$L_p$ Markov exponent of certain domains with cusps

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Abstract

In this paper we give sharp $L_p$ Markov type inequality for derivatives of polynomials for some family of domains with cusps.

1 Introduction

Let $P_n(\mathbb{R}^n)$ be the class of all algebraic polynomials in $m$ variables with real coefficients of degree $n$. Further, let $C(\Omega)$ be the real space of all real valued continuous functions $f$ defined on a compact set $\Omega \subset \mathbb{R}^n$ with the norm $\|f\|_{C(\Omega)} := \sup_{x \in \Omega} |f(x)|$, and let $L_{p, \mu}(\Omega)$, $1 \leq p \leq \infty$, be the space of all $\mu$-Lebesgue-measurable functions $f$ on $\Omega \subset \mathbb{R}^n$ such that $\|f\|_{L_{p, \mu}(\Omega)} := (\int_\Omega |f(x)|^p \mu(dx))^{1/p} < \infty$ if $1 \leq p < \infty$, and $L_{\infty, \mu} := C(\Omega)$. Set $L_p(\Omega) := L_{p, 1}(\Omega)$, $1 \leq p \leq \infty$. Here $\mu$ denotes an integrable weight defined on a set $\Omega \subset \mathbb{R}^n$ with the property that the set $\{x \in \Omega : W(x) = 0\}$ has $m$-dimensional Lebesgue measure 0. Moreover, $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Definition 1.1. We say that a compact set $\emptyset \neq E \subset \mathbb{R}^n$ satisfies $L_p$ Markov type inequality (or: is a $L_p$ Markov set) if there exist $\kappa, C > 0$ such that, for each polynomial $P \in P(\mathbb{R}^n)$ and each $\alpha \in \mathbb{N}_0^n$,

$$\|D^\alpha P\|_{L_p(\Omega)} \leq (C |\deg P|)^{\alpha_0} \|P\|_{L_p(\Omega)}, \quad (1)$$

where $D^\alpha P = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Clearly, by iteration, it is enough to consider in the above definition multi-indices $\alpha$ with $|\alpha| = 1$. The inequality (1) is a generalization of the classical Markov inequality:

$$\|P\|_{C([-1, 1])} \leq (\deg P)^2 \|P\|_{C([-1, 1])}.$$  

Markov-type inequalities play an important role in Approximation Theory since they are widely used for verifying inverse theorems of approximation. These inequalities and its various generalizations (restricted not only to nonpluripolar subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ but also their versions for pieces of semialgebraic sets or other "small" subsets of $\mathbb{R}^n$ (C^n)) found many applications in approximation theory, analysis, constructive function theory, but also in other branches of science (for example, in physics or chemistry).

In this paper we shall consider the following problem:

For a given $L_p$ Markov set $E$ determine $\mu_p(E) := \inf\{\kappa : E \text{ satisfies (1)}\}$.

The quantity $\mu_p(E)$ is called $L_p$ Markov exponent and was first considered by Baran and Pleśniak in [2] for $p = \infty$. This is related to the linear extension operator for $C^\infty$ functions with restricted growth of derivatives (see [8, 9]). For any compact set $E$ in $\mathbb{R}^n$ we have $\mu_p(E) \geq 2$. If $E$ is a fat convex subset of $\mathbb{R}^n$, then $\mu_p(E) = 2$. It is known that $L_\infty$ Markov exponent, for $Lip_\gamma$, $0 < \gamma < 1$ cuspidal domains in $\mathbb{R}^n$ is equal to $\frac{2}{\gamma}$ (see for instance, [4], [1], [6]). If $K \subset \mathbb{R}^n$ is a Lip\(_\text{p}_\gamma\), $0 < \gamma < 1$ cuspidal piecewise graph domain such that it is embedded in an affine image of the ball having one of its vertices on the boundary $\partial K$ of $K$, then $\mu_p(E) = \frac{2}{\gamma}$ for $1 \leq p < \infty$ (see [7]). Our goal is to establish $L_p$ Markov exponent of the following domains $\Psi_k := \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, 0 \leq y \leq x^2\}$, $T_k := \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, 0 \leq y \leq x\}$, $k \in \mathbb{N}$, $0 < a < b$. More precisely, we show that $\mu_p(\Psi_k) = \mu_p(T_k) = 2k$ and $\mu_p(\Lambda_k) = 2k + 1$ for every $k \in \mathbb{N}$, $1 \leq p < \infty$. Since none of the domains $\Psi_k, T_k$ and $\Lambda_k$ is cuspidal piecewise graph domain, the above results cannot be obtained using the methods of [7]. In particular, $\Lambda_k$ has a cusp at the origin that cannot be connected to the interior of $\Lambda_k$ by a straight line. However, these results are known in case of supremum norm (see [6]).

2 Auxiliary results

In order to verify our main results we shall need some auxiliary statements. The following inequalities play a central role in our considerations.
Lemma 2.1. Let $S := \{(t, y) \in \mathbb{R}^2 : 0 \leq t \leq 1, 0 \leq y \leq t\}$. For each $\alpha > -1$ and $\beta, \mu, \nu \geq 0$ there exists a positive constant $C$ such that

$$
\int_S t^\beta y^\alpha \frac{\partial Q(t, y)}{\partial t} |Q(t, y)|^\mu dtdy \leq Cn^{2\mu} \int_S t^\beta y^\alpha \frac{\partial Q(t, y)}{\partial t} |Q(t, y)|^\mu dtdy \leq Cn^{2\mu} \int_S t^\beta y^\alpha |Q(t, y)|^\mu dtdy \leq Cn^{2\mu} \int_S t^\beta y^\alpha |Q(t, y)|^\mu dtdy
$$

for every $Q \in \mathcal{P}_n(\mathbb{R}^2)$.

Proof. Let $L := \{(t, y) \in \mathbb{R}^2 : 0 \leq y \leq t, t \geq y + \frac{1}{2}\}$ and let $T := \{(t, y) \in \mathbb{R}^2 : \frac{1}{2} \leq t \leq 1, \frac{1}{2} \leq y \leq t\}$. Then

$$
\int_S t^\beta y^\alpha |Q(t, y)|^\mu dtdy \leq \int_T t^\beta y^\alpha |Q(t, y)|^\mu dtdy + 2^{\mu} \int_T |Q(t, y)|^\mu dtdy + \int_0^{\frac{1}{2}} t^\beta y^\alpha |Q(t, y)|^\mu dtdy
$$

and

$$
\int_T |Q(t, y)|^\mu dtdy \leq 2^{\alpha+\beta+1} \int_T t^\beta y^\alpha |Q(t, y)|^\mu dtdy.
$$

Now consider the integral over $L$. Using the change of variables $t - y = z$, we have

$$
\int_L t^\beta y^\alpha |Q(t, y)|^\mu dtdy = \int_{(0,1/2)^2} (y+z)^\beta |Q(y+z, y)|^\mu dzdy.
$$

Set $R(z, y) := Q(y+z, y)$. By Theorem 7.4 of [5] there exists a positive constant $A$ such that

$$
\int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta \frac{\partial R}{\partial z} |R(z, y)|^\mu dz \leq An^{2\mu} \int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |R(z, y)|^\mu dz,
$$

and

$$
\int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta \frac{\partial R}{\partial y} |R(z, y)|^\mu dy \leq An^{2\mu} \int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |R(z, y)|^\mu dy.
$$

Thus, by (4)-(9), and by the fact that $\frac{\partial R}{\partial z} = \frac{\partial Q}{\partial t} - \frac{\partial Q}{\partial y}$, we obtain the inequalities (2).

In order to prove the inequalities (3), it suffices to prove that there exists a positive constant $A'$ such that

$$
\int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |G(z, y)|^\mu dz \leq A'n^{2\mu} \int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |G(z, y)|^\mu dz,
$$

and

$$
\int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |G(z, y)|^\mu dy \leq A'n^{2\mu} \int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |G(z, y)|^\mu dy.
$$

for every $G \in \mathcal{P}_n(\mathbb{R}^2)$. This follows from the inequality (7.22) of [5].

In a similar way one can derive the following lemma.

Lemma 2.2. Fix $0 \leq \varepsilon < d$, and let $V := V_{\varepsilon, d} := \{(x, \eta) \in \mathbb{R}^2 : 0 \leq t \leq 1, ct \leq \eta \leq dt\}$. For each $\alpha > -1$ and $\beta, \mu, \nu \geq 0$ there exists a positive constant $C$ such that

$$
\int_V t^\beta y^\alpha \frac{\partial Q}{\partial t} |Q(t, \eta)|^\mu dtd\eta \leq Cn^{2\mu} \int_V t^\beta y^\alpha |Q(t, \eta)|^\mu dtd\eta \leq Cn^{2\mu} \int_V t^\beta y^\alpha |Q(t, \eta)|^\mu dtd\eta
$$

for every $Q \in \mathcal{P}_n(\mathbb{R}^2)$. \hfill \qed
3 Main results

This section addresses main theorems.

Theorem 3.1. Let k be a natural number, and let \( \Psi_k = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, 0 \leq y \leq x^{2k}\} \), \( \Upsilon_k = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, 0 \leq y \text{ sgn } x \leq |x|^{2k+1}\} \). Then for \( 1 \leq p \leq \infty \) we have \( \mu_p(\Psi_k) \leq \mu_p(\Upsilon_k) - 1 = 2k \).

Proof. First we prove that \( \mu_p(\Psi_k) = 2k \). It is clear that for each \( P \in \mathcal{P}_n(\mathbb{R}^2) \) there exist \( P_0, P_1 \in \mathcal{P}_n(\mathbb{R}^2) \) such that \( P(x, y) = P_0(x^2, y) + xP_1(x^2, y) \). Hence

\[
\int_{\Psi_k} |P(x, y)|^p dxdy = \int_{\Upsilon_k} |P(x^2, y)| + \sqrt{P_1(x^2, y)}|^p dt + \int_{\Upsilon_k} |P_0(x^2, y) - \sqrt{P_1(x^2, y)}|^p dt ddr,
\]

where \( S = \{(t, r) \in \mathbb{R}^2 : 0 \leq t \leq 1, 0 \leq r \leq t\} \). In (2) of Lemma 2.1, let \( \alpha = -\frac{1}{2} \), \( \beta = k - 1 \) to conclude that

\[
\int_{S} k^{k-1} |\frac{\partial P_1}{\partial t}(t, r)|^p \frac{2 \pi}{2 \pi} dt dr \leq C(\kappa_2)^p \int_{S} k^{k-1} |P_0(t, r)|^p \frac{2 \pi}{2 \pi} dt dr.
\]

Similarly,

\[
\int_{S} k^{k-1} |\frac{\partial P_1}{\partial r}(t, r)|^p \frac{2 \pi}{2 \pi} dt dr \leq C(\kappa_2)^p \int_{S} k^{k-1} |P_0(t, r)|^p \frac{2 \pi}{2 \pi} dt dr.
\]

Thus, by (12)-(14), and by the fact that \( \frac{\partial P_1}{\partial t}(x, y) = \frac{\partial P_1}{\partial t}(x^2, y) + x \frac{\partial P_1}{\partial t}(x^2, y) \), we have

\[
\int_{\Psi_k} |\frac{\partial P}{\partial y}(x, y)|^p dxdy \leq 2^p C^2(\kappa_2)^{kp} \int_{S} k^{k-1} |P_0(t, r)|^p \frac{2 \pi}{2 \pi} dt dr + 2^p C^2(\kappa_2)^{kp} \int_{S} k^{k-1} |\sqrt{P_1}(t, r)|^p \frac{2 \pi}{2 \pi} dt dr.
\]

From (12) and the inequality

\[
\|2f\|_{L^p(\Psi_k)} \leq 2^{p-1}(\|f - g\|_{L^p(\Psi_k)} + \|f + g\|_{L^p(\Psi_k)}),
\]

we see that

\[
\int_{S} k^{k-1} |P_0(t, r)|^p \frac{2 \pi}{2 \pi} dt dr \leq \int_{\Psi_k} |P(x, y)|^p dxdy
\]

(16)

\[
\int_{S} k^{k-1} |\sqrt{P_1}(t, r)|^p \frac{2 \pi}{2 \pi} dt dr \leq \int_{\Psi_k} |P(x, y)|^p dxdy.
\]

(17)

By using inequalities (15)-(17), we obtain

\[
\int_{\Psi_k} |\frac{\partial P}{\partial y}(x, y)|^p dxdy \leq 2^{p+1} C^2(\kappa_2)^{kp} \int_{\Psi_k} |P(x, y)|^p dxdy.
\]

(18)

In a similar way we can prove that

\[
\int_{\Psi_k} |\frac{\partial P}{\partial x}(x, y)|^p |x| dxdy \leq 2^{p+1} C^2(\kappa_2)^{kp} \int_{\Psi_k} |P(x, y)|^p |x| dxdy.
\]

(19)

We need now to consider \( \frac{\partial P}{\partial x} \). Using the change of variables \( y = \eta^{2k} \), we have

\[
\int_{\Psi_k} |\frac{\partial P}{\partial x}(x, y)|^p dxdy = \int_{S} 2k \eta^{2k-1} |\frac{\partial P}{\partial x}(x, \eta^{2k})|^p dxdy + \int_{S} 2k \eta^{2k-1} |\frac{\partial P}{\partial x}(-x, \eta^{2k})|^p dxdy.
\]

\[
\int_{\Psi_k} |P(x, y)|^p dxdy = \int_{S} 2k \eta^{2k-1} |P(x, \eta^{2k})|^p dxdy + \int_{S} 2k \eta^{2k-1} |P(-x, \eta^{2k})|^p dxdy.
\]
By Lemma 2.1, using \( \alpha = 0, \beta = 2k - 1 \),
\[
\int_S |\eta^{2k}| \frac{\partial P(x, \eta^{2k})}{\partial x} \, dx \, d\eta \leq C(2kn)^2 \int_S |\eta^{2k-1} P(x, \eta^{2k})| \, dx \, d\eta,
\]
\[
\int_S |\eta^{2k-1} \frac{\partial P(x, \eta^{2k})}{\partial x} \leq C(2kn)^2 \int_S |\eta^{2k-1} P(-x, \eta^{2k})| \, dx \, d\eta.
\]
Hence
\[
\int_{q_k} |\frac{\partial P}{\partial x}(x,y)|^p \, dx \, dy \leq C(2kn)^2 \int_{q_k} |P(x,y)|^p \, dx \, dy.
\]  
(20)

Similarly,
\[
\int_{q_k} |\frac{\partial P}{\partial x}(x,y)|^p |x| \, dx \, dy \leq C(2kn)^2 \int_{q_k} |P(x,y)|^p |x| \, dx \, dy.
\]  
(21)

By (18) and (20) we know that \( \mu_{(\Psi)} \leq 2k \) for \( 1 \leq p < \infty \). To prove the reverse inequality, define \( \Xi_n(x,y) = yP^{(n,\omega)}(1-x^2) \). Here \( P^{(n,\omega)} \) denotes the Jacobi polynomial of degree \( n \) associated with parameters \( \omega, \sigma \). Then
\[
\int_{q_k} |\frac{\partial \Xi}{\partial y}(x,y)|^p \, dx \, dy = \int_0^1 |P^{(n,\omega)}(t)|^p (1-t)^{k-1/2} \, dt,
\]
\[
\int_{q_k} |\Xi_n(x,y)|^p \, dx \, dy = \frac{1}{p+1} \int_0^1 |P^{(n,\omega)}(t)|^p (1-t)^{k+1/2} \, dt.
\]
It is known (see [10], Chap. VII) that
\[
\int_0^1 |P^{(n,\omega)}(x)|^p (1-x)^r \, dx \sim n^{\omega p - 2r - 2} \text{ whenever } 2r < \omega p - 2 + p/2.
\]  
(22)

If \( 2(p+1)k < \omega p - 1 + p/2 \), then by (22),
\[
\frac{\|\Xi\|_{L^p(W)}}{\|\Xi\|_{L^p(H)}} \sim n^{2k}.
\]

Hence \( \mu_{(\Psi)} \geq 2k \).

Now we wish to prove that \( \mu_{(\Psi)} = 2k + 1 \). Since \( T(x,y) = (x,xy) \) maps \( \Psi_k \) onto \( \Psi_k \),
\[
\int_{\Psi_k} |f(x,y)|^p \, dx \, dy = \int_{\Psi_k} |f(x,xy)|^p |x| \, dx \, dy.
\]  
(23)

Applying (19) and (21) to \( Q(x,y) := P(x,xy) \), we find that
\[
\int_{\Psi_k} |x \frac{\partial P}{\partial y}(x,xy)|^p |x| \, dx \, dy \leq 2^{p+1} C^2 (2kn)^{2k} \int_{\Psi_k} |P(x,y)|^p |x| \, dx \, dy,
\]  
(24)

In order to establish \( \mu_{(\Psi)} = 2k + 1 \) it will be enough to prove that there exists a positive constant \( B \) such that
\[
\int_{\Psi_k} |R(x,y)|^p |x| \, dx \, dy \leq B n^p \int_{\Psi_k} |xR(x,y)|^p |x| \, dx \, dy
\]  
(26)

for every \( R \in P_n(\mathbb{R}^2) \). If we write \( R(x,y) = R_0(x^2,y) + xR_1(x^2,y) \), then
\[
2 \int_{\Psi_k} |R(x,y)|^p |x| \, dx \, dy = \int_S kr^{k-1} |R_0(t,r^2) + \sqrt{t} R_1(t,r^2)|^p \, dt \, dr + \int_S kr^{k-1} |R_0(t,r^2) - \sqrt{t} R_1(t,r^2)|^p \, dt \, dr.
\]

By Lemma 2.1, we conclude that
\[
\int_S r^{k-1} |R_0(t,r^2)|^p \, dt \, dr \leq C n^p \int_S r^{k-1} |\sqrt{t} R_0(t,r^2)|^p \, dt \, dr,
\]
\[
\int_S r^{k-1} |\sqrt{t} R_1(t,r^2)|^p \, dt \, dr \leq C n^p \int_S r^{k-1} |t R_1(t,r^2)|^p \, dt \, dr.
\]
From this it follows that
\[ 2 \int_{\Omega} |R(x,y)|^p |x| \, dx \, dy \leq 2^p C n^p \left( \int_{\Omega} r^{k-1} |\sqrt{T} R_0(t,r^k)|^p \, dt \, dr + \int_{\Omega} r^{k-1} |R_i(t,r^k)|^p \, dt \, dr \right). \]

Therefore
\[ \int_{\Omega} |R(x,y)|^p |x| \, dx \, dy \leq 2^p C n^p \int_{\Omega} |xR(x,y)|^p |x| \, dx \, dy. \]

From (23)-(26), it follows that \( \mu_x(T_{k}) \leq 2k + 1 \). Let \( \Xi_k \) be given as above. If \( 2(p + 1)k < \omega_p - 2 - p/2 \), then
\[ \int_{\Omega} \frac{\partial P}{\partial x}(x,y) \right|_0^1 \, dx \, dy = \int_{\Omega} |P^{(n,o)}(t)|^p (1 - t)^{k} \, dt \sim n^{\omega_p - 2k - 2}, \]
\[ \int_{\Omega} |\Xi_k(x,y)|^p \, dx \, dy = \frac{1}{p + 1} \int_{\Omega} |P^{(n,o)}(t)|^p (1 + t)^{k} \, dt \sim n^{\omega_p - 2k - 2}. \]

Hence \( \mu_x(T_{k}) \geq 2k + 1 \). Thus \( \mu_x(T_{k}) = 2k + 1 \).

The second main theorem is as follows.

**Theorem 3.2.** Let \( k \) be a natural number. Fix \( 0 \leq a < b \), and let \( \Lambda_k := \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, ax^k \leq y \leq bx^k \} \). Then \( \mu_x(\Lambda_k) = 2k \) for every \( 1 \leq p \leq \infty \).

**Proof.** We note first that the \( L_\infty \) Markov exponent of \( \Lambda_k \) is known (see [6]). For \( 1 \leq p < \infty \), using the change of variables \( y = z^k \), we have
\[ \int_{\Omega} |P(x,y)|^p \, dx \, dy = \int_{\Omega} k z^{k-1} |P(x,z^k)|^p \, dx \, dz, \]
where \( c = \sqrt{a} \) and \( d = \sqrt{b} \). In (10) of Lemma 2.2, let \( \alpha = 0, \beta = k - 1 \) to conclude that
\[ \int \int_{V_{a,d}} z^{k-1} \frac{\partial P}{\partial x}(x,z^k) \right|_0^1 \, dx \, dz \leq C(kn)^{2p} \int \int_{V_{a,d}} z^{k-1} |P(x,z^k)|^p \, dx \, dz, \]
\[ \int \int_{V_{a,d}} z^{k-1} |z^{k-1} \frac{\partial P}{\partial y}(x,z^k)|^p \, dx \, dz \leq C(kn)^{2p} \int \int_{V_{a,d}} z^{k-1} |P(x,z^k)|^p \, dx \, dz \]
for every \( P \in \mathcal{P}_2(\mathbb{R}^2) \). Another application of Lemma 2.2 shows that
\[ \int \int_{V_{a,d}} z^{k-1} \frac{\partial P}{\partial y}(x,z^k) \right|_0^1 \, dx \, dz \leq C(kn)^{2p-2p} \int \int_{V_{a,d}} z^{k-1} |z^{k-1} \frac{\partial P}{\partial y}(x,z^k)|^p \, dx \, dz. \]

Thus, by (27)-(30), we have \( \mu_x(\Lambda_k) \leq 2k \).

To prove \( \mu_x(\Lambda_k) \geq 2k \), define \( U_n(x,y) = y P^{(n,o)}(1 - x) \). Then
\[ \int \int_{\Lambda_k} \frac{\partial U_n}{\partial y}(x,y) \right|_0^1 \, dx \, dy = (b - a) \int_0^1 |P^{(n,o)}(t)|^p (1 - t)^{k} \, dt, \]
\[ \int \int_{\Lambda_k} |U_n(x,y)|^p \, dx \, dy = \frac{b^{p+1} - a^{p+1}}{p + 1} \int_0^1 |P^{(n,o)}(t)|^p (1 - t)^{(p+1)k} \, dt. \]

Proceeding as before, whenever \( 2(p + 1)k < ap - 2 + p/2 \) we have
\[ \| U_n \|_{L^p(\Lambda_k)} \sim n^{2k}. \]

This completes the proof.

Similarly, one can prove the following theorem.

**Theorem 3.3.** Let \( k, l \) be natural numbers such that \( l < k \). Fix \( 0 \leq a < b \), and let \( \Lambda_{k,l} := \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, ax^k \leq y \leq bx^k \} \). Then \( \frac{d}{d} \leq \mu_x(\Lambda_{k,l}) \leq 2k \) for every \( 1 \leq p \leq \infty \).

Note that the above result is valid for \( p = \infty \) (see [6]).

In the last statement we turn our attention to more general types of cuspidal domains. Specifically, we replace \( x^k \) (in \( \Psi_k \)) by any convex function \( f \) such that \( f(0) = f'(0) = 0 \).
Theorem 3.4. Let $f$ be a real-valued convex function on the interval $[0, 1]$. Suppose that $f(0) = f'(0) = 0$, $f'(1) < \infty$ and $f(t) > 0$ for $t \in (0, 1)$. Let $K = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, 0 \leq y \leq f(x^2)\}$. Then for $1 \leq p < \infty$,

$$\mu_p(K) \leq \inf\{\varepsilon > 0 : \exists \zeta \in \mathbb{N} : \eta_{\varepsilon} n^2 \leq C(f(1/n^2)n)^{-1}\}. \quad (31)$$

Moreover, if there exists a constant $I > 0$ such that $I \cdot f(x) \geq f'(x)$ then the above inequality becomes an equality.

Proof. If $P_0, P_1 \in \mathcal{P}_n(\mathbb{R}^2)$, $P(x, y) = P_0(x^2, y) + xP_1(x^2, y)$, then

$$\int_K |P(x, y)|^p \, dx \, dy = \int_K \frac{|P_0(t, y) + \sqrt{t}P_1(t, y)|^p}{2\sqrt{t}} \, dt \, dy + \int_K \frac{|P_0(t, y) - \sqrt{t}P_1(t, y)|^p}{2\sqrt{t}} \, dt \, dy,$$

where $K' = \{(t, y) \in \mathbb{R}^2 : 0 \leq t, 1 \leq 0 \leq y \leq f(t)\}$. For $n \in \mathbb{N}$, let

$$K'_n = \{(t, y) \in \mathbb{R}^2 : \frac{1}{n^2} \leq t, 1 \leq 0 \leq y \leq f(t)\}.$$

It follows from (7.17) of [5] that for every $\alpha > -1$ there exists a positive constant $B$ such that if $H$ is a polynomial in one variable of degree at most $n$, then

$$\int_0^1 t^n |H(t)|^p \, dt \leq B \int_{1/n^2}^1 t^n |H(t)|^p \, dt.$$

Hence

$$\int_{K'_n} t^n |Q(t, y)|^p \, dt \, dy \leq B \int_{K'_n} t^n |Q(t, y)|^p \, dt \, dy$$

for every $Q$ in $\mathcal{P}_n(\mathbb{R}^2)$. Let $\eta_n = f(1/n^2)$, $\eta'_n = f'(1/n^2)$. Our assumptions guarantee that

$$T_n := \{(t, y) \in \mathbb{R}^2 : \frac{1}{n^2} \leq t, 1 \leq 0 \leq y - \frac{\eta'_n}{n^2} \leq \eta_n t - \frac{\eta'_n}{n^2} \} \subset K'_n.$$

For $n \geq 2$, define

$$L_n := \{(t, y) \in \mathbb{R}^2 : \frac{1}{n^2} \leq t, \frac{\eta'_n}{n^2} \leq y \leq \eta_n t - \frac{\eta'_n}{n^2} + f(1/4) \} \cap K'_n.$$

If we define $V := \{(t, y) \in \mathbb{R}^2 : 1/4 \leq t, f(1/4) \leq y \leq f(t)\}$, then $K'_n \subset L_n \cup T_n \cup V$ for $n \geq 2$. Hence

$$\int_{L_n} t^n |Q(t, y)|^p \, dt \, dy \leq \int_{L_n} t^n |Q(t, y)|^p \, dt \, dy + \int_{T_n} t^n |Q(t, y)|^p \, dt \, dy + \int_{V} t^n |Q(t, y)|^p \, dt \, dy.$$

Since $V$ is a locally Lipschitzian compact subset of $\mathbb{R}^2$, $\mu(V) = 2$ (see [3]). Hence there exists a positive constant $B_1$ such that

$$\max \left\{ \int_V |\frac{\partial Q}{\partial t}(t, y)|^p \, dt \, dy, \int_V |\frac{\partial Q}{\partial y}(t, y)|^p \, dt \, dy \right\} \leq B_1 (\deg Q)^{2p} \int_V |Q(t, y)|^p \, dt \, dy.$$

By the definition of $V$, we can write

$$\frac{1}{4} \int_V t^n |Q(t, y)|^p \, dt \, dy \leq \int_V |Q(t, y)|^p \, dt \, dy \leq 4^{n+1} \int_V t^n |Q(t, y)|^p \, dt \, dy.$$

By (37) and (38),

$$\max \left\{ \int_V t^n |\frac{\partial Q}{\partial t}(t, y)|^p \, dt \, dy, \int_V t^n |\frac{\partial Q}{\partial y}(t, y)|^p \, dt \, dy \right\} \leq 4^{n+2} B_1 (\deg Q)^{2p} \int_V t^n |Q(t, y)|^p \, dt \, dy.$$

Now consider the integral over $T_n$. Using similar ideas to those applied in the proof of Lemma 2.1, we can establish that there is a constant $B_2$, depending only on $\alpha$, such that

$$\int_{T_n} t^n |\frac{\partial Q}{\partial t}(t, y)|^p \, dt \, dy \leq B_2 (1 - 1/n^2)^{-p} (\deg Q)^{2p} \int_{T_n} t^n |Q(t, y)|^p \, dt \, dy,$$

$$\int_{T_n} t^n |\frac{\partial Q}{\partial y}(t, y)|^p \, dt \, dy \leq B_2 (\eta_n - \frac{\eta'_n}{4})^{-p} (\deg Q)^{2p} \int_{T_n} t^n |Q(t, y)|^p \, dt \, dy.$$

In order to deal with $L_n$, we define

$$L'_n := \{(t, y) \in \mathbb{R}^2 : \frac{1}{n^2} \leq t, 0 \leq y \leq \eta_n t - \frac{\eta'_n}{4} + f(1/4)\} \cap K'_n,$$

$$l(\theta) := \{(t, y) \in \mathbb{R}^2 : y = \theta\}, \quad \theta \in \mathbb{R}.$$
Then $l(\theta)$ intersects $L'_n$ along a single line segment of lengths not smaller then some positive constant depending only on $f$ for every $\theta \in [0, \eta_1 - \frac{\sqrt{2}}{4} + f(1/4)]$. Thus by using Theorem 7.4 of [5] along each of these segments implies that there exists a positive constant $B_2$ such that

$$
\int_{L'_n} t^n \frac{\partial Q}{\partial t}(t, y)|^p dt dy \leq B_2 (\deg Q)^{2p} \int_{L'_n} t^n |Q(t, y)|^p dt dy.
$$

By our assumptions, $L_n \subset L'_n \subset K_n$. Therefore

$$
\int_{L_n} t^n \frac{\partial Q}{\partial t}(t, y)|^p dt dy \leq B_2 (\deg Q)^{2p} \int_{K_n} t^n |Q(t, y)|^p dt dy.
$$

(42)

An illustration of $L_n$, $T_n$, $L'_n$, with $f(x) = x^2$, $n = 3$ is shown in Figure 1.

Using the change of variables $s = y - \eta_1 t + \frac{n}{2p}$ and proceeding as before, one can verify that

$$
\int_{L_n} t^n \left| \frac{\partial Q}{\partial t}(t, y) + \eta_1 \frac{\partial Q}{\partial y}(t, y) \right|^p dt dy \leq B_3 (\deg Q)^{2p} \int_{K_n} t^n |Q(t, y)|^p dt dy.
$$

(43)

From (42), (43) and the convexity of $x \mapsto x^p$ for $p \geq 1$,

$$
\int_{L_n} t^n \frac{\partial Q}{\partial y}(t, y)|^p dt dy \leq B_3 \left( \frac{2}{\eta_1 n} \right)^p (\deg Q)^{2p} \int_{K_n} t^n |Q(t, y)|^p dt dy.
$$

(44)

By (32), (33), (39), (40), (41), (42) and (44), there exists a positive constant $C$ such that

$$
\max \left\{ \int_{K} \left| \frac{\partial P}{\partial x}(x, y) \right|^p dx dy, \int_{K} \left| \frac{\partial P}{\partial y}(x, y) \right|^p dx dy \right\} \leq C \left( \frac{n}{\eta_1} \right)^p \int_{K} |P(x, y)|^p dx dy
$$

for every $P \in P_n(\mathbb{R}^2)$. Therefore,

$$
\mu_\mu(K) \leq \inf \{ \tau > 0 : \exists C > 0 \forall n \in \mathbb{N} \nabla f(\tau/n^2) \leq C \}
$$

To prove the reverse inequality, we shall use Jacobi polynomials $P_n^{(\alpha, \beta)}$. An easy computation leads to

$$
\int_{K_n} t^n |y P_n^{(\alpha, \beta)}(1-t)|^p dt dy = \int_{u_n}^{2} (f(1-t))^{p+1} \frac{t^n}{p+1} |P_n^{(\alpha, \beta)}(1-t)|^p dt = \int_{0}^{1} \frac{1}{p+1} \left( f(1-t) \right)^{p+1} \left( 1-t \right)^p |P_n^{(\alpha, \beta)}(1-t)|^p dt.
$$

(45)

Using the change of variables $t = \cos \theta$, we have

$$
\int_{K_n} t^n |y P_n^{(\alpha, \beta)}(1-t)|^p dt dy = \int_{u_n}^{2} \frac{1}{p+1} \left( f(1-\cos \theta) \right)^{p+1} \left( 1-\cos \theta \right)^p |P_n^{(\alpha, \beta)}(\cos \theta)|^p \sin \theta d\theta,
$$

(46)

where $u_n = \arccos(1 - 1/n^2)$. Applying certain properties of Jacobi polynomials $P_n^{(\alpha, \beta)}$ verified in [10], (7.32.5), p. 169, we conclude that there exists a natural number $n_0$ so that

$$
\int_{u_n}^{2} (f(1-\cos \theta))^{p+1} \left( 1-\cos \theta \right)^p |P_n^{(\alpha, \beta)}(\cos \theta)|^p \sin \theta d\theta \leq \frac{\Lambda n^{-p/2}}{\eta_1 n} \int_{u_n}^{2} (f(1-\cos \theta))^{p+1} \left( 1-\cos \theta \right)^p \theta^{-\alpha p-\beta p/2} \sin \theta d\theta
$$

(47)

for $n \geq n_0$ and appropriately adjusted constant $\Lambda$. The fact that $1 - \cos x \leq \sin^2 x$ for $-\pi/2 \leq x \leq \pi/2$ allows us to conclude that

$$
\int_{u_n}^{2} (f(1-\cos \theta))^{p+1} \left( 1-\cos \theta \right)^p \theta^{-\alpha p-\beta p/2} \sin \theta d\theta \leq \int_{u_n}^{2} \frac{1}{p+1} \left( f(1-\cos \theta) \right)^{p+1} \left( 1-\cos \theta \right)^p \theta^{-\alpha p-\beta p/2} \sin \theta d\theta. \quad (48)
$$

Since $\sin x \leq x$ for $x \geq 0$, we have

$$
\int_{u_n}^{2} \left( f(1-\cos \theta) \right)^{p+1} \theta^{-\alpha p-\beta p/2} d\theta \leq \int_{u_n}^{2} \left( f(1-\cos \theta) \right)^{p+1} \theta^{-\alpha p-\beta p/2} 2a+1 d\theta. \quad (49)
$$

Integration by parts gives us, for $-\alpha p - p/2 + 2a + 1 \neq -1$,

$$
\int_{u_n}^{2} \left( f(1-\cos \theta) \right)^{p+1} \theta^{-\alpha p-\beta p/2} 2a+1 d\theta = \left[ \frac{\left( f(1-\cos \theta) \right)^{p+1} \theta^{-\alpha p-\beta p/2}}{-\alpha p - p/2 + 2a + 2} \right]_{u_n}^{n/2} + \int_{u_n}^{2} \frac{1}{p+2}(f(1-\cos \theta))^{p+1} \theta^{-\alpha p-\beta p/2} 2a+2) d\theta = \frac{f'(1-\cos \theta)(1-\cos \theta) \sin \theta}{1-\cos \theta} d\theta. \quad (50)
$$
If $-1 \leq x \leq 1$, then $\sqrt{1-x^2} \arccos x \leq 2(1-x)$. Therefore,

$$\theta \sin \theta f'(1 - \cos \theta) \leq 2(1 - \cos \theta)f'(1 - \cos \theta).$$

Hence, by our assumption on $f$,

$$\theta \sin \theta f'(1 - \cos \theta) \leq 2I f(1 - \cos \theta).$$

(51)

If $\omega p + p/2 - 2\alpha - 2 > 0$, then by (51),

$$\int_{\omega}^{2} \frac{(p + 1)(1 - \cos \theta)^p}{\omega p + p/2 - 2\alpha - 2} f'(1 - \cos \theta) \sin \theta d\theta \leq \int_{\omega}^{2} \frac{2(1 + p/2 - 2\alpha - 2)}{\omega p + p/2 - 2\alpha - 2} d\theta.$$

(52)

Thus, by (50) and (52),

$$\int_{\omega}^{2} (1 - \cos \theta)^p \sin \theta d\theta \leq \frac{(\eta_n)^{p+1} \omega p + p/2 - 2\alpha - 2}{2(p + 1)(1 - \cos \theta)^p} \omega p + p/2 - 2\alpha - 2.$$

whenever $\omega p + p/2 - 2\alpha - 2 > 2I(p + 1)$. Hence, there exists $\Lambda_1$ such that

$$\int_{\omega}^{2} (1 - \cos \theta)^p \sin \theta d\theta \leq \Lambda_1(\eta_n)^{p+1} n^{\omega p + p/2 - 2\alpha - 2}.$$

(54)

Putting inequalities (45)-(54) and (33) together, we have

$$\int_{K'} t^n |p_n(x)|^2 (1 - t)^p dt d\gamma \leq \frac{\Lambda_1}{p + 1}(\eta_n)^{p+1} n^{\omega p + p/2 - 2\alpha - 2}.$$

(55)

By our assumption,

$$\int_{K'} t^n |P_n^{(a,0)}(1 - t)|^p dt d\gamma \geq \int_{0}^{2} f^{(l)}(t) \sin \theta \left| P_n^{(a,0)}(1 - t) \right|^p dt = \int_{0}^{2} f(t) \eta_n |P_n^{(a,0)}(1 - t)|^p dt.$$

(56)

By making the change of variable $t = \frac{\omega}{n^2}$, we obtain

$$\int_{0}^{2} f(t) \eta_n |P_n^{(a,0)}(1 - t)|^p dt = \frac{1}{2n^2} \int_{0}^{2} f(g_n(z))(d_n(z))^a |P_n^{(a,0)}(1 - g_n(z))^a| dz.$$

(57)

where $g_n(z) = \frac{\omega}{n^2}$. Again certain properties of Jacobi polynomials $P_n^{(a,0)}(x)$ play a role. By the formula of Mehler-Heine type (see [10], Theorem 8.1.1.)

$$\frac{1}{2n^2} \int_{0}^{2} f(g_n(z))(d_n(z))^a |P_n^{(a,0)}(1 - g_n(z))^a| dz \geq \frac{n^{\omega}}{4n^2} \int_{0}^{2} f(g_n(z))(d_n(z))^a (4(\omega/2)^a - J_n(z - J_n(\omega + 2))^a dz$$

for $\omega > 0$ all sufficiently large $n$. Here $J_n(z)$ is the Bessel functions of the first kind. Since

$$\min_{\gamma \in (0, \pi)} \cos \gamma \frac{\sin^2 \gamma}{\Gamma(\omega + 1)} \leq \min_{\gamma \in (0, \pi)} \frac{1}{\Gamma(\omega + 1)} \leq \frac{2}{4(\omega + 1)} \leq \frac{1}{\Gamma(\omega + 1)} = \frac{1}{\Gamma(\omega + 2)},$$

we have

$$\frac{1}{2n^2} \int_{0}^{2} f(g_n(z))(d_n(z))^a |P_n^{(a,0)}(1 - g_n(z))^a| dz \geq \frac{4(\omega - 1)}{4(\omega + 2)^2} \int_{0}^{2} f(g_n(z))(d_n(z))^a dz.$$

(58)

Then integration by parts shows that

$$\int_{0}^{2} f(g_n(z))(d_n(z))^a dz = \frac{2\eta_n}{n^{2a}(a + 1)} - \frac{1}{2n^2} \int_{0}^{2} \frac{z}{\alpha + 1} f'(g_n(z))(d_n(z))^a dz.$$

Since $If(x) \geq xf'(x)$, this leads to

$$\int_{0}^{2} f(g_n(z))(d_n(z))^a dz \geq \frac{2\eta_n}{n^{2a}(a + 1)} \geq \frac{2\eta_n}{n^{2a+1}(a + 1)}.$$

(59)

From (56), (57), (58) and (59) we see that

$$\int_{K'} t^n |P_n^{(a,0)}(1 - t)|^p dt d\gamma \geq \left( \frac{4(\omega - 1)}{4(\omega + 2)^2} \right)^\alpha \frac{2n^{\omega p - 2}\eta_n}{n^{2a+1}(a + 1)}.$$

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This last inequality together with (55) imply that there exists a positive constant \( \lambda \) depending only on \( f, \alpha \) and \( \omega \) such that, for each \( n \),

\[
\int_{K'} t^n |p_n^{(\alpha,\omega)}(1-t)|^p \, dt \, dy \geq \lambda \frac{n^p}{(\eta'_n)^p} \int_{K'} t^n |y| p_n^{(\alpha,\omega)}(1-t)|^p \, dt \, dy.
\]

Since

\[
\int_{K'} |p_n^{(\alpha,\omega)}(1-x^2)|^p \, dx \, dy = \int_{K'} t^{-1/2} |p_n^{(\alpha,\omega)}(1-t)|^p \, dt \, dy
\]

\[
\int_{K'} |y| p_n^{(\alpha,\omega)}(1-x^2)|^p \, dx \, dy = \int_{K'} t^{-1/2} |y| p_n^{(\alpha,\omega)}(1-t)|^p \, dt \, dy,
\]

it follows that \( \mu_f(K) \geq \inf\{\tau > 0 : \exists C > 0, \forall n \in \mathbb{N}, n^2 \leq C f'(1/n^2)n^\tau \} \). □

![Figure 1: \( L_3, T_3, L'_3 \), with \( f(x) = x^2 \) and the tangent line to the curve \( y = x^2 \) at the point \( \frac{1}{2} \).](image)

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**References**


