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The Weak TGA in p-Banach spaces

Pablo M. Berná[a] · Andrea García[a,b]

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Abstract

In this paper, we revisit the Weak Thresholding Greedy Algorithm (WTGA) and use it in the context of p-Banach spaces with two main objectives: to provide a natural and constructive way to define the algorithm, and to extend a series of results from Banach spaces over the field of real numbers to the case of p-Banach spaces over the complex field, by studying the best approximation of the WTGA in relation to approximation using polynomials with constant coefficients.

1 Introduction

Let us consider a p-Banach space $\mathbb{X}$ with $0 < p \leq 1$ over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, where a p-Banach space is a complete p-normed space, and a p-norm is a mapping $\|\cdot\| : \mathbb{X} \to [0, +\infty)$ satisfying the following conditions:

1. $\|f\| > 0$ for all $f \neq 0$;
2. $\|\lambda f\| = |\lambda| \, \|f\|$ for all $\lambda \in \mathbb{F}$ and $f \in \mathbb{X}$;
3. $\|f + g\|^p \leq \|f\|^p + \|g\|^p$ for all $f, g \in \mathbb{X}$.

Given now a p-Banach space with $0 < p \leq 1$, a Schauder basis $\mathcal{B} = (e_n)_{n\in\mathbb{N}}$ is any collection of vectors in $\mathbb{X}$ such that for every $f \in \mathbb{X}$, there is a unique sequence of scalars $(a_n(f))_{n\in\mathbb{N}}$ such that

$$\lim_{m\to+\infty} \left\| f - \sum_{j=1}^{m} a_j(f)\, e_j \right\| = 0.$$

That is, for every $f \in \mathbb{X}$, we have $f = \sum_{j=1}^{\infty} a_j(f)\, e_j$. Associated to a Schauder basis, we have the collection of biorthogonal functionals $\mathcal{B}^* = (e_n^*)_{n\in\mathbb{N}} \subseteq \mathbb{X}^*$ (called the dual basis) such that $e_n^*(e_m) = \delta_{n,m}$. Using these functionals,

$$e_j^*\left(\sum_{n=1}^{\infty} a_n(f)\, e_n \right) = a_j(f),$$

so we have the following identification

$$f = \sum_{n=1}^{\infty} e_n^*(f)\, e_n.$$

Given $m \in \mathbb{N}$ and $f \in \mathbb{X}$, $S_m(f)$ is the mth partial sum of f:

$$S_m[\mathcal{B}, \mathbb{X}](f) = S_m(f) := \sum_{n=1}^{m} e_n^*(f)\, e_n.$$

Using these partial sums, it is well known that $\mathcal{B}$ is a Schauder basis if and only if the partial sums are uniformly bounded (see [8, Proposition A.13, page 604]), that is,

$$\sup_m \|S_m\| < \infty.$$

We also assume that $\mathcal{B}$ is semi-normalized, that is,

$$0 < \inf_n \|e_n\| \leq \sup_n \|e_n\| < \infty.$$

[a] Departamento de Matemáticas, CUNEF Universidad, Madrid 28040, Spain
[b] Universidad San Pablo CEU, CEU Universities, Madrid 28003, Spain

Although we consider here the case of a p-Banach space, thanks to the theory developed by Aoki ([2]) and Rolewicz ([9]), we can extend the results we study to the case of quasi-Banach spaces.

In 1999, in [7], the authors introduced the *Thresholding Greedy Algorithm* $(G_m)_{m=1}^{\infty}$ (TGA, for short). To define the algorithm, we take $f \in \mathbb{X}$ and $m \in \mathbb{N}$. A *greedy sum* of order m is given by

$$G_m[\mathcal{B},\mathbb{X}](f) = G_m(f) := \sum_{n\in G} e_n^*(f)\, e_n,$$

where $G \subset \mathbb{N}$ is a finite set, called *a greedy set*, satisfying $|G| = m$ and

$$\min_{n\in G} |e_n^*(f)| \;\geq\; \max_{n\notin G} |e_n^*(f)|.$$

The collection $(G_m)_{m\in\mathbb{N}}$ is referred to as the *Thresholding Greedy Algorithm*.

Of course, there is a close connection between the greedy sums and the projection operator. Recall that for any finite set $A \subset \mathbb{N}$ and any $f \in \mathbb{X}$, the *projection operator* is defined as

$$P_A[\mathcal{B},\mathbb{X}](f) = P_A(f) := \sum_{n\in A} e_n^*(f)\, e_n.$$

Hence, given a finite greedy set G of $f \in \mathbb{X}$, we have

$$G_m(f) = P_G(f).$$

In this paper, we work with the *Weak Thresholding Greedy Algorithm* (WTGA, for short), which is a “relaxed” version of the TGA introduced by V. N. Temlyakov in [10] (and studied more recently in [3]).

Given a parameter $t \in (0,1]$, a set G is a t-greedy set of cardinality m if $|G| = m$ and

$$\min\left\{ |e_j^*(f)| : j \in B \right\} \;\geq\; t\, \max\left\{ |e_n^*(f)| : n \notin G \right\}.$$

We write $G \in \mathcal{G}(f,m,t)$. Hence, a (m,t)-greedy sum is the mapping

$$G_m^t[\mathcal{B},\mathbb{X}](f) = G_m^t(f) := \sum_{i\in G} e_i^*(f) e_i, \text{ for some } G \in \mathcal{G}(f,m,t).$$

Comparing the TGA with the WTGA for $t < 1$, we have a little bit more flexibility in building approximants, but the question is: how much does this flexibility affect the efficiency of the algorithm? Well, this effect is reflected in a multiplicative constant as in results found in [11] for the case of Banach spaces.

In this paper, we pursue the following three objectives:

1. Since there is no unique way to construct the (m,t)-greedy sums, we aim to provide a natural and constructive definition of them (Section 3).
2. To study the classical characterization of t-greedy bases in terms of properties such as super-democracy and unconditionality (Section 4).
3. To extend a series of results related to best approximation using polynomials with constan coefficients (Section 5).

2 Some known results in p-Banach spaces

In the context of p-Banach spaces, we do not have convexity, but we can make use of the so-called p-convexity, which we will employ in some of the results to be analyzed.

Lemma 2.1. *[1, Corollary 1.3] Let $\mathbb{X}$ be a p-Banach space for some $0 < p \leq 1$ and let $\mathcal{B}$ be a semi-normalized Schauder basis for $\mathbb{X}$. Then, for every sequence of scalars $(a_n)_{n\in A}$ with A a finite set of indices with $|a_j| \leq 1$ for every $j \in A$ and $g \in \mathbb{X}$,*

$$\left\| g + \sum_{j\in A} a_j e_j \right\| \leq A_p \sup \left\{ \left\| g + \sum_{n\in A} \varepsilon_n e_n \right\| : |\varepsilon_n| = 1\ \forall n \in A \right\},$$

where $A_p = (2^p - 1)^{1/p}$.

Using this p-convexity, we can also talk about unconditional bases.

Definition 2.1. Let $\mathcal{B}$ be a semi-normalized Schauder basis in a p-Banach space with $0 < p \leq 1$. We say that $\mathcal{B}$ is suppression-unconditional if there is $K_1 > 0$ such that for every finite set A and $f \in \mathbb{X}$,

$$\|f - P_A(f)\| \leq K_1 \|f\|. \tag{1}$$

Also, the basis is lattice unconditional if there is $K_2 > 0$ such that for every sequence $(c_n)_n$ verifying $\|(c_n)_n\|_\infty \leq 1$,

$$\left\| \sum_{n=1}^{\infty} c_n a_n e_n \right\| \leq K_2 \left\| \sum_{n=1}^{\infty} a_n e_n \right\|. \tag{2}$$

We denote by $K_s[\mathcal{B},\mathbb{X}] = K_s$ the least constant satisfying (1) and $K_u[\mathcal{B},\mathbb{X}] = K_u$ the least constant satisfying (2) and we have the following behaviour using the p-convexity (see [1, Proposition 1.7]):

$$K_s \le K_u \le B_p K_s, \tag{3}$$

where $B_p = 2^{1/p}A_p$ if $\mathbb{F} = \mathbb{R}$ and $B_p = 4^{1/p}A_p$ if $\mathbb{F} = \mathbb{C}$.

3 WTGA: a natural construction

As we can see in the definition of the WTGA, we can have different (m,t)-greedy sums for the same m and t. A simple example to show that for the WTGA we have the same problem is the following one.

Example 3.1. Let us illustrate that the construction of $G_m^t(f)$ may not be unique if no specific tie-breaking rule is imposed when several coefficients have equal or comparable magnitudes.

Consider $\mathbb{X} = \mathbb{R}^4$ with its canonical basis $\mathcal{B} = (e_n)_{n=1}^4$, and let $f = (1,1,1,1) = \sum_{n=1}^4 e_n$. Then $a_n = |e_n^*(f)| = 1$ for all n. For any $m \in \{1,2,3\}$ and $t \in (0,1]$, every subset $A \subset \{1,2,3,4\}$ with $|A| = m$ satisfies

$$\min_{j\in A} a_j = 1 \ge t \max_{n\notin A} a_n = t.$$

Hence, all such subsets A are admissible and thus $G_m^t(f) = \sum_{i\in A} e_i$ can be chosen in many different ways (for instance, $(1,1,0,0)$, $(1,0,1,0)$, or $(0,1,1,0)$ for $m = 2$). Only when we impose the natural-order tie-breaking rule of the iterative construction do we obtain a canonical choice, namely $\mathcal{N}_m^t(f) = \{1,2,\dots,m\}$.

Definition 3.1 (Iterative construction of the t-greedy sum with natural tie-break)**.** Let $\mathcal{B} = (e_n)_{n\in\mathbb{N}}$ be a semi-normalized Schauder basis of a p-Banach space $\mathbb{X}$ with $0 < p \le 1$ and coordinate functionals (e_n^*), and fix $t \in (0,1]$. For $f \in \mathbb{X}$ set $a_n := |e_n^*(f)|$ for $n \in \mathbb{N}$. We define the index set $\mathcal{N}_m^t(f)$ iteratively as follows.

Initialization: $S_0 := \emptyset$.

Step $k = 1,2,\dots,m$**:**

1. $M_k := \max\{a_n : n \notin S_{k-1}\}$.
2. $J_k := \{n \notin S_{k-1} : a_n \ge t\, M_k\}$.
3. $i_k := \min J_k$ (tie-break by the *natural order* of the indices).
4. $S_k := S_{k-1} \cup \{i_k\}$.

After m steps we set

$$\mathcal{N}_m^t(f) := S_m, \qquad G_m^t[\mathcal{B},\mathbb{X}](f) = G_m^t(f) := \sum_{n\in S_m} e_n^*(f)\, e_n.$$

Example 3.2. Let $\mathbb{X} = \mathbb{R}^4$ with the canonical basis $\mathcal{B} = (e_n)_{n=1}^4$ and consider

$$f = (1,\ 0.2,\ 0.9,\ 0.1), \qquad t = 0.7.$$

Then

$$a_1 = 1, \quad a_2 = 0.2, \quad a_3 = 0.9, \quad a_4 = 0.1.$$

Step 1. $S_0 = \emptyset$. The maximal coefficient is

$$M_1 = 1, \quad J_1 = \{n : a_n \ge 0.7 M_1\} = \{1,3\},$$

since $a_3 = 0.9 \ge 0.7$. By the natural tie-break, $i_1 = 1$. Then

$$S_1 = \mathcal{N}_1^t(f) = \{1\}, \qquad G_1^t(f) = e_1.$$

Step 2. Now $S_1 = \{1\}$. We compute

$$M_2 = \max\{a_n : n \notin S_1\} = 0.9,$$

and

$$J_2 = \{n \notin S_1 : a_n \ge 0.7\, M_2\} = \{n : a_n \ge 0.63\}.$$

Then $a_2 = 0.2$, $a_3 = 0.9$, $a_4 = 0.1$, so $J_2 = \{3\}$ and $i_2 = 3$. Hence

$$S_2 = \mathcal{N}_2^t(f) = \{1,3\}, \qquad G_2^t(f) = e_1 + 0.9 e_3.$$

Step 3. Now $S_2 = \{1, 3\}$. Then

$$M_3 = \max\{a_n : n \notin S_2\} = 0.2, \qquad J_3 = \{n \notin S_2 : a_n \geq 0.7 \times 0.2 = 0.14\} = \{2, 4\}.$$

The natural tie-break gives $i_3 = 2$. Thus

$$S_3 = \mathcal{N}_3^t(f) = \{1, 2, 3\}.$$

Step 4. Finally, with $S_3 = \{1, 2, 3\}$,

$$M_4 = \max\{a_n : n \notin S_3\} = 0.1, \qquad J_4 = \{n \notin S_3 : a_n \geq 0.7 \times 0.1 = 0.07\} = \{4\}.$$

Hence $i_4 = 4$ and

$$S_4 = \mathcal{N}_4^t(f) = \{1, 2, 3, 4\}, \qquad G_4^t(f) = f.$$

4 t-greedy bases

One of the most desirable properties when working with an approximation algorithm $(T_m)_m$ is that the algorithm produces the best approximation; that is, that there exists a positive constant such that, for all $f \in \mathbb{X}$ and $m \in \mathbb{N}$,

$$\|f - T_m(f)\| \approx \sigma_m(f),$$

where

$$\sigma_m[\mathcal{B}, \mathbb{X}](f) = \sigma_m(f) := \inf_{y : |\operatorname{supp}(y)| \leq m} \|f - y\|,$$

and, given $g \in \mathbb{X}$, $\operatorname{supp}(g) = \{n \in \mathbb{N} : e_n^*(g) \neq 0\}$.

In the context of the TGA, in [7], we find *greedy* bases, where a basis is greedy if there is $C > 0$ such that for every element $f \in \mathbb{X}$,

$$\|f - P_G(f)\| \leq C\sigma_{|G|}(f),$$

whenever G is a finite greedy set of f. In that case, greedy bases are characterized as those bases that are democratic and unconditional (see the following papers to know more about greedy-like bases [6, 7, 12]).

Definition 4.1. Let $\mathcal{B}$ be a semi-normalized Schauder basis in a p-Banach space $\mathbb{X}$ with $0 < p \leq 1$. We say that the basis is super-democratic if there is $C > 0$ such that

$$\left\| \sum_{n \in A} \varepsilon_n e_n \right\| \leq C \left\| \sum_{j \in B} \eta_j e_j \right\|, \tag{4}$$

for every pair of finite sets A and B of the same cardinality and $|\varepsilon_n| = |\eta_j| = 1$ for every $n \in A$ and $j \in B$. We denote by $\Delta[\mathcal{B}, \mathbb{X}] = \Delta$ the least constant satisfying (4) and we say that $\mathcal{B}$ is Δ-super-democratic. Also, impossing the condition $A \cap B = \emptyset$, we say that $\mathcal{B}$ is Δ_s-disjoint-super-democratic.

Remark 1. Of course, in general,

$$\Delta_s \leq \Delta \leq \Delta_s^2.$$

In our case, we will talk about t-greedy bases.

Definition 4.2. Let $\mathcal{B}$ be a semi-normalized Schauder basis in a p-Banach space $\mathbb{X}$ with $0 < p \leq 1$. We say that $\mathcal{B}$ is t-greedy if there is $C > 0$ such that

$$\|f - P_G(f)\| \leq C\,\sigma_m(f),\ \forall G \in \mathcal{G}(f, m, t), \forall m \in \mathbb{N} \text{ and } \forall f \in \mathbb{X}. \tag{5}$$

The least constant satisfying (5) is denoted by $C_{g,t}[\mathcal{B}, \mathbb{X}] = C_{g,t}$ and we say that $\mathcal{B}$ is $C_{g,t}$-greedy.

Theorem 4.1. *Let $\mathcal{B} = (e_n)_n$ be a Schauder basis in a p-Banach space $\mathbb{X}$ with $0 < p \leq 1$ and let $t \in (0, 1]$. The following are equivalent:*

a) $\mathcal{B}$ is t-greedy.

b) $\mathcal{B}$ is suppression-unconditional and disjoint-super-democratic.

Moreover,

$$\max\{K_s, \Delta_s\} \leq C_{g,t} \leq \left(K_s^p + \left(\frac{A_p B_p \Delta_s K_s}{t} \right)^p \right)^{1/p}.$$

Proof. Assume a) and prove, first, unconditionality. The technique that we use is common in this context. Take $f \in \mathbb{X}$ and consider A a finite set in the support of f of cardinality m and define the element

$$y := (f - P_A(f)) + \sum_{n\in A}(\alpha - e_n^*(f))e_n,$$

where $\alpha > \left(\max_{n\in A}|e_n^*(f)| + t\max_{n\notin A}|e_n^*(f)|\right)$. Hence, one (m,t)-greedy sum for y is $G_m^t(y) = \sum_{n\in A}\alpha e_n$ since

$$\min_{n\in A}|e_n^*(y)| \geq \alpha - |e_n^*(f)| \geq t\max_{n\notin A}|e_n^*(f)| = t\max_{n\notin A}|e_n^*(y)|.$$

Thus,

$$\|f - P_A(f)\| = \|y - G_m^t(y)\| \leq C_{g,t}\sigma_m(y) \leq C_{g,t}\left\|y - \sum_{n\in A}\alpha e_n\right\| = \|f\|.$$

Hence, the basis is suppression-unconditional with $K_s \leq C_{g,t}$. Now, we show the disjoint-super-democracy. For that, we take two disjoint finite sets A and B and $\delta > 0$ to define the element

$$h := \sum_{n\in A}\varepsilon_n e_n + (1+\delta)\sum_{j\in B}\eta_j e_j,$$

where $|\varepsilon_n| = |\eta_j| = 1$ for every $n \in A$ and $j \in B$. Then, if $m := |B|$, we can take $\mathcal{N}_m^t(h) = B$ since

$$\min_{i\in B}|e_i^*(h)| = (1+\delta) > t = t\max_{j\notin B}|e_j^*(h)|.$$

Thus, denoting $P_B(h) = G_m^t(h)$,

$$\left\|\sum_{n\in A}\varepsilon_n e_n\right\| = \|h - G_m^t(h)\| \leq C_{g,t}\left\|h - \sum_{n\in A}\varepsilon_n e_n\right\| = C_{g,t}(1+\delta)\left\|\sum_{j\in j}\eta_j e_j\right\|.$$

Taking $\delta \to 0$, we obtain the super-democracy with $\Delta_s \leq C_{g,t}$.

Assume now b) and take $f \in \mathbb{X}$, $m \in \mathbb{N}$ and $\varepsilon > 0$. Take $y \in \mathbb{X}$ such that

$$\|f - y\| < \sigma_m(f) + \varepsilon, \tag{6}$$

with $B := \mathrm{supp}(y)$. Then, for a (m,t)-greedy sum $G_m^t(f)$, we can write

$$\|f - G_m^t(f)\|^p \leq \|P_{(A_m^t(f)\cup B)^c}(f)\|^p + \|P_{B\setminus A_m^t(f)}(f)\|^p,$$

where $A_m^t(f) := \mathrm{supp}(G_m^t(f))$, that is, $A_m^t(f) \in \mathcal{G}(f,m,t)$. Of course,

$$\|P_{(A_m^t(f)\cup B)^c}(f)\| = \|P_{(A_m^t(f)\cup B)^c}(f-y)\| \leq K_s\|f-y\|. \tag{7}$$

On the other hand, using p-convexity and disjoint-super-democracy with $\eta \equiv \{\mathrm{sign}(e_n^*(f))\}_n$,

$$\|P_{B\setminus A_m^t(f)}(f)\| \leq A_p\Delta\max_{j\in B\setminus A_m^t(f)}|e_j^*(f)|\left\|\sum_{n\in A_m^t(f)\setminus B}\eta_n e_n\right\| \leq \frac{A_p\Delta}{t}\min_{j\in A_m^t(f)\setminus B}|e_j^*(f)|\left\|\sum_{n\in A_m^t(f)\setminus B}\eta_n e_n\right\|. \tag{8}$$

Now, we can select a sequence of real numbers $(\lambda_n)_n \in [0,1]$ such that

$$\min_{j\in A_m^t(f)\setminus B}|e_j^*(f)| = \lambda_n|e_n^*(f-y)|, n \in A_m^t(f)\setminus B.$$

Then, using (3),

$$\min_{j\in A_m^t(f)\setminus B}|e_j^*(f)|\left\|\sum_{n\in A_m^t(f)\setminus B}\eta_n e_n\right\| = \left\|\sum_{n\in A_m^t(f)\setminus B}\lambda_n|e_n^*(f-y)|e_n\right\| \leq B_pK_s\|f-y\|.$$

Thus,

$$\|f - G_m^t(f)\| \leq \left(K_s^p + \left(\frac{A_pB_p\Delta_sK_s}{t}\right)^p\right)^{1/p}\|f-y\| \underset{(6)}{\leq} \left(K_s^p + \left(\frac{A_pB_p\Delta_sK_s}{t}\right)^p\right)^{1/p}(\sigma_m(f)+\varepsilon).$$

Taking now $\varepsilon \to 0$,

$$\|f - G_m^t(f)\| \leq \left(K_s^p + \left(\frac{A_pB_p\Delta_sK_s}{t}\right)^p\right)^{1/p}\sigma_m(f),$$

so the proof is over. □

5 Polynomials with constant coefficients

Since around the year 2017, one of the "most peculiar" properties that has been observed regarding best approximation is its equivalence when using polynomials with constant coefficients. Specifically, in [4], the authors prove that a semi-normalized Schauder basis in a Banach space $\mathbb{X}$ is $C_{g,1}$-greedy if and only if, for every $G_m^1(f)$, $m \in \mathbb{N}$ and $f \in \mathbb{X}$,

$$\|f - G_m^1(f)\| \approx \inf\left\{\|f - \sum_{n\in C}\alpha e_n\| : \alpha \in \mathbb{R},\, |C| = m\right\}.$$

As we can see, studying the best approximation using any linear combination of the form $y = \sum_{n\in C} a_n e_n$ is equivalent to using polynomials with constant coefficients $y' = \sum_{n\in C}\alpha e_n$. Later, in 2019, the authors of [5] provided additional properties related to this characterization, though still within the context of Banach spaces over the field of real numbers.

Our aim here is to study the extension of these results to the setting of p-Banach spaces over the field $\mathbb{F}$ (which includes the complex numbers), carrying out a quantitative analysis of the constants involved in the characterization of t-greedy bases.

Our purpose here is to prove two results.

Theorem 5.1. *Let $\mathcal{B}$ be a semi-normalized Schauder basis in a p-Banach space $\mathbb{X}$ with $0 < p \le 1$ and $t \in (0,1]$. The following are equivalent:*

a) $\mathcal{B}$ is $C_{g,t}$-greedy.

b) There exists $C_1 = C_1(t,p) > 0$ such that for every (m,t)-greedy sum $G_m^t(f)$, $m \in \mathbb{N}$ and $f \in \mathbb{X}$,

$$\|f - G_m^t(f)\| \le C_1 \inf\left\{\|f - \sum_{n\in C}\alpha e_n\| : \alpha \in \mathbb{F},\, |C| = m\right\}.$$

c) There exists $C_2 = C_2(t,p) > 0$ such that for every (m,t)-greedy sum $G_m^t(f)$, $m \in \mathbb{N}$ and $f \in \mathbb{X}$,

$$\|f - G_m^t(f)\| \le C_2 \inf\left\{\|f - \sum_{n\in C}\beta e_n\| : |C| = m\right\}, \tag{9}$$

where $\beta = \min_{n\in A_m^t(f)} |e_n^*(f)|$ and $A_m^t(f) = supp(G_m^t(f))$.

Proof. Of course, a) implies b) and b) implies c). Now, we study the implication c) $\Rightarrow$ a). For that, the same proof of Theorem 4.1 works to show that if (9) is satisfied, then the basis is unconditional and super-democratic with

$$\max\{K_s, \Delta_s\} \le C_2.$$

Hence, the basis is t-greedy with

$$C_{g,t} \le C_2\left(1 + \left(\frac{A_p B_p C_2}{t}\right)^p\right)^{1/p}.$$

□

Now, we want to study an stronger result showing that if (9) is satisfied for some $0 < t \le 1$, then the basis is s-greedy for every $0 < s \le 1$.

Theorem 5.2. *Let $\mathcal{B}$ be a semi-normalized Schauder basis in a p-Banach space $\mathbb{X}$ with $0 < p \le 1$. The following are equivalent:*

a) There exists $s \in (0,1]$ and $C_1(s) > 0$ such that for every (m,s)-greedy sum $G_m^s(f)$, $m \in \mathbb{N}$ and $f \in \mathbb{X}$,

$$\|f - G_m^s(f)\| \le C_1(s) \inf\left\{\|f - \sum_{n\in C}\beta e_n\| : \beta \in \mathbb{F}, |C| = m\right\}.$$

b) The basis is t-greedy for all $t \in (0,1]$.

Proof. Of course, b) implies a) trivially. Here, we show the implication a) $\Rightarrow$ b). For that, we take $f \in \mathbb{X}$, $t \in (0,1]$, $m \in \mathbb{N}$ and $G_m^t(f)$ a (t,m)-greedy sum of f with $A_m^t(f)$ as support. Also, for $\varepsilon > 0$, take $y \in \mathbb{X}$ such that

$$\|f - y\| < \sigma_m(f) + \varepsilon. \tag{10}$$

We can write $f - G_m^t(f)$ as follows:

$$f - G_m^t(f) = f - P_{A_m^t(f)\cup B}(f) + P_{B\setminus A_m^t(f)}(f).$$

Applying the p-convexity, we only need to show the existence of $\lambda \ge 1$ and $C(s,t,p) > 0$ such that

$$\left\| f - P_{A_m^t(f)\cup B}(f) + \lambda\gamma \sum_{n\in B\setminus A_m(f)} \eta_n e_n \right\| \le C(s,t,p)\|f - y\|,$$

for any collection $(\eta_n)_{n\in B\setminus A_m^t(f)}$ such that $|\eta_n|=1$ for every n and $\gamma := \max_{j\in B\setminus A_m^t(f)}|e_j^*(f)|$.

To this end, we consider two possible cases. The first one: let $t\geq s$. We prove that

$$\left\| f-P_{A_m^t(f)\cup B}(f)+\frac{t}{s}\gamma\sum_{n\in B\setminus A_m(f)}\eta_n e_n\right\| \leq A_p(C_3(s))^2\|f-y\|,$$

for any collection $(\eta_n)_{n\in B\setminus A_m^t(f)}$ such that $|\eta_n|=1$ for every n and $\gamma := \max_{j\in B\setminus A_m^t(f)}|e_j^*(f)|$. For that, we define the element

$$h := f-P_B(f)+\frac{t}{s}\gamma\sum_{n\in B\setminus A_m^t(f)}\eta_n e_n.$$

It is routine to check that

$$\min_{j\in A_m^t(f)\setminus B}|e_j^*(h)|\geq s\max_{j\notin A_m^t(f)\setminus B}|e_j^*(h)|,$$

so, taking $n:=|A_m^t(f)\setminus B|$,

$$h-G_n^s(h)=f-P_{A_m^t(f)\cup B}(h)+\frac{t}{s}\gamma\sum_{n\in B\setminus A_m^t(f)}\eta_n e_n.$$

Hence,

$$\begin{aligned}\left\| f-P_{A_m^t(f)\cup B}(h)+\frac{t}{s}\gamma\sum_{n\in B\setminus A_m^t(f)}\eta_n e_n\right\| &\leq C_1(s)\inf\left\{\|h-\sum_{n\in C}\alpha e_n\| : \alpha\in\mathbb{F}, |C|=m\right\}\\ &\leq C_1(s)\left\| h-\frac{t}{s}\gamma\sum_{n\in B\setminus A_m^t(f)}\eta_n e_n\right\| = C_1(s)\|f-P_B(f)\|.\end{aligned}$$

Now, we define the element $g := f-y+\sum_{n\in B}ke_n$, where

$$k = s\max_{n\notin B}|e_n^*(f)|+\max_{n\in B}|e_n^*(f-y)|.$$

Then, it is clear that

$$\min_{n\in B}|k+e_n^*(f-y)|\geq s\max_{n\notin B}|e_n^*(f-y)|,$$

so

$$\|f-P_B(f)\|=\|g-G_{|B|}^s(g)\|\leq C_1(s)\|f-y\|.$$

Thus, we obtain that

$$\left\| f-P_{A_m^t(f)\cup B}(h)+\frac{t}{s}\gamma\sum_{n\in B\setminus A_m^t(f)}\eta_n e_n\right\| \leq (C_1(s))^2\|f-y\|,$$

so applying the p-convexity,

$$\|f-G_m^t(f)\|\leq A_p(C_1(s))^2\|f-y\| \underset{(10)}{\leq} A_p(C_1(s))^2(\sigma_m(f)+\varepsilon).$$

Taking now $\varepsilon\to 0$, the basis is t-greedy with with constant $C_{g,t}\leq A_p(C_1(s))^2$.

Now, considering the case $s>t$, we will show that

$$C_{g,t}\leq C_1(s)\left(1+(C_1(s))^p\left(1+\left(\frac{A_pB_pC_2(s)}{t}\right)^p\right)(1+(C_1(s))^p)\right)^{1/p}$$

Following the same decomposition as before, we divide $\left\| f-P_{A_m^t(f)\cup B}(f)+\gamma\sum_{n\in B\setminus A_m^t(f)}\eta_n e_n\right\|$ in three terms:

$$\left\| f-P_{A_m^t(f)\cup B}(f)+\gamma\sum_{n\in B\setminus A_m^t(f)}\eta_n e_n\right\|^p \leq \|f-P_B(f)\|^p+\|P_{A_m^t(f)\setminus B}(f)\|^p+\left\|\gamma\sum_{n\in B\setminus A_m^t(f)}\eta_n e_n\right\|^p.$$

Using the unconditionality condition given by Theorem 4.1 and 5.1,

$$\|P_{A_m^t(f)\setminus B}(f)\|=\|P_{A_m^t(f)}(f-P_B(f))\|\leq C_1(s)\left(1+\left(\frac{A_pB_pC_2(s)}{t}\right)^p\right)^{1/p}\|f-P_B(f)\|.$$

We denote by $\mathcal{C}:=C_1(s)\left(1+\left(\frac{A_pB_pC_2(s)}{t}\right)^p\right)^{1/p}$. On the other hand, we can define the element

$$w := P_{A_m^t(f)\setminus B}(f)+\frac{t}{s}\gamma\sum_{n\in B\setminus A_m^t(f)}\eta_n e_n,$$

where, of course, a (k,s)-greedy sum of w is $G_k^s(w) = P_{A_m^t(f)\setminus B}(f)$ with $k = |A_m^t(f)\setminus B|$. Then,

$$\left\| \frac{t}{s}\gamma \sum_{n\in B\setminus A_m^t(f)} \eta_n e_n \right\| \leq C_1(s)\|P_{A_m^t(f)\setminus B}(f)\|.$$

Consequently,

$$\left\| f - P_{A_m^t(f)\cup B}(f) + \gamma \sum_{n\in B\setminus A_m^t(f)} \eta_n e_n \right\|^p \leq (1+\mathcal{C}^p(1+(C_1(s))^p))\,\|f-P_B(f)\|^p.$$

Now, as we have done in the previous case,

$$\|f-P_B(f)\| \leq C_1(s)\|f-y\|,$$

thus

$$\left\| f - P_{A_m^t(f)\cup B}(f) + \gamma \sum_{n\in B\setminus A_m^t(f)} \eta_n e_n \right\|^p \leq (C_1(s))^p\,(1+\mathcal{C}^p(1+(C_1(s))^p))\,\|f-y\|^p,$$

so applying the p-convexity,

$$\|f-G_m^t(f)\| \leq C_1(s)\,(1+\mathcal{C}^p(1+(C_1(s))^p))^{1/p}\,\|f-y\| \underset{(10)}{\leq} C_1(s)\,(1+\mathcal{C}^p(1+(C_1(s))^p))^{1/p}\,(\sigma_m(f)+\varepsilon),$$

and hence, taking again $\varepsilon\to 0$, the basis is t-greedy with constant

$$C_{g,t} \leq C_1(s)\,(1+\mathcal{C}^p(1+(C_1(s))^p))^{1/p}\,.$$

□

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