/n} < \frac{1}{R_0} = l_1 \qquad \text{and} \qquad \limsup_{n\to\infty} |\delta_{n,3}|^{1/n} < \frac{1}{R_0} = l_1. \tag{33}$$

Applying

$$\limsup_{n\to\infty} |\Delta_{n+1,2}|^{1/n} = l_2 l_0 < l_1^2,$$

(31), and (33), it follows from (32) that

$$\limsup_{n\to\infty} |\Delta_{n,1}^2 - \Delta_{n+1,1}\Delta_{n-1,1}|^{1/n}$$

$$\le \max\{\limsup_{n\to\infty}|\Delta_{n+1,2}|^{1/n}, \limsup_{n\to\infty}|2\delta_{n,2}\langle F,\varphi_n\rangle|^{1/n}, \limsup_{n\to\infty}|\delta_{n,3}\langle F,\varphi_{n+1}\rangle|^{1/n}, \limsup_{n\to\infty}|\delta_{n,2}|^{2/n}\}$$

$$< l_1^2. \tag{34}$$

According to a result of Hadamard [9], namely any sequence of complex numbers $\{d_n\}$ such that

$$\limsup_{n\to\infty}|d_n|^{1/n} = 1, \qquad \limsup_{n\to\infty}|d_{n+1}d_{n-1}-d_n^2|^{1/n} < 1$$

has the regular limit

$$\lim_{n\to\infty}|d_n|^{1/n} = 1,$$

we set $d_n := \Delta_{n,1}/l_1^n$ and it follows from (31) and (34) that

$$\lim_{n\to\infty}|\Delta_{n,1}|^{1/n} = l_1 = \frac{1}{R_0} \neq 0. \tag{35}$$

This implies that there exists $n_0 \in \mathbb{N}$ such that $\langle F,\varphi_{n+1}\rangle = \Delta_{n+1,1} \neq 0$ for all $n \ge n_0$. Set

$$Q_{n,1}^{\mu}(z) := c_{n,0}z - c_{n,1}.$$

By definition, $\langle Q_{n,1}^{\mu}F,\varphi_{n+1}\rangle = 0$, equivalently

$$0 = c_{n,0}\langle zF,\varphi_{n+1}\rangle - c_{n,1}\langle F,\varphi_{n+1}\rangle.$$

It is clear that for all $n \ge n_0$, if $c_{n,0} = 0$, then $c_{n,1} = 0$. Because $Q_{n,1}^{\mu} \not\equiv 0$, $\deg Q_{n,1} = 1$ for all $n \ge n_0$. From this, since $Q_{n,1}$ are monic ($c_{n,0} = 1$), for all $n \ge n_0$,

$$c_{n,1} = \frac{\langle zF,\varphi_{n+1}\rangle}{\langle F,\varphi_{n+1}\rangle}$$

and $Q_{n,1}^{\mu}$ are unique.

Next, let us show (30). From Lemma 2.4 and a similar argument used in (16), we obtain

$$\Delta_{n+2,2} = \begin{vmatrix} \langle zF,\varphi_{n+2}\rangle & \langle F,\varphi_{n+2}\rangle \\ \langle z^2F,\varphi_{n+2}\rangle & \langle zF,\varphi_{n+2}\rangle \end{vmatrix} = \begin{vmatrix} \langle Q_1F,\varphi_{n+2}\rangle & \langle F,\varphi_{n+2}\rangle \\ \langle zQ_1F,\varphi_{n+2}\rangle & \langle zF,\varphi_{n+2}\rangle \end{vmatrix}$$

$$= \begin{vmatrix} \langle Q_1F,\varphi_{n+2}\rangle & \langle F,\varphi_{n+2}\rangle \\ \langle Q_1F,\varphi_{n+1}\rangle + \delta_{n+1,1} & \langle F,\varphi_{n+1}\rangle + \delta_{n+1,2} \end{vmatrix} = \begin{vmatrix} \langle Q_1F,\varphi_{n+2}\rangle & \langle F,\varphi_{n+2}\rangle \\ \langle Q_1F,\varphi_{n+1}\rangle & \langle F,\varphi_{n+1}\rangle \end{vmatrix} + \begin{vmatrix} \langle Q_1F,\varphi_{n+2}\rangle & \langle F,\varphi_{n+2}\rangle \\ \delta_{n+1,1} & \delta_{n+1,2} \end{vmatrix}$$

$$= \begin{vmatrix} \langle zF,\varphi_{n+2}\rangle & \langle F,\varphi_{n+2}\rangle \\ \langle zF,\varphi_{n+1}\rangle & \langle F,\varphi_{n+1}\rangle \end{vmatrix} + \begin{vmatrix} \langle Q_1F,\varphi_{n+2}\rangle & \langle F,\varphi_{n+2}\rangle \\ \delta_{n+1,1} & \delta_{n+1,2} \end{vmatrix}, \tag{36}$$

where the polynomial Q_1 is defined as in the proof of Lemma 2.3, and

$$\limsup_{n\to\infty}|\delta_{n+1,1}|^{1/n} < \frac{1}{R_1} \qquad \text{and} \qquad \limsup_{n\to\infty}|\delta_{n+1,2}|^{1/n} < \frac{1}{R_0}.$$

Then, from (36),

$$|c_{n+1,1} - c_{n,1}| = \left| \frac{\begin{vmatrix} \langle zF,\varphi_{n+2}\rangle & \langle F,\varphi_{n+2}\rangle \\ \langle zF,\varphi_{n+1}\rangle & \langle F,\varphi_{n+1}\rangle \end{vmatrix}}{\langle F,\varphi_{n+1}\rangle\langle F,\varphi_{n+2}\rangle} \right| \le \left| \frac{\Delta_{n+2,2}}{\langle F,\varphi_{n+1}\rangle\langle F,\varphi_{n+2}\rangle} \right| + \left| \frac{\begin{vmatrix} \langle Q_1F,\varphi_{n+2}\rangle & \langle F,\varphi_{n+2}\rangle \\ \delta_{n+1,1} & \delta_{n+1,2} \end{vmatrix}}{\langle F,\varphi_{n+1}\rangle\langle F,\varphi_{n+2}\rangle} \right|. \tag{37}$$

By (35),

$$\limsup_{n\to\infty}\left| \frac{\Delta_{n+2,2}}{\langle F,\varphi_{n+1}\rangle\langle F,\varphi_{n+2}\rangle} \right|^{1/n} = \frac{l_2 l_0}{l_1^2} < 1$$

and

$$b := \limsup_{n\to\infty}\left| \frac{\begin{vmatrix} \langle Q_1F,\varphi_{n+2}\rangle & \langle F,\varphi_{n+2}\rangle \\ \delta_{n+1,1} & \delta_{n+1,2} \end{vmatrix}}{\langle F,\varphi_{n+1}\rangle\langle F,\varphi_{n+2}\rangle} \right|^{1/n} < \frac{R_0}{R_1}. \tag{38}$$

Combining these and (37), we obtain

$$\limsup_{n\to\infty}|c_{n+1,1} - c_{n,1}|^{1/n} \le \max\left\{ \frac{l_2 l_0}{l_1^2}, b \right\} < 1$$

Therefore, $\sum_n |c_{n+1,1} - c_{n,1}|$ is convergent. Let $\lambda := \lim_{n\to\infty} c_{n,1}$ and

$$Q_1(z) := (z - \lambda).$$

Consequently,

$$\limsup_{n\to\infty} \|Q^{\mu}_{n,1} - Q_1\|^{1/n} = \limsup_{n\to\infty} |c_{n,1} - \lambda|^{1/n} \le \max\left\{\frac{l_2 l_0}{l_1^2}, b\right\}. \tag{39}$$

However, if

$$\max\left\{\frac{l_2 l_0}{l_1^2}, b\right\} = b,$$

then by (9), (10), (38), and (39),

$$\frac{R_0}{R_1} = \limsup_{n\to\infty} \|Q^{\mu}_{n,1} - Q_1\|^{1/n} \le b < \frac{R_0}{R_1}$$

which is impossible. Therefore,

$$\max\left\{\frac{l_2 l_0}{l_1^2}, b\right\} = \frac{l_2 l_0}{l_1^2},$$

which means that (39) implies (30). This completes the proof. □

Proof of Corollary 1.5. From the proof of Lemma 2.7, we recall that the equation (11) implies

$$\Phi_n(0) = c\overline{b}^n + O(|b|^n \delta^n), \qquad n \ge n_0,$$

where $c \ne 0$. By the Nevai-Totik Theorem (see, e.g., [10, Theorem 7.1.3]),

$$\frac{1}{R_0} = l_1 = \limsup_{n\to\infty} |\alpha_n|^{1/n} = \limsup_{n\to\infty} |\Phi_n(0)|^{1/n} = |b| < 1. \tag{40}$$

This means that the corresponding μ belongs to $\hat{\mathbf{S}}$.

From (36) replacing F by D^{-1} and $n+1$ by n, we obtain

$$\Delta_{n+1,2} = \begin{vmatrix} \langle zD^{-1}, \varphi_{n+1}\rangle & \langle D^{-1}, \varphi_{n+1}\rangle \\ \langle zD^{-1}, \varphi_n\rangle & \langle D^{-1}, \varphi_n\rangle \end{vmatrix} + O(\eta^n), \tag{41}$$

where $\eta < 1/(R_0 R_1)$. From Lemmas 2.5 and 2.7,

$$\begin{aligned}
\overline{\begin{vmatrix} \langle zD^{-1}, \varphi_{n+1}\rangle & \langle D^{-1}, \varphi_{n+1}\rangle \\ \langle zD^{-1}, \varphi_n\rangle & \langle D^{-1}, \varphi_n\rangle \end{vmatrix}} &= \begin{vmatrix} \varphi^{(1)}_{n+1}(0)D(0) + \varphi_{n+1}(0)D^{(1)}(0) & \varphi_{n+1}(0)D(0) \\ \varphi^{(1)}_{n}(0)D(0) + \varphi_{n}(0)D^{(1)}(0) & \varphi_{n}(0)D(0) \end{vmatrix} \\
&= (D(0))^2 \kappa_n \kappa_{n+1} (\Phi_{n+1}(0)\Phi_n(0)) \left(\frac{\Phi^{(1)}_{n+1}(0)}{\Phi_{n+1}(0)} - \frac{\Phi^{(1)}_{n}(0)}{\Phi_{n}(0)}\right) \\
&= (D(0))^2 \kappa_n \kappa_{n+1} (\Phi_{n+1}(0)\Phi_n(0)) \left(\frac{|c|^2 |b|^{2n}}{\overline{b}} + O(\delta^n)\right).
\end{aligned} \tag{42}$$

Using the equations (8), (40), and (42), the equation (41) implies that

$$l_2 \le \max\{|b|^4, |b|^2\delta, \eta\}.$$

If $\max\{|b|^4, |b|^2\delta, \eta\} = \eta$, then by Theorem 1.1,

$$R_1 = \frac{l_1}{l_2} \ge \frac{l_1}{\eta} > l_1 R_0 R_1 = R_1,$$

which is impossible. Hence,

$$R_1 = \frac{l_1}{l_2} \ge \frac{|b|}{\max\{|b|^4, |b|^2\delta\}} > \frac{1}{|b|}. \tag{43}$$

On the other side, (11) implies that $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = b$. From Theorem 1.4, b^{-1} is a singularity of D^{-1} and D^{-1} is holomorphic in $\mathbb{B}_{|b|^{-1}}$. By (43), b^{-1} is the only single pole of D^{-1} in a neighborhood of $\overline{\mathbb{B}_{|b|^{-1}}}$. This proves the first part of the corollary.

Now, let us prove the second part of our corollary. Assume further that $\delta < |b|^2$. Because $\max\{|b|^4, |b|^2\delta, \eta\} = |b|^4$ (from above, the maximum cannot be η), by (41) and (42),

$$\Delta_{n+1,2} = \left[\overline{(D(0))}^2 \kappa_n \kappa_{n+1} \overline{(\Phi_{n+1}(0)\Phi_n(0))} \left(\frac{|c|^2|b|^{2n}}{b}\right)\right] + O(\varepsilon^n), \tag{44}$$

where $\varepsilon < |b|^4$. From (11),

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim_{n\to\infty}\left|\frac{\Phi_{n+2}(0)}{\Phi_{n+1}(0)}\right| = |b|,$$

which further implies that

$$\lim_{n\to\infty}|\Phi_n(0)|^{1/n} = |b|. \tag{45}$$

Therefore, from (8) and (45),

$$\lim_{n\to\infty}\left|(\overline{D(0)})^2\kappa_n\kappa_{n+1}(\overline{\Phi_{n+1}(0)\Phi_n(0)})\left(\frac{|c|^2|b|^{2n}}{b}\right)\right|^{1/n} = |b|^4.$$

It is not difficult to show that this and (44) imply that

$$l_2 = \lim_{n\to\infty}|\Delta_{n+1,2}|^{1/n} = |b|^4.$$

Again, from this, and the equations (3) and (40), we obtain

$$R_1 = \frac{l_1}{l_2} = \frac{|b|}{|b|^4} = \frac{1}{|b|^3}.$$

This completes the proof. □

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