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On simple approximations to simple curves [1]

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Abstract

We show that the property of a planar parametric curve of being simple, is preserved by an approximation when the curve is piecewise smooth and generalized regular, provided that the errors to the curve and its tangent vectors are sufficiently small. To this purpose, we provide also an explicit error threshold.

Keywords: piecewise smooth curve, simple curve, injectivity, generalized regularity, approximation, piecewise polynomial interpolation.

In this note we give a partial answer to the following basic question: when is the injectivity of a planar parametric curve, i.e., the property of being a *simple curve*, preserved by a general approximation method? We mean an approximation that is not explicity constructed to be shape preserving, such as are, for example, variation diminishing approximations in the field of computer aided geometric design [7].

The problem is relevant, for example, to the setting of moment computations over planar regions. Indeed, suppose that we have to compute the integral of a bivariate polynomial over a region whose boundary is a Jordan curve, and that we are able to give an accurate piecewise polynomial approximation of the boundary. Then, integrating the polynomial over the approximate region becomes trivial using Green's formula, provided that its boundary is still a simple curve, since any x-primitive (or y-primitive) becomes a piecewise polynomial univariate function along the curve. This idea has been used to construct algebraic cubature formulas in [11], by resorting to spline interpolation. But we may also consider using the recent software package `chebfun`, which can approximate curves at machine precison by piecewise Chebyshev interpolation in an efficient and automatic way (cf. [2, 9, 10]).

With no loss of generality, we consider curves parametrized on $[0, 1]$

$$P(t) = (x(t), y(t)) \ , \ \ t \in [0, 1] \tag{1}$$

where $P(\cdot)$ is continuous, and injective on $[0, 1]$ (*simple open curve*), or injective on $[0, 1)$ and $(0, 1]$ with $P(0) = P(1)$ (*simple closed curve*). Moreover, we assume that $P(\cdot)$ is *piecewise* C^1, i.e., there is at most a finite number of *breakpoints* $P(t_i)$ where $P'_+(t_i) := \lim_{t\to t_i^+} P'(t) \neq P'_-(t_i) := \lim_{t\to t_i^-} P'(t)$; for a closed curve, $P(0) = P(1)$ is considered a breakpoint if $P'_+(0) \neq P'_-(1)$.

For convenience, as is usual, we shall include global continuity in the notion of piecewise C^1 parametric curve. The space $PC^1([0, 1]; S)$, for brevity PC^1, of piecewise C^1 parametric curves on the partition of $[0, 1]$ generated by a fixed finite set of parameter breakpoints $S = \{t_i\}$, is endowed with the norm

$$\|P\|_{PC^1} := \max\left\{\|P\|_{L^\infty}, \|P'\|_{L^\infty}\right\} \tag{2}$$

where $\|Q\|_{L^\infty} := \max\{\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}\}$ for $Q(t) = (q_1(t), q_2(t))$ piecewise continuous.

Definition 1: A *singular point* is a point $P(t^*)$ such that $P'_+(t^*) = (0, 0)$ or $P'_-(t^*) = (0, 0)$. A *cusp* is a breakpoint $P(t_i)$ such that $P'_+(t_i) = -kP'_-(t_i)$ for some $k > 0$ (i.e., the left and right tangent vectors have opposite directions).

Definition 2: We say that a curve in $PC^1([0,1];S)$ is *generalized regular* if it has *no singular points* and *no cusps*.

Observe that, in the case when the curve has no breakpoints, generalized regularity coincides with classical regularity as defined, e.g., in [12], namely: the tangent vector is never the zero vector.

We can now state and prove the main result.

Theorem 1. *Let $P(t) = (x(t), y(t))$, $t \in [0,1]$, be a simple and generalized regular curve in $PC^1([0,1];S)$. Then, any (closed if P is closed) approximating curve $\tilde{P}(t) = (\tilde{x}(t), \tilde{y}(t))$ in $PC^1([0,1];S)$ is simple itself, provided that the error $\|P - \tilde{P}\|_{PC^1}$ is sufficiently small.*

We give two proofs of the theorem. The first is essentially qualitative, working by contradiction with some typical arguments of differential topology (see, e.g., [8, Thm. 1.7]). The second is constructive, giving an estimate (even though not always easy to apply in practice) of the radius of a sufficient approximation neighborhhood.

In order to treat properly the breakpoints, we shall resort to some basic results of nonsmooth analysis. We recall that the Clarke generalized gradient of a piecewise C^1 univariate function f at a point t, say $\partial f(t)$, is the convex hull of the directional derivatives at such a point

$$\partial f(t) = co\{f'_+(t), f'_-(t)\} \tag{3}$$

(cf. [6, Ch. 2]), that is the interval with the left and right derivatives as extrema, that reduces to one point when the function is differentiable in the classical sense. Then, for a planar piecewise C^1 parametric curve, we may define at each point $P(t)$ a generalized tangent vector as the Cartesian product of the generalized gradients,

$$\partial P(t) := \partial x(t) \times \partial y(t)$$

which is a Cartesian rectangle at the breakpoints (possibly degenerating into a horizontal or vertical segment).

First Proof. Assume that the conclusion of the theorem is false. Then, there exists a sequence of PC^1 curves, say $\{P_n\}$, with $\lim \|P_n - P\|_{PC^1} = 0$, that are not simple, i.e., for every n there exist $u_n, v_n \in [0,1)$, or $u_n, v_n \in (0,1]$, $u_n \neq v_n$, such that

$$P_n(u_n) = P_n(v_n)\ .$$

By resorting possibly to subsequences, we may assume that $\lim u_n = u$ and $\lim v_n = v$ exist; since $\lim \|P_n - P\|_{L^\infty} = 0$, we have that $\lim P_n(u_n) = P(u) = P(v) = \lim P_n(v_n)$. If $P(u)$ is a breakpoint, since it is not singular and it is not a cusp, the angle between the left and right tangent vectors is less than π. By a suitable rotation of the coordinates (which clearly doesn't affect the property of a curve of being simple or not), we may assume that the left and right tangent vectors are both in the upper (or lower) half-plane, i.e., that $(0,0) \notin \partial P(u)$ either when $P(u)$ is a smooth point or when it is a breakpoint.

Now, if the curve is open, we have that $u = v$, whereas if the curve is closed we may have either $u = v$, or $u = 1$ and $v = 0$, or $u = 0$ and $v = 1$. Consider without loss of generality the case that either $u = v$ or $u = 1$ and $v = 0$, and define $\hat{v}_n = v_n$ if $v \neq 0$, $\hat{v}_n = v_n + 1$ if $u = 1$ and $v = 0$ (i.e., $\lim \hat{v}_n = u$). Extend P (and P_n) to $[0, 2]$ as $\hat{P}(t) = P(t)$, $t \in [0, 1]$ and $\hat{P}(t) = P(t - 1)$, $t \in (1, 2]$ (the extension being still piecewise C^1). Applying the Hermite-Genocchi formula to the first divided differences (cf., e.g., [1]), we can write

$$(0,0) \equiv \frac{\hat{P}_n(u_n) - \hat{P}_n(\hat{v}_n)}{u_n - v_n} = \int_0^1 \hat{P}_n'(tu_n + (1-t)\hat{v}_n)\, dt$$

$$= \int_0^1 \hat{P}'(tu_n + (1-t)\hat{v}_n)\, dt + E_n = \frac{\hat{P}(u_n) - \hat{P}(\hat{v}_n)}{u_n - \hat{v}_n} + E_n$$

where the vector sequence E_n tends to zero since $\|E_n\|_\infty \leq \|P_n' - P'\|_{L^\infty}$. Now, if $P(u)$ is not a breakpoint, it is not singular, in view of the generalized regularity condition: taking the limit as $n \to \infty$ we get the contradiction $\hat{P}'(u) = P'(u) = (0, 0)$. If $P(u)$ is a breakpoint, by the mean-value theorem for generalized gradients (cf. [6, Thm. 2.3.7]) we can write

$$(0,0) \equiv \frac{\hat{P}(u_n) - \hat{P}(\hat{v}_n)}{u_n - \hat{v}_n} + E_n \in \partial\hat{x}(\tau_n) \times \partial\hat{y}(\sigma_n) + E_n \tag{4}$$

where τ_n, σ_n belong to the open interval with endpoints u_n and $\hat{v}_n$. As $n \to \infty$ in (4) we get the contradiction $(0, 0) \in \partial\hat{P}(u) = \partial P(u)$. $\square$

Second Proof. We begin with the case of an open curve. The key observation is that the parametrization is injective if and only if

$$g(t_1, t_2) := |P[t_1, t_2]|^2 = \frac{|P(t_2) - P(t_1)|^2}{|t_2 - t_1|^2} = (x[t_1, t_2])^2 + (y[t_1, t_2])^2 > 0 \tag{5}$$

for $(t_1, t_2) \in [0, 1]^2$, $t_1 \neq t_2$ (where $f[t_1, t_2]$ denotes the first divided difference of a function f, and $|V|$ is the Euclidean norm of a vector $V \in \mathbb{R}^2$). Notice that the function g is defined and continuous on $[0, 1]^2$ off the diagonal. We shall now show that under our assumptions g can be extended to the whole square $[0, 1]^2$ and is bounded away from zero.

Indeed, by the generalized mean-value thorem for generalized gradients (cf. [6, Thm. 2.3.7]), there exist $\xi, \eta \in (t_1, t_2)$ such that $x[t_1, t_2] \in \partial x(\xi)$ and $y[t_1, t_2] \in \partial y(\eta)$. Now, fix $\tau \in [0, 1]$. Since the curve is piecewise C^1, it is easy to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x[t_1, t_2] \in \partial x(\tau) + [-\varepsilon, \varepsilon]$ and $y[t_1, t_2] \in \partial y(\tau) + [-\varepsilon, \varepsilon]$ for $|t_1 - \tau| \leq \delta$ and $|t_2 - \tau| \leq \delta$. Moreover, since the function g is invariant under rotations of the coordinates (x, y), and $P(\tau)$ is not singular and is not a cusp in view of the generalized regularity condition, reasoning as in the first proof it is not restrictive to assume (up to a suitable rotation of the coordinates when $P(\tau)$ is a breakpoint) that $(0, 0) \notin \partial P(\tau)$, i.e., at least one of the intervals $\partial x(\tau)$, $\partial y(\tau)$ does not contain zero. Assume, for simplicity, that $0 \notin \partial x(\tau)$, thus $(x[t_1, t_2])^2 \in co\{(x_+'(\tau) \pm \varepsilon)^2, (x_-'(\tau) \pm \varepsilon)^2\}$. It follows that we can extend the definition of g to the diagonal preserving positivity by setting

$$g(\tau, \tau) := \liminf_{(t_1,t_2)\to(\tau,\tau)} g(t_1, t_2) > 0 \tag{6}$$

since $g(\tau, \tau) \geq \min\{(x_+'(\tau))^2, (x_-'(\tau))^2\} > 0$, and hence we obtain an everywhere positive and lower semicontinuous function on $[0, 1]^2$. Then we have

$$m := \min_{(t_1,t_2)\in[0,1]^2} g(t_1, t_2) > 0 \tag{7}$$

by the extremum theorem for semicontinuous functions; see, e.g., [5].

Consider now

$$\tilde{g}(t_1,t_2) := \left|\tilde{P}[t_1,t_2]\right|^2 \geq |g(t_1,t_2) - e(t_1,t_2)| \tag{8}$$

for $(t_1,t_2) \in [0,1]^2$, $t_1 \neq t_2$, where

$$e(t_1,t_2) = \left|(\tilde{x}[t_1,t_2])^2 - (x[t_1,t_2])^2 + (\tilde{y}[t_1,t_2])^2 - (y[t_1,t_2])^2\right|$$

and the estimate

$$\begin{aligned} |(\tilde{x}[t_1,t_2])^2 - (x[t_1,t_2])^2| &= |(\tilde{x}[t_1,t_2] - x[t_1,t_2])(\tilde{x}[t_1,t_2] + x[t_1,t_2])| \\ &\leq |\tilde{x}[t_1,t_2] - x[t_1,t_2]|^2 + 2|x[t_1,t_2]|\,|\tilde{x}[t_1,t_2] - x[t_1,t_2]| \, . \end{aligned}$$

Using the Hermite-Genocchi formula we obtain the bounds

$$|\tilde{x}[t_1,t_2] - x[t_1,t_2]| \leq \int_0^1 |(\tilde{x}' - x')(st_1 + (1-s)t_2)|\, ds \leq \|\tilde{x}' - x'\|_{L^\infty}$$

and

$$|x[t_1,t_2]| \leq \int_0^1 |x'(st_1 + (1-s)t_2)|\, ds \leq \|x'\|_{L^\infty}$$

Proceeding similarly with the y variables, we get finally the bound

$$|e(t_1,t_2)| \leq 2\varepsilon^2 + 4c\varepsilon \; , \;\; \varepsilon := \|P - \tilde{P}\|_{PC^1} \; , \;\; c := \|P\|_{PC^1}$$

and thus, solving the inequality $2\varepsilon^2 + 4c\varepsilon < m$, in view of (7)-(8) we can ensure that $\tilde{g}(t_1,t_2) > 0$, i.e., injectivity of $\tilde{P}$, as soon as the inequality

$$\varepsilon < \sqrt{c^2 + m/2} - c \tag{9}$$

is satisfied.

We consider now the case of a closed curve. First, we extend P (and $\tilde{P}$) to $[0,2]$ by setting $\hat{P}(t) = P(t)$, $t \in [0,1]$ and $\hat{P}(t) = P(t-1)$, $t \in (1,2]$, and we define

$$g(t_1,t_2) := \max_{j,k \in \{0,1\}} \left|\hat{P}[t_1 + j, t_2 + k]\right|^2 = \max_{j,k \in \{0,1\}} \frac{\left|\hat{P}(t_2 + k) - \hat{P}(t_1 + j)\right|^2}{|t_2 + k - (t_1 + j)|^2} \tag{10}$$

for $(t_1,t_2) \in [0,1]^2 \setminus J$, where $J := \{(t_1,t_2) : \, t_1 = t_2\} \cup \{(0,1),(1,0)\}$, which is a positive and continuous function. Reasoning as above with generalized gradients, we can show that g can be extended to the whole $[0,1]^2$ by

$$g(\sigma,\tau) := \liminf_{(t_1,t_2) \to (\sigma,\tau)} g(t_1,t_2) > 0$$

for every $(\sigma,\tau) \in J$. In such a way g becomes a positive and lower semicontinuous function on $[0,1]^2$, and thus has a positive minimum, say m. The rest of the proof proceeds as above via the Hermite-Genocchi formula, leading to the estimate (9) of the error threshold which ensures that $\tilde{P}$ is a simple curve. □

Remark 1. The kind of approximation in Theorem 1 is completely general. Indeed, it is only required that not only the curve, but also its tangent vectors are (piecewise) approximated. This means that the result can be applied for example to piecewise polynomial or trigonometric approximation, under suitable smoothness assumptions, and in general to any approximation process which guarantees convergence in PC^1 (the only constraint being that the approximating curve is closed if the original one is, a property that is guaranteed by any interpolation method including the endpoints of the parameter interval).

It is worth mentioning here two popular polynomial-based interpolation methods, namely spline interpolation (cf., e.g., [3]), and (piecewise) Chebyshev(-Lobatto) interpolation (which is at the core of the recent software package `chebfun`, cf. [2, 9]). Using for example complete cubic spline interpolation with maximum stepsize h in each subinterval of smoothness, we get convergence in PC^1 of order $\mathcal{O}(h^3)$, as soon as the curve is piecewise C^4, by a classical result [3, Ch. 5]. On the other hand, piecewise Chebyshev-Lobatto interpolation of degree n guarantees convergence in PC^1 for functions that are piecewise $C^{3+\alpha}$, $\alpha > 0$, with order $\mathcal{O}(n^{-\alpha})$, in view of classical results concerning convergence of such process in Sobolev spaces; cf., e.g., [4, §5.5.3].

Example: Consider the case of the unit circle, parametrized by the angle as $P(t) = (\cos 2\pi t, \sin 2\pi t)$, $t \in [0, 1]$. It is immediate to see that $g(t_1, t_2)$ in (10) is $4\pi^2$ times the squared ratio of the lengths of the chord $P(t_2)-P(t_1)$ and of the corresponding shortest circle arc. Moreover, extension of g to the diagonal gives the squared Euclidean norm of the tangent vector, $g(t, t) = |P'(t)|^2 = 4\pi^2$. Hence, in (9) we have $c = \|P'\|_\infty = 2\pi$ and $m = 4\pi^2 \cdot 4/\pi^2 = 16$ (the minimal chord/arc ratio being $2/\pi$), that is the approximating curve is simple as soon as

$$\varepsilon = \|P - \tilde{P}\|_{C^1} < \sqrt{4\pi^2 + 8} - 2\pi = 0.607...$$

This entails that, for example, if we approximate the circle by a complete cubic spline interpolant $\tilde{P}$ with constant stepsize h, by the classical estimate $\|P - \tilde{P}\|_{C^1} \le h^3 \|P^{(4)}\|_\infty/24$ (cf. [3, Ch. 5]), the spline curve will be surely a Jordan curve as soon as $h < h_0 = \sqrt[3]{24(\sqrt{4\pi^2 + 8} - 2\pi)/(2\pi)^4} = 0.210...$, i.e., if we use at least $[1/h_0] + 1 = 5$ equispaced interpolation points.

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