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A Simple Recipe for Modelling a $d-$cube by Lissajous curves

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Abstract

We give a simple recipe for Lissajous curves that for (certain) numerical purposes can serve as a proxy for the cube $[-1,1]^d$.

For $\mathbf{a} \in \mathbb{Z}_{>0}^d$ we let

$$\ell_{\mathbf{a}}(t) := (\cos(a_1 t), \cos(a_2 t), \cdots, \cos(a_d t)),\ t \in \mathbb{R}. \tag{1}$$

denote the associated Lissajous curve with *frequencies* $a_1, \cdots, a_d$. We note that such curves are given by the fundamental parameter interval $t \in [0, \pi]$.

The recent articles [2] and [3] discuss the use of such Lissajous curves as a proxy for the cube $[-1,1]^d$, for the purposes of quadrature, polynomial approximation of a function $f \in C[-1,1]^d$ and so-called hyperinterpolation. An emphasis of these articles is on the optimality of the fequencies $\mathbf{a}$. In this paper we give a simple recipe for choosing the frequencies that, although not optimal, can be used for all of the above purposes. We remark that a reader interested in this topic might also consult the very general results of Dencker and Erb [6].

Recipe: Let $n_1, n_2, \cdots, n_d \in \mathbb{Z}_{>0}$ be pairwise co-prime positive integers and let $N := \prod_{i=1}^d n_i$. We let

$$N_i := \frac{N}{n_i},\ 1 \le i \le d \tag{2}$$

and use the notation

$$\mathbf{a_n} := (N_1, N_2, \cdots, N_d)$$

to denote the d-tuple of such frequencies. □

Example. For $d = 2$ and $n \in \mathbb{Z}_{>0}$, the choice of $n_1 = n + 1$ and $n_2 = n$ results in the frequencies $\mathbf{a_n} = (n, n + 1)$, i.e., those of the underlying curve for the Padova points (cf. [1]). □

Below we show the plots of two 3d Lissajous curves. The one on the left is chosen according to the recipe with $n_1 = 3$, $n_2 = 4$ and $n_3 = 5$. The one on the right with the given frequencies. One sees that the first curve is well-distributed within $[-1,1]^3$ while the second exhibits a "concentration" phenomenom. Care must indeed be taken with the choice of the frequncies!

We now proceed to give the properties of Lissajous curves selected according to our Recipe. We use the notation $K := [-1,1]^d$.

Proposition 0.1. *The Lissajous curves $\ell_{\mathbf{a_n}}(t)$ are well positioned with respect to the Dubiner distance. Specifically, for every $\mathbf{x} \in K$*

$$\min_{0 \le t \le \pi} d_K(\mathbf{x}, \ell_{\mathbf{a_n}}(t)) \le \pi \frac{1}{\min_{1 \le i \le d} n_i}.$$

Here $d_K(\mathbf{x}, \mathbf{y})$ is the Dubiner distance

$$\begin{aligned} d_K(\mathbf{x}, \mathbf{y}) &:= \sup\{\frac{1}{deg(p)} |\cos^{-1}(p(\mathbf{y})) - \cos^{-1}(p(\mathbf{x}))| \,:\, deg(p) \ge 1,\ ||p||_K \le 1\} \\ &= \max_{1 \le j \le d} |\cos^{-1}(y_j) - \cos^{-1}(x_j)| \end{aligned}$$

as discussed and shown in [4, 5].

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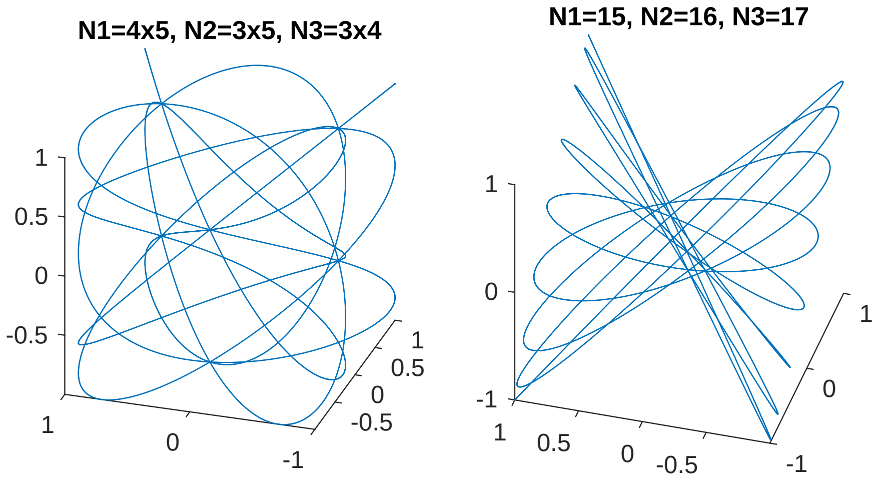

Figure 1: Two Lissajous Curves

Proof. Write $\mathbf{x} \in [-1,1]^d$ as $\mathbf{x} = \cos(\boldsymbol{\theta})$, $\theta_j \in [0,\pi]$, $1 \le j \le d$. Let $m_i \in \mathbb{Z}_{\ge 0}$, $1 \le i \le d$, be such that

$$\left|\frac{2\pi m_i}{n_i} - \theta_i\right| \le \frac{\pi}{n_i}, \ \frac{2\pi m_i}{n_i} \in [0,\pi], \ 1 \le i \le d.$$

Note that then $m_i < n_i$. Since by assumption the n_i are co-prime, the Chinese Remainder Theorem guarantees the existence of $m \in \mathbb{Z}$ such that

$$m \equiv m_i \bmod n_i, \quad 1 \le i \le d.$$

We set $t := 2\pi m/N$. Then

$$\begin{aligned} N_i t &= \frac{N}{n_i}\left(\frac{2\pi m}{N}\right) \\ &= 2\pi \frac{m}{n_i} \\ &= 2\pi\left(\frac{m_i + k_i n_i}{n_i}\right) \text{ for some } k_i \in \mathbb{Z} \\ &= \frac{2\pi m_i}{n_i} + 2k_i\pi. \end{aligned}$$

Hence, for this value of t,

$$\begin{aligned} d_K(\mathbf{x}, \ell_{\mathbf{a_n}}(t)) &= \max_{1\le i\le d} |\cos^{-1}(\cos(N_i t)) - \cos^{-1}(x_j)| \\ &= \max_{1\le i\le d} |\cos^{-1}(\cos(\frac{2\pi m_i}{n_i} + 2k_i\pi)) - \cos^{-1}(x_j)| \\ &= \max_{1\le i\le d} |\cos^{-1}(\cos(\frac{2\pi m_i}{n_i})) - \cos^{-1}(x_j)| \\ &= \max_{1\le i\le d} |\frac{2\pi m_i}{n_i} - \theta_i| \\ &\le \max_{1\le i\le d} \frac{\pi}{n_i} \\ &= \pi \frac{1}{\min_{1\le i\le d} n_i}. \end{aligned}$$

Thus, substituting t by $t \bmod \pi$, if necessary, we have our result. □

Proposition 0.2. *Suppose that we have a sequence of indices $\mathbf{n}^{(n)} \in \mathbb{Z}^d_{>0}$ which satisfy the condition of the Recipe and are such that there is some constant $\alpha > 2$ such that*

$$\min_{1\le i\le d} n_i^{(n)} \ge \alpha n, \quad n = 1, 2, \cdots.$$

Then the collection of Lissajous curves $\{\ell_{\mathbf{a}_\mathbf{n}^{(n)}} : n = 1, 2, \cdots\}$ forms a Norming Set for polynomials on $K = [-1, 1]^d$, in that for all polynomials $p(\mathbf{x})$, setting $n = \deg(p)$,

$$\max_{\mathbf{x}\in K} |p(\mathbf{x})| \le \sec(\pi/\alpha) \max_{0\le t\le\pi} |p(\ell_{\mathbf{a}_\mathbf{n}^{(n)}}(t))|.$$

Proof. Assume for simplicity that $\|p\|_K = 1$ and let $\mathbf{x} \in K$ be a point such that $|p(\mathbf{x})| = 1$. Multiplying by -1 if necessary, we may assume that $p(\mathbf{x}) = 1$. By Proposition 0.1 there is a value of $t \in [0, \pi]$ such that for $\mathbf{y} := \ell_{\mathbf{n}^{(n)}}(t)$,

$$d_K(\mathbf{x}, \mathbf{y}) \le \frac{\pi}{\alpha n},$$

which implies that

$$\frac{1}{n} |\cos^{-1}(p(\mathbf{y})) - \cos^{-1}(p(\mathbf{x}))| \le \frac{\pi}{\alpha n}.$$

But, as $p(\mathbf{x}) = 1$, $\cos^{-1}(p(\mathbf{x})) = 0$ and so we have

$$\cos^{-1}(p(\mathbf{y})) \le \frac{\pi}{\alpha} < \frac{\pi}{2}.$$

Then, since the inverse cosine function is monotonically decreasing, we obtain

$$p(\mathbf{y}) \ge \cos(\pi/\alpha) > 0$$

and hence,

$$\|p\|_K = 1 \le \sec(\pi/\alpha) p(\mathbf{y}) \le \sec(\pi/\alpha) \max_{0\le t\le\pi} |p(\ell_{\mathbf{a}_\mathbf{n}^{(n)}}(t))|.$$

□

There is also a quadrature formula with respect to the product Chebyshev measure:

$$d\mu_K := \frac{1}{\pi^d} \prod_{j=1}^d \frac{1}{\sqrt{1 - x_j^2}} dx_j.$$

Proposition 0.3. *For the frequency tuple $\mathbf{n} \in \mathbb{Z}^d_{>0}$ satisfying the condition of the Recipe, let*

$$m := \min_{1\le i\ne j\le d} n_i + n_j.$$

Then for all polynomials $p(\mathbf{x})$ with $\deg(p) \le m - 1$,

$$\int_{[-1,1]^d} p(\mathbf{x}) d\mu_K = \frac{1}{\pi} \int_0^\pi p(\ell_{\mathbf{a}_\mathbf{n}}(t)) dt. \tag{3}$$

Proof. Proposition 1 of [3] shows that there is quadrature formula (3) if and only if

$$\nexists\, 0 \ne \mathbf{b} \in \mathbb{Z}^d, \ \sum_{i=1}^d |b_i| \le m$$

such that

$$\sum_{i=1}^d N_i b_i = 0,$$

i.e., there are no "small" solutions of the homogeneous linear diophantine equation $\sum_{i=1}^d N_i x_i = 0$.

Let us suppose then that for $\mathbf{b} \in \mathbb{Z}^d$, $\sum_{i=1}^d N_i b_i = 0$. We will show that then necessarily $\sum_{i=1}^d |b_i| \ge m$. To see this, first note that, by construction n_i divides evenly into N_j, for $j \ne i$ while $\gcd(n_i, N_i) = 1$. Then write, for each $1 \le i \le d$,

$$N_i b_i = -\sum_{j\ne i} N_j b_j.$$

Since n_i divides into the right it must also divide into the left and hence, as n_i and N_i are co-prime, b_i is divisible by n_i. Consequently, if $b_i \ne 0$, $|b_i| \ge n_i$, $1 \le i \le d$. Since clearly, at least two of the b_i are non-zero, we have

$$\sum_{i=1}^d |b_i| \ge \min_{1\le i\ne j\le d} n_i + n_j = m,$$

as claimed. □

Corollary 0.4. *If $\mathbf{n} \in \mathbb{Z}^d_{>0}$ is a tuple satisfying the condition of the Recipe and is such that $n_i \ge n$, $1 \le i \le d$, then there is a Quadrature Formula (3) for $m = 2n$.*

Proof. We need only note that then in Proposition 0.3 $m \ge 2n + 1$, as we can have at most one index $n_i = n$. □

We remark that once we have a quadrature rule with accuracy $2n$ we can discretize the univariate integral in (3) to obtain a discrete Hyperinterpolation formula. We refer the reader to [2, 3] for further details.

We also remark that if the n_i are all $O(n)$ then the frequencies N_i are all $O(n^{d-1})$ and this order is optimal to have a Quadrature Formula (3), as also discussed in [2, 3].

References

[1] L. Bos, M. Caliari, S. De Marchi, M. Vianello and Y. Xu, Bivariate Lagrange interpolation at the Padua points: the generating curve approach, J. Approx. Theory 143 (2006), 15–25.

[2] L. Bos, S. De Marchi and M. Vianello, Trivariate polynomial approximation on Lissajous curves, IMA J. Numer. Anal. 37 (2017), 519–541.

[3] L. Bos, S. De Marchi and M. Vianello, Polynomial approximation on Lissajous curves in the d-cube, Appl. Numer. Math., published online January 26, 2017 .

[4] L. Bos, N. Levenberg and S. Waldron, Metrics Associated to Multivariate Polynomial Inequalities, in Advances in Constructive Approximation, Vanderbilt 2003, M. Neamtu and E.B. Saff eds., Nashboro Press, 2004.

[5] L. Bos, N. Levenberg and S. Waldron, Pseudometrics, Distances and Multivariate Polynomial Inequalities, J. Approx. Theory 153 (2008), 80–96.

[6] P. Dencker and W. Erb, Multivariate polynomial interpolation on Lissajous-Chebyshev curves, arXiv:1511.04564v1.