PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Proceedings of Multivariate Approximation honoring Len Bos 60th birthday, Volume 7 · 2014 · Pages 16–21

Hölder continuity of the Green function, Markov-type inequality and a capacity related to HCP

Mirosław Baran[a] · Leokadia Bialas-Ciez[b]

Abstract

Let V_E be the pluricomplex Green function associated to a compact subset E of $\mathbb{C}^N$. The well known Hölder Continuity Property (HCP) of E means that there exist constants $B > 0, \gamma \in (0,1]$ such that $V_E(z) \le B\,\mathrm{dist}(z,E)^\gamma$. It turns out that this condition is equivalent to a Vladimir Markov type inequality, i.e. $\|D^\alpha P\|_E \le M^{|\alpha|}(\deg P)^{m|\alpha|}\,(|\alpha|!)^{1-m}\,\|P\|_E$, where $m, M > 0$ are independent of the polynomial P of N variables and $\|\cdot\|_E$ is the supremum norm on E. This equivalence has some interesting implications, e.g. for convex bodies in $\mathbb{R}^N$, for uniformly polynomially cuspidal sets and for some disconnected compact sets. Moreover, we give a definition of a capacity related to HCP and we prove some its basic properties. This allows improving an estimate from below of the L-capacity of sets with (HCP). The paper is mostly based on the talk given during the Workshop on Multivariate Approximation in honor of Prof. Len Bos 60th birthday, and rests on the article [3].

Let E be a compact set in $\mathbb{C}^N$. The pluricomplex Green's function (with pole at infinity) of E can be defined by

$$V_E(z) := \sup\{u(z) \,:\, u \in \mathcal{L}_N \ \text{ and } \ u \le 0 \text{ on } E\}, \quad z \in \mathbb{C}^N,$$

where $\mathcal{L}_N$ is the Lelong class of all plurisubharmonic functions in $\mathbb{C}^N$ of logarithmic growth at the infinity, i.e.

$$\mathcal{L}_N := \{u \in PSH(\mathbb{C}^N) \,:\, u(z) - \log\|z\|_2 \le \mathcal{O}(1) \ \text{ as } \|z\|_2 \to \infty\}$$

(for background information, see [12]). Here $\|z\|_2$ stands for the Euclidean norm in $\mathbb{K}^N$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. In the univariate case V_E coincides with the Green's function g_E of the unbounded component of $\hat{\mathbb{C}} \setminus E$ with logarithmic pole at infinity (as usual $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$).

Let V_E^* be the standard upper regularization of V_E. By Siciak's theorem, either $V_E^* \in \mathcal{L}_N$ or $V_E^* \equiv +\infty$. It is equivalent to the fact that E is a *non-pluripolar* or *pluripolar* set, respectively (*non-polar* or *polar* for $N = 1$). For a non-polar set E, V_E^* coincides with the Green's function g_E of the unbounded component of $\hat{\mathbb{C}} \setminus E$ with logarithmic pole at infinity (as usual $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$).

If we define the L-*capacity* of E to be $C(E) = \liminf\limits_{\|z\|_2 \longrightarrow \infty} \frac{\|z\|_2}{\exp V_E^*(z)}$, then E is a pluripolar set if and only if $C(E) = 0$. In the one-dimensional space, $C(E)$ equals the logarithmic capacity of E.

A set E is L-regular if $\lim\limits_{w \longrightarrow z} V_E^*(w) = 0$ for every $z \in E$. Siciak has proved that this is equivalent to the continuity of V_E in the whole space $\mathbb{C}^N$. Therefore, L-regularity, i.e. the continuity of V_E is the global property of V_E determined by the behaviour of V_E only near E.

Another global property of the set E that depends only on the behaviour of V_E near E is the Hölder continuity property of the pluricomplex Green's function V_E.

Definition 1. Let $\gamma \in (0,1]$, $B > 0$. A compact set $E \subset \mathbb{C}^N$ admits the *Hölder continuity property of the pluricomplex Green's function* V_E ($E \in HCP(\gamma, B)$ in short) if for every $z \in \mathbb{C}^N$

$$V_E(z) \ \le \ B\ \mathrm{dist}(z,E)^\gamma. \tag{1}$$

In order to investigate the behavior of V_E near E, we define

$$V_E^\bullet(z) := \sup\{V_E(x - w) :\ x \in E,\ \|w\|_2 \le \|z\|_2\},\ z \in \mathbb{C}^N,$$

that is, a radial modification of V_E. The definition and main properties of $V_E^\bullet$ were presented by M. Baran, L. Bialas-Ciez, *Comparison principles for compact sets in $\mathbb{C}^N$ with HCP and Markov properties* during the Conference on Several Complex Variables

[a]Jagiellonian University, Faculty of Mathematics and Computer Science, Institute of Mathematics, 30-348 Kraków, Łojasiewicza 6, Poland, Miroslaw.Baran@im.uj.edu.pl
[b]Jagiellonian University, Faculty of Mathematics and Computer Science, Institute of Mathematics, 30-348 Kraków, Łojasiewicza 6, Poland, Leokadia.Bialas-Ciez@im.uj.edu.pl

on the occasion of Professor Józef Siciak's 80th birthday, Kraków, 4-8 July 2011. We set out (without proofs) the following examples:

- if E is a unit ball in $\mathbb{C}^N$ (with respect to a fixed complex norm) then $V_E^\bullet(z) = \log(1 + ||z||_2/C(E))$,
- if E is a convex symmetric body in $\mathbb{R}^N$ then $V_E^\bullet(z) = \log h(1 + ||z||_2/(2C(E)))$, where $h(t) = t + \sqrt{t^2-1}$ for $t \geq 1$.
- if E is a polar set then $V_E^\bullet(0) = 0$, $V_E^\bullet|_{\mathbb{C}^N\setminus\{0\}} \equiv +\infty$.

For the non-polar sets we can obtain a very important fact which is derived from Prop.1.4 in [20] (cf. [6, Th.2.1c)]):

Proposition 2. *If E is a non-pluripolar compact subset of $\mathbb{C}^N$ and*

$$\rho_E(r) := V_E^\bullet(z) \quad for \quad ||z||_2 = r,$$

then $t \longmapsto \rho_E(e^t)$ is an increasing convex function.

Remark 3. The function ρ_E has the following basic properties:

a) $\rho_{a+\lambda E}(r) = \rho_E(\lambda^{-1}r)$, $a \in \mathbb{C}^N$, $\lambda > 0$,
b) $\rho_{E\times F}(r) = \max(\rho_E(r), \rho_F(r))$,
c) $\rho_E(r) - \log r$ is a decreasing function and tends to $-\log C(E)$ as $r \to \infty$,
d) ρ_E is increasing, continuous on $(0, +\infty)$ and consequently, $0 = \rho_E(0) \leq \lim_{r\to 0+} \rho_E(r)$. Therefore, L-regularity is equivalent to the equality $\lim_{r\to 0+} \rho_E(r) = 0$.

It seems appropriate to mention here five equivalents for the Hölder continuity property HCP.

Proposition 4. *If E is a compact subset of $\mathbb{C}^N$ and $\gamma \in (0, 1]$ then the following statements are equivalent:*

(i) $\exists B_1 \geq 1$ $E \in HCP(\gamma, B_1)$,
(ii) $\exists B_2 \geq 1$ $\rho_E(r) \leq B_2 r^\gamma$ *for* $r \geq 0$,
(iii) $\exists B_3 \geq 1$ $|\rho_E(r) - \rho_E(s)| \leq B_3 |r - s|^\gamma$ *for* $r, s \geq 0$,
(iv) $\exists B_4 \geq 1$ $\Phi_E(z) \leq 1 + B_4 \operatorname{dist}(z, E)^\gamma$ *for* $z \in \mathbb{C}^N$, $\operatorname{dist}(z, E) \leq 1$,
(v) $\exists B_5 \geq 1$ $|V_E(z) - V_E(w)| \leq B_5 ||z - w||_2^\gamma$ *for* $z, w \in \mathbb{C}^N$,
(vi) $\forall R > 0$ $\exists B_6 \geq 1$ $|\Phi_E(z) - \Phi_E(w)| \leq B_6 ||z - w||_2^\gamma$ *for* $z, w \in E_R := \{z \in \mathbb{C}^N : \operatorname{dist}(z, E) \leq R\}$.

Moreover, in the equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (v)$ we have $B_1 = B_2 = B_3 = B_5$.

One can easily prove that HCP implies the *A. Markov inequality*, i.e. there exist constants $m \geq 1$, $M > 0$ such that for every polynomial P of N variables

$$\| \, ||\operatorname{grad} P||_2 \|_E \leq M (\deg P)^m \, ||P||_E, \tag{2}$$

where $\deg P$ is the total degree of the polynomial P, i.e. the highest degree of its monomials. If E admits inequality (2) then it is said to be a *Markov set* and we write $E \in AMI(m, M)$.

An interesting question is whether there exists a relationship between the A. Markov inequality and the behaviour of the Green's function near the considered set. Every Markov set $E \subset \mathbb{C}$ is non-polar (see [6]) and every Markov set $E \subset \mathbb{R}$ is L-regular (see [7]). It seems that A. Markov inequality (2) implies Hölder continuity property but a proof is an open problem mentioned e.g. in [19]. Actually, even the question about L-regularity of Markov sets in the general case remains open.

By Theorem 3.5 in [14] we can obtain

Proposition 5. *If $E \subset \mathbb{C}$, n is a positive integer, $k \in \{1, \dots, n\}$ and there exists $M_k = M_k(E)$, $m > 0$ such that for every polynomial P of degree at most n*

$$||P^{(k)}||_E \leq M_k \, n^{mk} \, ||P||_E \tag{3}$$

then $M_k \geq B^k/[(k!)^{m-1}]$ for certain constant $B > 0$ depending only on the set E.

We concentrated in [3] on a generalization of an inequality proved by A. Markov's younger brother, V. Markov. He discovered in 1892, after a very detailed investigation, a precise but intricate estimate for the k-th derivative of polynomials (see e.g. [21]): *for any polynomial P of degree not greater than n*

$$||P^{(k)}||_{[-1,1]} \leq T_n^{(k)}(1) \, ||P||_{[-1,1]} = \frac{n^2[n^2-1]...[n^2-(k-1)^2]}{1 \cdot 3 \cdot \ldots \cdot (2k-1)} \, ||P||_{[-1,1]} \tag{4}$$

where $T_n(x) = \cos(n \arccos x)$ is the n-th Chebyshev polynomial (for $k = 1$ it was proved by A. Markov in 1889).

Inequality (4) inspired us to consider a new type of Markov inequality (see [3]). Let $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 6. Fix $m \geq 1$, $M > 0$. A compact set $E \subset \mathbb{C}^N$ admits the *V. Markov inequality* ($E \in VMI(m, M)$ in short) if for every $\alpha \in \mathbb{N}_0^N$, $P \in \mathcal{P}(\mathbb{C}^N)$

$$\|D^\alpha P\|_E \leq M^{|\alpha|} \frac{(\deg P)^{m|\alpha|}}{(|\alpha|!)^{m-1}} \|P\|_E \tag{5}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_N$, $D^\alpha \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \ldots \partial z_N^{\alpha_N}}$ for $\alpha = (\alpha_1, \ldots, \alpha_N)$.

In other words, (5) is a version of inequality (3) (and also its analogue in higher dimensional space) with the strongest possible constants M_k (compare with [5] where best Markov exponents were studied). Wiesław Ple?iak was the first to propose inequalities similar to (5), see eg. [18].

Remark 7. If $E \in AMI(m_1, M_1)$ and if we fix an arbitrary $\delta \in (0, 1)$ then for every polynomial P of degree at most n and for all $|\alpha| \leq n^\delta$, inequality (5) holds with $m = \frac{m_1 - \delta}{1 - \delta}$ and $M = M_1$. In the particular case of $m_1 = 1$, we get $AMI(1, M_1) \Leftrightarrow VMI(1, M)$.

In the general case, we do not know whether or not the V. Markov inequality is equivalent to that of A. Markov. However, we can show that the Hölder continuity property is equivalent to (5).

Theorem 8. ([3, Th.2.9]) *If E is a compact subset of $\mathbb{C}^N$, $0 < \gamma \leq 1 \leq m$, $B, M > 0$ then*

$$E \in HCP(\gamma, B) \implies E \in VMI(m, M) \quad with \quad m = 1/\gamma, \quad M = \sqrt{N}\,(B\gamma e)^{1/\gamma}$$

$$E \in VMI(m, M) \implies E \in HCP(\gamma, B) \quad with \quad \gamma = 1/m, \quad B = M^\gamma N^\gamma m.$$

Moreover, if $E \in VMI(m, M)$, then $C(E) \geq e^{-m} \frac{1}{NM}$. Hence, if $E \in HCP(\gamma, B)$, then $C(E) \geq \left(N^{3/2}(B\gamma e^2)^{1/\gamma}\right)^{-1}$.

The last estimate can be slightly improved, see Corollary 18.

As a consequence of the above theorem, the well known open problem concerning the conjectured implication $AMI \Rightarrow HCP$ is equivalent to a new question of whether AMI implies VMI. The first problem regards the properties related to the notions in two different fields: the pluricomplex Green's function and polynomials, whereas the new question is formulated only in terms of derivatives of polynomials. Observe that by the Zahariuta-Siciak formula:

$$V_E(z) = \log \sup\left\{\left(\frac{|P(z)|}{\|P\|_E}\right)^{1/n} : P \in \mathcal{P}_n(\mathbb{C}^N),\ n \geq 1,\ P|_E \not\equiv 0\right\} \quad \text{for} \quad z \in \mathbb{C}^N,$$

the Hölder continuity property HCP can be easily described by a polynomial condition. However, this condition requests a control on polynomials not only on the considered set E but also on its neighborhood independent of the degree of polynomials. In contrast, AMI request a control on polynomials on a neighborhood of E depending on the degree of polynomials (see [19, Th.3.3, condition (ii)]). As for VMI, inequality (5) concerns the behaviour of derivatives of polynomials only on the set E, similarly to AMI.

Due to Theorem 8 given above, we can give new, somewhat unexpected equivalents to the Hölder continuity property of the pluricomplex Green's function:

Corollary 9. ([3, Corollary 2.10]) *If E is a compact subset of $\mathbb{C}^N$ and $\gamma \in (0, 1]$ then the following conditions are equivalent:*

(i) $E \in HCP(\gamma, B_1)$ *with some* $B_1 \geq 1$,

(ii) $\exists B_2 > 0$ $\forall z_0 \in E$ $\forall j \in \{1, \ldots, N\}$ $\forall \zeta \in \mathbb{C}$ *such that* $|\zeta| \leq 1$ *we have*

$$V_E(z_0 + \zeta e_j) \leq B_2 |\zeta|^\gamma,$$

(iii) $\exists M_3 > 0$ $\forall j \in \{1, \ldots, N\}$ $\forall P \in \mathcal{P}(\mathbb{C}^N)$ $\forall k \in \mathbb{N}$ *we have*

$$\|D^{ke_j} P\|_E \leq M_3^k \frac{(\deg P)^{k/\gamma}}{k!^{\frac{1}{\gamma} - 1}} \|P\|_E,$$

where $e_1, \ldots, e_N$ are the canonical vectors in $\mathbb{C}^N$: $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the value 1 in the jth entry.

It seems to be rather surprising that condition (ii) in Corollary 6 that holds only in N canonical directions, is sufficient to guarantee the Hölder continuity property of V_E in all directions.

If E is a compact subset of $\mathbb{R}^N \subset \mathbb{R}^N + i\mathbb{R}^N = \mathbb{C}^N$ and inequality (1) holds for $x \in \mathbb{R}^N$ then it holds for all $z \in \mathbb{C}^N$ (see [3, Cor.4.4]). As a consequence, we obtain

Proposition 10. ([3, Example 4.7]) *There exists an absolute constant $\mathcal{B}$ such that for all dimensions N and for all convex bodies $E \subset \mathbb{R}^N$ the following inequality holds:*

$$V_E(z) \leq \mathcal{B}(dist(z, E)/C(E))^{1/2}, \ z \in \mathbb{C}^N.$$

In particular, these sets belong to $VMI(2, \sqrt{N}\mathcal{B}^2 e^2/[4\,C(E)])$.

Recall a definition of a class of UPC sets introduced by Pawłucki and Pleśniak [15] who have shown its importance in approximation theory. In particular, they have proved a deep result (cf. [15, Cor. 6.5]) that every fat compact subanalytic subset of $\mathbb{R}^N$ belongs to this class (see also [16]).

Let $s \geq 1$, $S > 0$ and $d \in \{1, 2, \ldots\}$.

Definition 11. A compact set $E \subset \mathbb{R}^N$ is called *uniformly polynomially cuspidal* ($E \in UPC(s, S, d)$ in short) if for every $x_0 \in E$ we can find a polynomial mapping $\varphi : \mathbb{R} \to \mathbb{R}^N$ of degree at most d such that $\varphi(1) = x_0$ and

$$\mathrm{dist}(\varphi(t), \mathbb{R}^N \setminus E) \geq S(1-t)^s \quad \text{for} \quad t \in [0,1].$$

It is rather difficult to find the optimal constant s in the last inequality. However, calculations are much simpler for the following modification of the above definition.

Definition 12. (cf. [2]) Let v be a fixed unit vector in $\mathbb{R}^N$. A compact set $E \subset \mathbb{R}^N$ is called *uniformly polynomially cuspidal in direction v* ($E \in UPC_v(s, S, d)$ in short) if for every $x_0 \in E$ we can find a polynomial mapping $\varphi : \mathbb{R} \to \mathbb{R}^N$ of degree at most d such that $\varphi(1) = x_0$ and

$$\mathrm{dist}_v(\varphi(t), \mathbb{R}^N \setminus E) \geq S(1-t)^s \quad \text{for} \quad t \in [0,1].$$

Here $\mathrm{dist}_v(x, \mathbb{R}^N \setminus E) := \sup\{r \geq 0 : [x - rv, x + rv] \subset E\}$.

If $E \in UPC(s, S, d)$ then $E \in UPC_v(s, S, d)$ for every unit vector v. An open problem is whether conditions $E \in UPC_{v_j}(s_j, S_j, d_j)$, $j = 1, \ldots, N$, $v_1, \ldots, v_N$ that are linearly independent imply $E \in UPC(s, S, d)$ with some S, s, d. We conjecture this is true for $N = 2$ but not for $N \geq 3$.

As a consequence of Th.8 given above, we obtain the following theorem that essentially improves earlier result by Pawłucki and Pleśniak [15, Th.4.1] (see also [17]).

Theorem 13. ([3, Corollary 4.11]) *If $E \in UPC_{e_j}(s_j, S_j, d_j)$, $j = 1, \ldots, N$ then there exists a constant B such that $E \in HCP(\gamma, B)$ with $\gamma = 1/(2 \min_j s_j)$. In particular, if $E \in UPC(s, S, d)$ then $E \in HCP(1/(2s), B)$.*

Although the definition of HCP is simple, its verification for particular sets can be very complicated (see e.g. [1, 10, 11]). The Carleson-Totik criterion (see [9, Th.1.2, Th.1.7]) merits mentioning here. It gives an equivalent condition for HCP expressed in terms of capacities in a similar way to Wiener's criterion for L-regularity. This criterion can be used for proving HCP for a large family of sets. However, the Carleson-Totik criterion holds only in the univariate complex case (or in $\mathbb{R}$) and the equivalence is valid under certain additional assumption on sets e.g. for sets satisfying an exterior cone condition. In this context, Theorem 8 given above (i.e. Th.2.9 in [3]) provides a useful tool for showing HCP especially when sets do not satisfy the assumptions of the criterion mentioned above. We give some examples of such an application of Th.8.

The first example regards certain onion type sets in the complex plane that are very useful in a problem concerning local and global Markov's properties (see [8]).

Example 14. ([3, Prop.5.1]) *Let $(a_j)_j$ be a strictly decreasing sequence of positive numbers such that $a_1 = 1$, $a_j \to 0$ as $j \to \infty$ and let $\varphi_j \in (0, \frac{\pi}{2})$ for $j = 1, 2, \ldots$. Put*

$$C_j := \{a_j e^{it} \; : \; t \in [\varphi_j, 2\pi]\} \qquad for \quad j = 1, 2, \ldots,$$

$$E := \{0\} \cup \bigcup_{j=1}^{\infty} C_j.$$

If $|1 - e^{i\varphi_j}| \leq a_{j+1}$ for $j = 1, 2, \ldots$ then $E \in HCP(\frac{1}{6}, B)$ for some $B > 0$.

Example 15. ([3, Prop.5.2]) *Let $\mu \geq 2$, $b \in (0, \sqrt{2} - 1)$ and let $(a_j)_j$, $(r_j)_j$ be sequences of positive numbers such that*

$$a_1 = 2, \quad r_1 = 1, \quad a_j = r_j + r_j^2, \quad r_j = b\, r_{j-1}^{\mu} \; for \quad j \geq 2.$$

Then the set E defined by

$$E := \{0\} \cup \bigcup_{j=1}^{\infty} E_j, \quad E_j := \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N \; : \; |z_1 - a_j| \leq r_j, |z_2| \leq r_j, \ldots, |z_N| \leq r_j\}$$

satisfies $E \in HCP(\frac{1}{2+\mu}, B)$ with some $B > 0$.

In order to study the best constants in Hölder continuity property of the pluricomplex Green's function, we define a new capacity $H(E)$ related to this property. This capacity leads us to an estimate of the L-capacity of the sets with HCP stronger than this one given in Th.8.

Definition 16. If $\gamma \in (0,1]$ is fixed then for E being a compact subset of $\mathbb{C}^N$ put

$$H_\gamma(E) := \left(\inf_{r>0} \frac{r^\gamma}{\gamma e \rho_E(r)}\right)^{1/\gamma} = \inf_{r>0} \frac{r}{(\gamma e \rho_E(r))^{1/\gamma}} = 1/(B(\gamma)\gamma e)^{1/\gamma},$$

where $B(\gamma) = \sup_{r>0} \frac{\rho_E(r)}{r^\gamma}$. We define the *H-capacity* (or *Hölder capacity*) by the following formula

$$H(E) := \sup_{0<\gamma\leq 1} H_\gamma(E).$$

The set function $E \mapsto H(E)$ is called a *capacity* because of its properties similar to features of known capacities:

- $H(a+sE) = sH(E)$ for $a \in \mathbb{C}^N, s>0$ since $\rho_{a+sE}(r) = \rho_E(r/s)$ (cf. Remark 3), and so $H_\gamma(a+sE) = sH_\gamma(E)$,
- $H_\gamma(E\times F) = \min(H_\gamma(E), H_\gamma(F))$ and $H(E\times F) = \sup_{0<\gamma\leq 1} \min(H_\gamma(E), H_\gamma(F))$ in view of the property $\rho_{E\times F}(r) = \max(\rho_E(r), \rho_F(r))$,
- $H(F) \leq H(E)$ if $\rho_E \leq \rho_F$ (where E and F are subsets of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively, for two integers N_1, N_2 that can be distinct),
- if E is a unit ball in $\mathbb{C}^N$ (with respect to a given norm) then $H(E) = C(E)H(\overline{\mathbb{D}}) = C(E)$ because of the formula $\rho_E(r) = \log(1 + r/C(E)) = \rho_{C(E)\overline{\mathbb{D}}}(r)$ and Remark 19,
- if E is a unit ball in $\mathbb{R}^N$ then $H(E) = 2C(E)H([-1,1]) = C(E)$ thanks to $\rho_E(r) = \rho_{[-1,1]}(r/(2C(E)))$ and Remark 19. The same formula is true if E is the standard simplex in $\mathbb{R}^N$, i.e. $E = \{x \in \mathbb{R}^N \mid x_1, \dots, x_N \geq 0,\ x_1 + \dots + x_N \leq 1\}$.

The idea of the Hölder capacity has appeared as an answer to a question posed by R.Eggink about the behaviour of the right-hand side of the estimate of $C(E)$ as a function of $\gamma \in (0,1]$ (see Th.8).

It is clear that $B(\gamma)$ is the best constant in HCP in view of Proposition 4 (see (ii)) and $H_\gamma(E) > 0$ if and only if $E \in HCP(\gamma, B)$. If $H(E) > 0$ then $\gamma(E)$, the Hölder exponent of E, is equal to

$$\gamma(E) = \sup\{\gamma \in (0,1] : \ H_\gamma(E) > 0\}.$$

Theorem 17. *If E is a compact subset of $\mathbb{C}^N$ then $C(E) \geq H_\gamma(E)$ for an arbitrary $\gamma \in (0,1]$ and thus $C(E) \geq H(E)$.*

Proof. We can assume that $H_\gamma(E) > 0$, in particular E is an L-regular set. Let us recall that $\rho_E(r) - \log r \searrow -\log C(E)$ as $r \to \infty$, which implies $\frac{r}{\exp \rho_E(r)} \nearrow C(E)$. In particular, $\frac{r}{\exp \rho_E(r)} \leq C(E)$ for all $r > 0$. Since $\rho_E([0,+\infty)) = [0,+\infty)$, we can take $r \in \rho_E^{-1}(1/\gamma)$. Then

$$H_\gamma(E) \leq \frac{r}{(\gamma e \rho_E(r))^{1/\gamma}} = \frac{r}{e^{1/\gamma}} = \frac{r}{\exp \rho_E(r)} \leq C(E)$$

which completes the proof. □

By Th.8, for $E \in HCP(\gamma, B)$, we have $N^{3\gamma/2}\gamma e^2 B \cdot C(E)^\gamma \geq 1$. We can essentially improve this estimate taking into account the first inequality given in Th.17:

Corollary 18. If E is a compact subset of $\mathbb{C}^N$ and $E \in HCP(\gamma, B)$ for some $\gamma \in (0,1]$ and $B > 0$, then

$$\gamma e B \cdot C(E)^\gamma \geq 1.$$

The above inequality gives not only an estimate of the capacity but also a bound for the constant B in the Hölder Continuity Property in the dependance on the capacity of E (compare with [20]). The estimates of the constant B are important from the numerical point of view, especially as the best exponent in HCP (and in AMI) is not known.

Remark 19.

(i) $H(\overline{\mathbb{D}}) = \lim_{\gamma\to 0+} H_\gamma(\overline{\mathbb{D}}) = 1 = C(\overline{\mathbb{D}})$ and so $H(E) = C(E)$ for all complex balls (see the first property of the H-capacity listed above).

(ii) $\lim_{\gamma\to 0+} H_\gamma([-1,1]) = \frac{1}{2} = C([-1,1])$ and thus $H(E) = C(E)$ for all real balls.

Indeed, for $\gamma \in (0,1)$ we consider $f_\gamma(r) = \frac{\log(1+r)}{r^\gamma}$. Since $\lim_{r\to 0+} f_\gamma(r) = \lim_{r\to+\infty} f_\gamma(r) = 0$, we have $\sup_{r>0} f_\gamma(r) = f_\gamma(r_\gamma)$ with r_γ such that $f_\gamma'(r_\gamma) = 0$. In other words, $\gamma \log(1+r_\gamma) = \frac{r_\gamma}{1+r_\gamma}$. The inequality

$$\gamma r_\gamma \geq \gamma \log(1+r_\gamma) \geq \frac{r_\gamma}{1+r_\gamma}$$

implies that

$$r_\gamma \geq 1/\gamma - 1$$

and so

$$\lim_{\gamma\to 0+} r_\gamma = +\infty.$$

Hence

$$H_\gamma(\overline{\mathbb{D}}) = \frac{r_\gamma}{(\gamma e \log(1+r_\gamma))^{1/\gamma}}$$

$$= \frac{r_\gamma}{(1+r_\gamma)^{1+1/r_\gamma}} \exp\big((1+1/r_\gamma)\log(1+1/r_\gamma)\log(1+r_\gamma)\big) \longrightarrow 1 \quad \text{as} \quad \gamma \to 0+.$$

Similarly, we can verify (ii). If $\gamma \in (0, \frac{1}{2})$ and r_γ is a solution of the equation $\frac{1}{\gamma} = \sqrt{1+\frac{2}{r}}\log h(1+r) \geq 2$ where $h(t) = t + \sqrt{t^2-1}$, then $r_\gamma \geq \frac{1}{2\gamma} - 2$ and

$$H_\gamma([-1,1]) = \frac{r_\gamma}{h(1+r_\gamma)^{\sqrt{1+2/r_\gamma}}} \exp\left(\frac{1}{2}\sqrt{1+\frac{2}{r_\gamma}}\log h(1+r_\gamma)\log\left(1+\frac{2}{r_\gamma}\right)\right) \longrightarrow \frac{1}{2}$$

as $\gamma \to 0+$.

Proposition 20. *The function* $(0,1) \ni \gamma \longmapsto H_\gamma(\overline{\mathbb{D}})$ *is decreasing.*

Proof. Let $g(r) = (1+1/r)\log(1+r)$. It is easy to verify that $g'(r) = -\frac{1}{r^2}\log(1+r) + \frac{1}{r} > 0$ and $g : (0,+\infty) \longrightarrow (1,+\infty)$ is increasing. We have

$$H_\gamma(\overline{\mathbb{D}}) = g^{-1}(1/\gamma)e^{-1/\gamma}(1+1/g^{-1}(1/\gamma))^{1/\gamma}.$$

If we put

$$\varphi(x) = g^{-1}(x)e^{-x}(1+1/g^{-1}(x))^x$$

then

$$\log\psi(t) = \log\varphi(g(t)) = \log t - g(t) + g(t)\log(1+1/t),$$

which gives

$$\frac{\psi'(t)}{\psi(t)} = g'(t)\log(1+1/t) > 0$$

and the proof is completed. □

Similarly, we can check that there exists $\gamma_0 > 0$ such that the function $(0,\gamma_0) \ni \gamma \longmapsto H_\gamma([-1,1])$ is decreasing.

Acknowledgment. We wish to express our gratitude to Raimondo Eggink for his stimulating question.
The work was partially supported by the NCN grant No. 2013/11/B/ST1/03693.

References

[1] V. V. Andrievski. The highest smoothness of the Green function implies the highest density of a set. *Ark. Mat.*, 42:217–238, 2004.

[2] M. Baran. Markov inequality on sets with polynomial parametrization. *Ann. Polon. Math.*, 60:69–79, 1994.

[3] M. Baran, L. Bialas-Ciez. Hölder continuity of the Green function and Markov brothers' inequality. *Constr. Approx.*, 40:121–140, 2014.

[4] M. Baran, L. Bialas-Ciez. Product property for capacities in $\mathbb{C}^N$. *Ann. Polon. Math.*, 106:19–29, 2012.

[5] M. Baran, L. Bialas-Ciez, B. Milówka. On the best exponent in Markov inequality. *Potential Anal.*, 38:635–651, 2013.

[6] L. Bialas-Ciez. Markov sets in $\mathbb{C}$ are not polar. *Bull. Pol. Acad. Sci.-Math.*, 46:83–89, 1998.

[7] L. Bialas-Ciez, R. Eggink. L-regularity of Markov sets and of m-perfect sets in the complex plane. *Constr. Approx.*, 27:237–252, 2008.

[8] L. Bialas-Ciez, R. Eggink. Equivalence of the global and local Markov inequalities in the complex plane. Submitted.

[9] L. Carleson, V. Totik. Hölder continuity of Green's functions. *Acta Sci. Math., Szeged*, 70:557–608, 2004.

[10] M. Kosek. Hölder continuity property of filled-in Julia sets in $\mathbb{C}^n$. *P. Am. Math. Soc.*, 125:2029–2032, 1997.

[11] M. Kosek. Hölder continuity property of composite Julia sets. *B. Pol. Acad. Sci.-Math.*, 46:391–399, 1998.

[12] M. Klimek. Pluripotential Theory. London Mathematical Society Monographs New Series 6, Clarendon Press, Oxford, 1991.

[13] P. Lelong, L. Gruman. Entire Functions of Several Complex Variables. Springer Verlag, Berlin, Heidelberg, 1986.

[14] B. Milówka. Markov's inequality and a generalized Pleśniak condition. *East J. Approx.*, 11:291–300, 2005.

[15] W. Pawłucki, W. Pleśniak. Markov's inequality and C^∞ functions on sets with polynomial cusps. *Math. Ann.*, 275:467–480, 1986.

[16] R. Pierzchała. UPC condition in polynomially bounded o-minimal structures. *J. Approx. Theory*, 132:25–33, 2005.

[17] R. Pierzchała. Siciak's extremal function of non-UPC cusps. *J. Math. Pures Appl.*, 94:451-469, 2010.

[18] W. Pleśniak. Quasianalytic functions in the sense of Bernstein. *Dissert. Math. (Rozprawy Mat.)*, 147, 1977.

[19] W. Pleśniak. Markov's inequality and the existence of an extension operator for C^∞ functions. *J. Approx. Theory*, 61(1):106–117, 1990.

[20] C. Pommerenke. On the derivative of a polynomial. *Michigan Math. J.*, 6:373–375, 1959.

[21] A. Shadrin. Twelve proofs of the Markov inequality. *Approx. Theory*: a volume dedicated to Borislav Bojanov, Prof. Drinov Acad. Publ. House, Sofia, 2004, 233-298.