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Estimates in variation for multivariate sampling-type operators

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Abstract

We prove some estimates with respect to the Tonelli variation for the multidimensional generalized sampling operators and for a class of sampling-Kantorovich type operators in terms of the Tonelli integrals. As a consequence, we obtain an estimate for the total variation of the same operators.

1 Introduction

In this paper we state some estimates with respect to the Tonelli variation for the multidimensional *generalized sampling operators* in terms of the Tonelli integrals and, as a consequence, we obtain an estimate for the total variation of the same operators. Moreover we also investigate some estimates for the so called *mixed sampling Kantorovich operators*, again in terms of the Tonelli integrals and of the total variation functional.

In order to frame the results obtained in this paper in the context of the state of the art of the specific field, we introduce the families of operators considered and we motivate both their role and the results, obtained so far in the scientific literature.

The role and the importance of the generalized sampling series (operators) are well known from the work of the famous german mathematician P.L. Butzer and his school at RWTH-Aaachen, where they were introduced and studied, around the 1980s, as a mathematical tool designed to weaken the well-known and strong assumptions of the classical Whittaker-Kotelnikov-Shannon sampling theorem ([21, 22, 23]); see also e.g., ([40, 41, 16]).

Subsequently, in [18, 19, 12], the multidimensional version of these operators was introduced and studied, while several years later in [28], the multidimensional version of the so called *sampling-Kantorovich operators* ([13]) has been introduced, also because their importance from the point of view of applications was understood, in particular as a quasi-interpolation method for the reconstruction of digital images [32, 33, 15, 36, 25, 37, 29, 27]; see also, e.g., [13, 38, 3, 6, 30, 24, 26].

The generalized sampling operators are defined as

$$(S_w f)(\mathtt{t}) := \sum_{\mathtt{k}\in\mathbb{Z}^N} f\left(\frac{\mathtt{k}}{w}\right)\chi(w\mathtt{t}-\mathtt{k}), \tag{1}$$

for every $\mathtt{t} = (t_1, \dots, t_N) \in \mathbb{R}^N$, $\mathtt{k} = (k_1, \dots, k_N) \in \mathbb{Z}^N$ and $w > 0$, where $\chi : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a *kernel* and $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ is the function to be approximated.

While estimates and convergence (pointwise, uniform or in norm) for the operators (1) have been investigated using a kind of "direct" approach, as concerns the same results with respect to the variation, the situation appears very delicate to treat and, in case of the Tonelli variation, the approach is not "direct". Indeed, in [7] in order to obtain the convergence with respect to the Tonelli variation, it has been suitable to pass through the convergence of a family of operators, called mixed sampling-Kantorovich operators (later defined), in a suitable subspace of $L^p(\mathbb{R}^N)$ (i.e., precisely the space $\Lambda^p(\mathbb{R}^N)$ recalled in Section 3), exploiting a relation between the partial derivatives of the multidimensional sampling series and the multidimensional sampling-Kantorovich type operators acting on the partial derivatives of the function. In addition, the authors used product kernels of *averaged type*, that is kernels of the form

$$\bar{\chi}_m(\mathtt{t}) := \prod_{i=1}^{N} \bar{\chi}_{i,m}(t_i) \tag{2}$$

where

$$\bar{\chi}_{i,m}(t) := \frac{1}{m}\int_{-\frac{m}{2}}^{\frac{m}{2}} \chi_i(t+v)\,dv,$$

for some $m \in \mathbb{N}$, and where $\chi_i : \mathbb{R} \longrightarrow \mathbb{R}$ is a one-dimensional kernel for every $i = 1, \dots, N$.

In case of product kernels of averaged type, the corresponding multivariate generalized sampling series and mixed sampling-Kantorovich operators associated to $\bar{\chi}_m$ will be defined as

$$(\bar{S}^m_w f)(\mathtt{t}) := \sum_{\mathtt{k}\in\mathbb{Z}^N} f\left(\frac{\mathtt{k}}{w}\right)\bar{\chi}_m(w\mathtt{t}-\mathtt{k}), \quad \mathtt{t}\in\mathbb{R}^N,\ w>0, \tag{3}$$

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and

$$(\bar{K}_{w,j}f)(\mathtt{t}) := \sum_{\mathtt{k}\in\mathbb{Z}^N} \left[w \int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} f\left(\frac{k_1}{w},\ldots,u\ldots,\frac{k_N}{w}\right) du \right] \bar{\chi}_m(w\mathtt{t}-\mathtt{k}), \tag{4}$$

respectively.

In [8] an estimate with respect to the Tonelli variation for the Kantorovich sampling operators (not-mixed) with product-type averaged kernels, has been provided.

The main results of the present paper consist of Theorems 1 and 2 of Section 3. Namely in Theorem 1 we prove that the operators $\bar{S}^m_w f$ map $BV_\Lambda(\mathbb{R}^N)$ in $BV(\mathbb{R}^N)$ and

$$\|V^j[\bar{S}^m_w f]\|_{L^1(\mathbb{R}^{N-1})} \leq \prod_{i=1}^N \|\chi_i\|_{L^1} \|V^j[f]\|_{\ell^1(\Sigma_w^{N-1})}$$

for every $j=1,\ldots,N$, $w>0$, $m\in\mathbb{N}$, $f\in BV_\Lambda(\mathbb{R}^N)$, where $BV_\Lambda(\mathbb{R}^N)$ is a subspace of $BV(\mathbb{R}^N)$ introduced in Section 2.
As a consequence,

$$V[\bar{S}^m_w f] \leq \prod_{i=1}^N \|\chi_i\|_{L^1} \sum_{j=1}^N \|V^j[f]\|_{\ell^1(\Sigma_w^{N-1})}.$$

While the first inequality represents an estimate for the Tonelli integrals of the generalized sampling operators (with averaged kernels) in terms of the Tonelli integrals of the function f, the second inequality gives an estimate of the total variation of the same operators in terms of a finite sum of a kind of Tonelli integrals of f. In both the inequalities, the Tonelli integrals of the function f are calculated with respect to the measure of the space where the function f lives, i.e. a space which is related to the uniform partition compatible with the structure of the sampling type operators (having an admissible partition with a uniform sampling scheme).

As concerns Theorem 2, we have reached similar results for the mixed sampling-Kantorovich operators (with averaged kernels), i.e., $\bar{K}^m_{w,j}f\in BV(\mathbb{R}^N)$ for every $j=1,\ldots,N$, $w>0$, $m\in\mathbb{N}$, whenever $f\in BV_\Lambda(\mathbb{R}^N)$ and

$$\|V^i[\bar{K}^m_{w,j}f]\|_{L^1(\mathbb{R}^{N-1})} \leq \frac{m+1}{m} \prod_{i=1}^N \|\chi_i\|_{L^1(\mathbb{R})} \|V^j[f]\|_{\ell^1(\Sigma_w^{N-1})},$$

for every $i,j=1,\ldots,N$.
Moreover, as a consequence, we obtain the following estimate for the total variaton of $\bar{K}^m_{w,j}f$,

$$V[\bar{K}^m_{w,j}f] \leq \frac{m+1}{m} \prod_{i=1}^N \|\chi_i\|_{L^1(\mathbb{R})} \sum_{j=1}^N \|V^j[f]\|_{\ell^1(\Sigma_w^{N-1})}.$$

In conclusion, in this paper we solve the still unsolved problem of obtaining estimates in variation for the families of operators (3), (4) and a fundamental step to get these results are the estimates in terms of the Tonelli integrals of the operators involved and of the function f.
Note that, in case of a step function f with compact support (see Remark 2 of Section 3), the above estimates bring to variation-diminishing type properties for the considered operators. This property is very important in the reconstruction of digital images, where variation diminishing plays a fundamental role as a smoothing filter on the original image (see, e.g., [8]).

2 Notations and preliminaries

In the multidimensional setting of $\mathbb{R}^N$, we will use the usual notation $\mathtt{t}=(t_1,\ldots,t_N)$ for a vector of $\mathbb{R}^N$; moreover we will denote $\mathtt{t}=(\mathtt{t}'_j,t_j)$, where $\mathtt{t}'_j=(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_N)\in\mathbb{R}^{N-1}$, and also $\mathtt{t}=(\mathtt{t}''_{j,i},t_j,t_i)$ with $\mathtt{t}''_{j,i}=(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{i-1},t_{i+1},\ldots,t_N)$. Finally $\alpha\mathtt{t}=(\alpha t_1,\ldots,\alpha t_N)$ and, for $\alpha\neq 0$, $\frac{\mathtt{t}}{\alpha}=\left(\frac{t_1}{\alpha},\ldots,\frac{t_N}{\alpha}\right)$. In order to study a "real" multidimensional setting we assume $N>1$: nevertheless the one-dimensional case has already been explored in [6, 8] (see Remark 1).

We will study estimates in variation for the classical *generalized sampling operators* (1) and for a Kantorovich version of such operators recently introduced in [7], named *mixed sampling-Kantorovich operators*.

The latter are obtained replacing the value $f\left(\frac{\mathtt{k}}{w}\right)$ with the integral mean computed with respect to the j—th variable, hence leading to a "mixed" Kantorovich version of $(S_w)_w$, that is,

$$(K_{w,j}f)(\mathtt{t}) := \sum_{\mathtt{k}\in\mathbb{Z}^N} \left[w \int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} f\left(\frac{k_1}{w},\ldots,u\ldots,\frac{k_N}{w}\right) du \right] \chi(w\mathtt{t}-\mathtt{k}), \tag{5}$$

for every $\mathtt{t}\in\mathbb{R}^N$, $w>0$ and $j=1,\ldots,N$ ([7, 9]). We refer to [13] for the introduction of the generalized sampling-Kantorovich operators in the one-dimensional case, where the reader can find the reason for the association of the name of Kantorovich with the generalized sampling series.

In both the definitions, χ is a product kernel, that is, $\chi(\mathtt{t})=\prod_{i=1}^N \chi_i(\mathtt{t})$, $\mathtt{t}\in\mathbb{R}^N$, is the product of one-dimensional kernels $\chi_i:\mathbb{R}\to\mathbb{R}$, $i=1,\ldots,N$, that fulfill the following assumptions:

(I) χ_i is continuous and such that $\sum_{k\in\mathbb{Z}} \chi_i(t-k)=1$, for every $t\in\mathbb{R}$;

(II) $A_{\chi_i} := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_i(u-k)| < +\infty$, where the convergence of the series $\sum_{k\in\mathbb{Z}} |\chi_i(u-k)|$ is uniform on the compact subsets of $\mathbb{R}$.

We point out that the product function χ is a kernel itself, i.e., it satisfies the (multidimensional) conditions

(i) χ is continuous and such that $\sum_{\mathtt{k}\in\mathbb{Z}^N} \chi(\mathtt{t}-\mathtt{k})=1$, for every $\mathtt{t}\in\mathbb{R}^N$;

(ii) $A_\chi := \sup_{\mathtt{u}\in\mathbb{R}^N} \sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(\mathtt{u}-\mathtt{k})| < +\infty$, where the convergence of the series $\sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(\mathtt{u}-\mathtt{k})|$ is uniform on the compact subsets of $\mathbb{R}^N$.

The above assumptions on kernels are absolutely natural in case of discrete operators (see, e.g., [19, 45, 15, 34, 14, 17, 31]).

It is immediate to see that, with the above assumptions, both the operators are well defined if, for example, f is bounded (and therefore in particular if f is of bounded variation). Indeed if $|f(\mathtt{t})|\le L$, for every $\mathtt{t}\in\mathbb{R}^N$, $w>0$,

$$|(S_w f)(\mathtt{t})| \le L \sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(w\mathtt{t}-\mathtt{k})| \le LA_\chi$$

and, for every $j=1,\dots,N$,

$$|(K_{w,j}f)(\mathtt{t})| \le \sum_{\mathtt{k}\in\mathbb{Z}^N}\left[w\int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} \left| f\left(\frac{k_1}{w},\dots,u\dots,\frac{k_N}{w}\right)\right| du \right] |\chi(w\mathtt{t}-\mathtt{k})| \le L\sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(w\mathtt{t}-\mathtt{k})| \le LA_\chi.$$

Throughout this paper, we will consider product kernels of *averaged type*, that is kernels of the form (2) where $\chi_i:\mathbb{R}\longrightarrow\mathbb{R}$ is a one-dimensional kernel for every $i=1,\dots,N$ (i.e., satisfying (I) and (II)).

Of course $\bar{\chi}_m$ is a kernel itself and

$$\begin{aligned}\|\bar{\chi}_{i,m}\|_1 &= \int_{\mathbb{R}} \left| \frac{1}{m}\int_{-\frac{m}{2}}^{\frac{m}{2}} \chi_i(t+v)\,dv \right| dt \le \frac{1}{m}\int_{-\frac{m}{2}}^{\frac{m}{2}} \int_{\mathbb{R}} |\chi_i(t+v)|\,dt\,dv \\ &= \|\chi_i\|_1\end{aligned} \tag{6}$$

and also $\|\bar{\chi}_m\|_1 \le \prod_{i=1}^N \|\chi_i\|_1$.

It is easy to provide examples of product kernels of averaged type. Among them, it is easy to see that the central B-splines of order $n\in\mathbb{N}$ (see, e.g., [20, 43]), defined as

$$M_n(x) := \frac{1}{(n-1)!}\sum_{i=0}^{n} (-1)^i \binom{n}{i}\left(\frac{n}{2}+x-i\right)_+^{n-1}, \quad x\in\mathbb{R},$$

where $(x)_+ := \max\{x,0\}$ denotes "the positive part" of $x\in\mathbb{R}$, are kernels (they satisfy conditions (I) and (II)) of averaged type since

$$\bar{M}_{n,1}(t) = M_{n+1}(t), \quad t\in\mathbb{R},$$

for every $n\in\mathbb{N}$, i.e., the averaged kernel with $m=1$ generated by a central B-spline of order n is a B-spline itself of order $n+1$. Therefore the product kernel $\mathcal{M}_1^n(\mathtt{t}) := \prod_{i=1}^N \bar{M}_{n,1}(t_i) = \prod_{i=1}^N M_{n+1}(t_i)$, $\mathtt{t}\in\mathbb{R}^N$, is an example of a product kernel of averaged type to which our results can be applied.

We refer to [7, 8] for other examples of product averaged type kernels.

From now on, we will deal with multivariate generalized sampling series and mixed sampling-Kantorovich operators associated to $\bar{\chi}_m$, as defined in (3) and (4), i.e.,

$$(\bar{S}_w^m f)(\mathtt{t}) := \sum_{\mathtt{k}\in\mathbb{Z}^N} f\left(\frac{\mathtt{k}}{w}\right)\bar{\chi}_m(w\mathtt{t}-\mathtt{k}), \ \ \mathtt{t}\in\mathbb{R}^N, \ w>0,$$

and

$$(\bar{K}_{w,j}f)(\mathtt{t}) := \sum_{\mathtt{k}\in\mathbb{Z}^N}\left[w\int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} f\left(\frac{k_1}{w},\dots,u\dots,\frac{k_N}{w}\right) du \right]\bar{\chi}_m(w\mathtt{t}-\mathtt{k}),$$

respectively.

Notice that $\bar{\chi}_m$ is differentiable and, obviously,

$$\frac{\partial \bar{\chi}_m}{\partial t_j}(\mathtt{t}) = \frac{1}{m}\prod_{i\ne j}\bar{\chi}_{i,m}(t_i)\left[\chi_j\left(t_j+\frac{m}{2}\right)-\chi_j\left(t_j-\frac{m}{2}\right)\right], \ \ \mathtt{t}\in\mathbb{R}^N.$$

As usual, $V_{[a,b]}[f] = \sup\sum_{i=1}^n |f(x_i)-f(x_{i-1})|$, where the supremum is taken over all the possible partitions $a=x_0<x_1<\dots<x_n=b$ of the interval $[a,b]$, is the Jordan variation of f over $[a,b]$ and $V_{\mathbb{R}}[f] := \sup_{[a,b]\subset\mathbb{R}} V_{[a,b]}[f]$ denotes the Jordan variation of f over $\mathbb{R}$.

By $M(\mathbb{R}^N)$ will denote the space of all the measurable and bounded functions $f:\mathbb{R}^N\to\mathbb{R}$.

The concept of Jordan variation was extended in the multidimensional frame in several directions and several definitions were proposed in the literature (see, e.g., [11]). One of the most used concepts, that is very convenient working with approximation results, is the Tonelli variation (see [42, 39, 44]). In order to recall the definition of BV-functions in the sense of Tonelli, we introduce the auxiliary notation

$$V^j[f](\mathtt{x}'_j) := V_{\mathbb{R}}[f(\mathtt{x}'_j,\cdot)], \;\; \mathtt{x}'_j\in\mathbb{R}^{N-1},$$

so that $V^j[f]:\mathbb{R}^{N-1}\to\mathbb{R}$, $j=1,\dots,N$, where $f(\mathtt{x}'_j,\cdot)$ are the $j-th$ sections of f.

Definition 2.1. A function $f\in M(\mathbb{R}^N)$ is said to be *of bounded variation* (in the sense of Tonelli) if the $j-th$ sections $f(\mathtt{x}'_j,\cdot)$ are of bounded variation on $\mathbb{R}$ for a.e. $\mathtt{x}'_j\in\mathbb{R}^{N-1}$ and $V^j[f]\in L^1(\mathbb{R}^{N-1})$, for every $j=1,\dots,N$. We denote by $BV(\mathbb{R}^N)$ the space of functions of bounded variation.

For the sake of completeness, we now recall how to compute the Tonelli variation of a function f, according to its classical definition, although in this paper we will always use its integral representation (7).

Given $I=\prod_{i=1}^N[a_i,b_i]$ and $j=1,\dots,N$ one considers the $(N-1)$—dimensional integrals (Tonelli integrals)

$$\Phi_I^j(f) := \int_{[\mathtt{a}'_j,\mathtt{b}'_j]} V_{[a_j,b_j]}[f(\mathtt{x}'_j,\cdot)]d\mathtt{x}'_j,$$

and their Euclidean norm $\Phi_I(f) := \left\{\sum_{j=1}^N(\Phi_I^j(f))^2\right\}^{\frac{1}{2}}$, where $\Phi_I(f)=+\infty$ if $\Phi_I^j(f)=+\infty$, for some $j=1,\dots,N$.

Then

$$V_I[f] := \sup\sum_{k=1}^m \Phi_{J_k}(f),$$

where the supremum is taken over all the finite families of N—dimensional intervals $\{J_1,\dots,J_m\}$ which form partitions of I, is the variation of f on $I\subset\mathbb{R}^N$. Passing to the supremum over all the intervals $I\subset\mathbb{R}^N$, we obtain the variation of f on $\mathbb{R}^N$, i.e.,

$$V[f] := \sup_{I\subset\mathbb{R}^N} V_I[f].$$

We now recall the notion of absolute continuity in the sense of Tonelli.

Definition 2.2. A function $f:\mathbb{R}^N\to\mathbb{R}$ is locally absolutely continuous in the sense of Tonelli ($f\in AC_{\mathrm{loc}}(\mathbb{R}^N)$) if, for every interval $I=\prod_{i=1}^N[a_i,b_i]$ and for every $j=1,2,\dots,N$, the j-th section of f, $f(\mathtt{x}'_j,\cdot):[a_j,b_j]\to\mathbb{R}$, is absolutely continuous for almost every $\mathtt{x}'_j\in[\mathtt{a}'_j,\mathtt{b}'_j]$.

It is well known that, if $f\in AC(\mathbb{R}^N) := BV(\mathbb{R}^N)\cap AC_{\mathrm{loc}}(\mathbb{R}^N)$, ($f$ is absolutely continuous) then the variation of f has the following integral representation, i.e.,

$$V[f] = \int_{\mathbb{R}^N} |\nabla f(\mathtt{x})|\, d\mathtt{x} \tag{7}$$

(see, e.g., [39, 44, 35, 10]). For approximation results by means of discrete and integral operators we refer, e.g., to [2, 4, 1, 5].

Working with the multidimensional version of the generalized sampling series $(S_w f)_{w>0}$ in the L^p frame, it is natural to introduce a suitable subspace of $L^p(\mathbb{R}^N)$, namely the space $\Lambda^p(\mathbb{R}^N)$: indeed in [15] (see also [12] for $N=1$) it is proved that, in such subspace, it is possible to achieve convergence in L^p for $(S_w f)_{w>0}$. Due to the natural link between convergence in variation of a function f and convergence in L^1 of its partial derivatives, in order to obtain convergence results in the space $BV(\mathbb{R}^N)$ by means of the generalized sampling series, it is natural to work again within the subspace $\Lambda^1(\mathbb{R}^N)$ (see [7]). On the other side, since the definition of the mixed sampling-Kantorovich operators is very close to that one of $(S_w f)_{w>0}$, it is natural to expect that convergence in L^p holds for $(K_{w,j} f)_{w>0}$ in the same subspace ([7]). For all these reasons, in the present paper $\Lambda^p(\mathbb{R}^N)$ will play a crucial role: let us now recall its definition ([15, 7] and [12] for $N=1$).

We first recall the concept of *admissible partition* over the i-th axis, i.e., a partition $\Sigma_i := (x_{i,j_i})_{j_i\in\mathbb{Z}}$ such that

$$0<\underline{\Delta} := \min_{i=1,\dots,N}\inf_{j_i\in\mathbb{Z}}(x_{i,j_i}-x_{i,j_i-1}) \le \max_{i=1,\dots,N}\sup_{j_i\in\mathbb{Z}}(x_{i,j_i}-x_{i,j_i-1}) =: \overline{\Delta} < +\infty.$$

A sequence $\Sigma=(\mathtt{x}_{\mathtt{j}})_{\mathtt{j}\in\mathbb{Z}^N}\subset\mathbb{R}^N$, $\mathtt{x}_{\mathtt{j}}=(x_{1,j_1},\dots,x_{N,j_N})$, $\mathtt{j}=(j_1,\dots,j_N)\in\mathbb{Z}^N$, is an *admissible sequence* if it is the cartesian product of admissible partitions $\Sigma_i=(x_{i,j_i})_{j_i\in\mathbb{Z}}$.

Let us fix an admissible sequence Σ: then the $l^p(\Sigma)$-norm of $f:\mathbb{R}^N\to\mathbb{R}$ is defined as

$$\|f\|_{l^p(\Sigma)} := \left\{\sum_{\mathtt{j}\in\mathbb{Z}^N}\sup_{\mathtt{x}\in Q_{\mathtt{j}}}|f(\mathtt{x})|^p\Delta_{\mathtt{j}}\right\}^{\frac{1}{p}}, \quad 1\le p<+\infty,$$

where $Q_{\mathtt{j}}=\prod_{i=1}^N[x_{i,j_i-1}-x_{i,j_i}[$ and $\Delta_{\mathtt{j}} := \prod_{i=1}^N(x_{i,j_i}-x_{i,j_i-1})$ is the volume of $Q_{\mathtt{j}}$. The subspace $\Lambda^p(\mathbb{R}^N)$ is defined as

$$\Lambda^p(\mathbb{R}^N) := \{f\in M(\mathbb{R}^N):\; \|f\|_{l^p(\Sigma)}<+\infty, \text{ for every admissible sequence } \Sigma\}.$$

It can be proved (see [12, 15]) that $\Lambda^p(\mathbb{R}^N)$ is a proper linear subspace of $L^p(\mathbb{R}^N)$ and that it contains, for example, all the measurable functions with compact support. We refer to [15] and to [12] for other properties concerning such space. Among the admissible sequences, the sampling grid will play an important role: let us denote it as Σ_w^N, i.e., Σ_w^N is the cartesian product of $\left(\frac{k_j}{w}\right)_{k_j\in\mathbb{Z}}$, $j=1,\dots,N$.

For convenience, we also introduce the following notation: we will denote by $BV_\Lambda(\mathbb{R}^N)$ the space of functions $f\in M(\mathbb{R}^N)$ such that the $j-th$ sections $f(\mathtt{x}_j',\cdot)$ are of bounded variation on $\mathbb{R}$ for a.e. $\mathtt{x}_j'\in\mathbb{R}^{N-1}$ and $V^j[f]\in\Lambda^1(\mathbb{R}^{N-1})$, for every $j=1,\dots,N$.

Of course, $BV_\Lambda(\mathbb{R}^N)$ is a subspace of $BV(\mathbb{R}^N)$ and, for example, it contains all the functions of bounded variation with compact support.

3 Main results

Our first result will be an estimate in variation for the generalized sampling operators.

Theorem 3.1. *If $f\in BV_\Lambda(\mathbb{R}^N)$, then $\bar{S}_w^m f\in BV(\mathbb{R}^N)$, $w>0$, $m\in\mathbb{N}$, and*

$$\|V^j[\bar{S}_w^m f]\|_{L^1(\mathbb{R}^{N-1})}\le\prod_{i=1}^N\|\chi_i\|_{L^1(\mathbb{R})}\|V^j[f]\|_{\ell^1(\Sigma_w^{N-1})} \tag{8}$$

for every $j=1,\dots,N$.

As a consequence,

$$V[\bar{S}_w^m f]\le\prod_{i=1}^N\|\chi_i\|_{L^1(\mathbb{R})}\sum_{j=1}^N\|V^j[f]\|_{\ell^1(\Sigma_w^{N-1})}. \tag{9}$$

Proof. We can write, for every $j=1,\dots,N$, $\mathtt{t}\in\mathbb{R}^N$, $w>0$,

$$\frac{\partial\bar{S}_w^m f}{\partial t_j}(\mathtt{t})=\frac{w}{m}\sum_{\mathtt{k}\in\mathbb{Z}^N}f\left(\frac{\mathtt{k}}{w}\right)\prod_{i\neq j}\bar{\chi}_{i,m}(wt_i-k_i)\left[\chi_j\left(wt_j-k_j+\frac{m}{2}\right)-\chi_j\left(wt_j-k_j-\frac{m}{2}\right)\right].$$

Moreover, since f is in particular bounded, by (II) for each one-dimensional kernel $\bar{\chi}_{i,m},\chi_j$,

$$\left|\frac{\partial\bar{S}_w^m f}{\partial t_j}(\mathtt{t})\right|\le\frac{w}{m}L\sum_{\mathtt{k}\in\mathbb{Z}^N}\prod_{i\neq j}|\bar{\chi}_{i,m}(wt_i-k_i)|\quad\left[\left|\chi_j\left(wt_j-k_j+\frac{m}{2}\right)\right|+\left|\chi_j\left(wt_j-k_j-\frac{m}{2}\right)\right|\right]\le\frac{2w}{m}L\prod_{i\neq j}A_{\bar{\chi}_{i,m}}A_{\chi_j}.$$

This implies that $\bar{S}_w^m f\in AC_{loc}(\mathbb{R}^N)$.

Again, it is possible to write, putting in the second series $\tilde{k}_i=k_i$ for $i\neq j$ and $\tilde{k}_j=k_j+m$,

$$\begin{aligned}
\frac{\partial\bar{S}_w^m f}{\partial t_j}(\mathtt{t})&=\frac{w}{m}\sum_{\mathtt{k}\in\mathbb{Z}^N}f\left(\frac{\mathtt{k}}{w}\right)\prod_{i\neq j}\bar{\chi}_{i,m}(wt_i-k_i)\cdot\chi_j\left(wt_j-k_j+\frac{m}{2}\right)+\\
&-\frac{w}{m}\sum_{\tilde{\mathtt{k}}\in\mathbb{Z}^N}f\left(\frac{\tilde{\mathtt{k}}_j'}{w},\frac{\tilde{k}_j-m}{w}\right)\prod_{i\neq j}\bar{\chi}_{i,m}(wt_i-\tilde{k}_i)\cdot\chi_j\left(wt_j-\tilde{k}_j+\frac{m}{2}\right)\\
&=\frac{w}{m}\sum_{\mathtt{k}_j'\in\mathbb{Z}^{N-1}}\sum_{k_j\in\mathbb{Z}}\left[f\left(\frac{\mathtt{k}_j'}{w},\frac{k_j}{w}\right)-f\left(\frac{\mathtt{k}_j'}{w},\frac{k_j-m}{w}\right)\right]\prod_{i\neq j}\bar{\chi}_{i,m}(wt_i-k_i)\cdot\chi_j\left(wt_j-k_j+\frac{m}{2}\right)\\
&\le\frac{w}{m}\sum_{\mathtt{k}_j'\in\mathbb{Z}^{N-1}}\sum_{k_j\in\mathbb{Z}}V_{\left[\frac{k_j-m}{w},\frac{k_j}{w}\right]}\left[f\left(\frac{\mathtt{k}_j'}{w},\cdot\right)\right]\prod_{i\neq j}\bar{\chi}_{i,m}(wt_i-k_i)\cdot\chi_j\left(wt_j-k_j+\frac{m}{2}\right)\\
&=w\sum_{\mathtt{k}_j'\in\mathbb{Z}^{N-1}}V_{\mathbb{R}}\left[f\left(\frac{\mathtt{k}_j'}{w},\cdot\right)\right]\prod_{i\neq j}\bar{\chi}_{i,m}(wt_i-k_i)\cdot\chi_j\left(wt_j-k_j+\frac{m}{2}\right).
\end{aligned}$$

Therefore there holds

$$\begin{aligned}
\int_{\mathbb{R}^N}\left|\frac{\partial\bar{S}_w^m f}{\partial t_j}(\mathtt{t})\right|d\mathtt{t}&\le w\sum_{\mathtt{k}_j'\in\mathbb{Z}^{N-1}}V_{\mathbb{R}}\left[f\left(\frac{\mathtt{k}_j'}{w},\cdot\right)\right]\int_{\mathbb{R}^N}\prod_{i\neq j}\bar{\chi}_{i,m}(wt_i-k_i)\cdot\chi_j\left(wt_j-k_j+\frac{m}{2}\right)d\mathtt{t}\\
&=\sum_{\mathtt{k}_j'\in\mathbb{Z}^{N-1}}\frac{1}{w^{N-1}}V^j[f]\left(\frac{\mathtt{k}_j'}{w}\right)\prod_{i\neq j}\|\bar{\chi}_{i,m}\|_{L^1(\mathbb{R})}\cdot\|\chi_j\|_{L^1(\mathbb{R})}\le\|V^j[f]\|_{\ell^1(\Sigma_w^{N-1})}\prod_{i=1}^N\|\chi_i\|_{L^1(\mathbb{R})},
\end{aligned}$$

by (6). By Proposition 1 of [7], $V[\bar{S}_w^m f]=\int_{\mathbb{R}^N}\left|\nabla\bar{S}_w^m f(\mathtt{t})\right|d\mathtt{t}$ and therefore, by the previous inequality we have that $\bar{S}_w^m f$ are in $BV(\mathbb{R}^N)$: being also all the sections of $\bar{S}_w^m f$ of bounded variation and locally absolutely continuous, $V^j[\bar{S}_w^m f](\mathtt{x}_j')=$

$\int_{\mathbb{R}} \left| \frac{\partial \bar{S}_w^m f}{\partial t_j} \left(\mathtt{x}'_j, u\right) \right| du$, a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}$, and therefore

$$\|V^j[\bar{S}_w^m f]\|_{L^1(\mathbb{R}^{N-1})} \leq \prod_{i=1}^{N} \|\chi_i\|_{L^1(\mathbb{R})} \|V^j[f]\|_{\ell^1(\Sigma_w^{N-1})}.$$

The estimate (9) follows immediately by the previous one and the obvious inequality $V[\bar{S}_w^m f] \leq \sum_{j=1}^{N} \|V^j[\bar{S}_w^m f]\|_{L^1(\mathbb{R}^{N-1})}$. □

We point out that, of course, the main part of the above result is the first estimate: indeed in case of positive kernels χ_i, $i = 1, \ldots, N$, since the L^1–norm of χ_i turns out to be 1, inequality (8) becomes

$$\|V^j[\bar{S}_w^m f]\|_{L^1(\mathbb{R}^{N-1})} \leq \|V^j[f]\|_{\ell^1(\Sigma_w^{N-1}))}.$$

Notice that $\|V^j[\bar{S}_w^m f]\|_{L^1(\mathbb{R}^{N-1})}$ is the Tonelli integral of $\bar{S}_w^m f$, $\Phi^j_{\mathbb{R}^N}(\bar{S}_w^m f)$: in other words, we have a variation diminishing-type estimate for the L^1–norm of the sections of the operators in terms of the $\ell^1(\Sigma_w^{N-1})$–norm of the sections of the function. Working with the generalized sampling operators, it is natural to expect that we cannot obtain an estimate in terms of the L^1–norm of the sections of f, since their L^1–norm can just be estimated in terms of the ℓ^1–norm of f (see [15]) on the sampling grid: we could think to the ℓ^1–norm of the variation of the section of f as a kind of Tonelli integral in the setting of $BV_\Lambda(\mathbb{R}^N)$, instead of the classical Tonelli integrals in $BV(\mathbb{R}^N)$.

We now prove a similar estimate in variation for the mixed sampling-Kantorovich operators $(\bar{K}_{w,j}^m f)_{w>0}$, $j = 1, \ldots, N$.

Theorem 3.2. *If $f \in BV_\Lambda(\mathbb{R}^N)$, then $\bar{K}_{w,j}^m f \in BV(\mathbb{R}^N)$ for every $j = 1, \ldots, N$, $w > 0$, $m \in \mathbb{N}$, and*

$$\|V^i[\bar{K}_{w,j}^m f]\|_{L^1(\mathbb{R}^{N-1})} \leq \frac{m+1}{m} \prod_{l=1}^{N} \|\chi_l\|_{L^1(\mathbb{R})} \|V^j[f]\|_{\ell^1(\Sigma_w^{N-1})}, \tag{10}$$

for every $i, j = 1, \ldots, N$.
As a consequence,

$$V[\bar{K}_{w,j}^m f] \leq \frac{m+1}{m} \prod_{l=1}^{N} \|\chi_l\|_{L^1(\mathbb{R})} \sum_{j=1}^{N} \|V^j[f]\|_{\ell^1(\Sigma_w^{N-1})}.$$

Proof. Let us first consider the case $i = j$.

Since f is in particular bounded, $\frac{\partial \bar{K}_{w,j}^m f}{\partial t_j}(\mathtt{t})$ exists for every $\mathtt{t} \in \mathbb{R}^N$. Indeed, for every $j = 1, \ldots, N$, $\mathtt{t} \in \mathbb{R}^N$, there holds

$$\frac{\partial \bar{K}_{w,j}^m f}{\partial t_j}(\mathtt{t}) = \frac{w^2}{m} \sum_{\mathtt{k} \in \mathbb{Z}^N} \left[\int_{[\frac{k_j}{w}, \frac{k_j+1}{w}]} f\left(\frac{\mathtt{k}'_j}{w}, u\right) du \right] \prod_{i \neq j} \bar{\chi}_{i,m}(wt_i - k_i) \left[\chi_j\left(wt_j - k_j + \frac{m}{2}\right) - \chi_j\left(wt_j - k_j - \frac{m}{2}\right) \right], \tag{11}$$

and hence, using (II), for every $\mathtt{t} \in \mathbb{R}^N$,

$$\left| \frac{\partial \bar{K}_{w,j}^m f}{\partial t_j}(\mathtt{t}) \right| \leq \frac{w}{m} L \sum_{\mathtt{k} \in \mathbb{Z}^N} \prod_{i \neq j} |\bar{\chi}_{i,m}(wt_i - k_i)| \left[\left| \chi_j\left(wt_j - k_j + \frac{m}{2}\right) \right| + \left| \chi_j\left(wt_j - k_j - \frac{m}{2}\right) \right| \right] \leq \frac{2w}{m} L \prod_{i \neq j} A_{\bar{\chi}_{i,m}} A_{\chi_j} < +\infty. \tag{12}$$

This implies that $\bar{K}_{w,j}^m f \in AC_{loc}(\mathbb{R}^N)$ and therefore we have, for a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}$,

$$\int_{\mathbb{R}} \left| \frac{\partial \bar{K}_{w,j}^m f}{\partial t_j}(\mathtt{x}'_j, u) \right| du = \sup_{[a,b] \in \mathbb{R}} \int_a^b \left| \frac{\partial \bar{K}_{w,j}^m f}{\partial t_j}(\mathtt{x}'_j, u) \right| du = \sup_{[a,b] \in \mathbb{R}} V_{[a,b]}[\bar{K}_{w,j}^m f(\mathtt{x}'_j, \cdot)] = V^j[\bar{K}_{w,j}^m f](\mathtt{x}'_j). \tag{13}$$

Now, putting in the second series of (11) $\tilde{k}_i = k_i$ for $i \neq j$ and $\tilde{k}_j = k_j + m$, we can write

$$\begin{aligned} \frac{\partial \bar{K}_{w,j}^m f}{\partial t_j}(\mathtt{t}) = & \frac{w^2}{m} \sum_{\mathtt{k} \in \mathbb{Z}^N} \left[\int_{[\frac{k_j}{w}, \frac{k_j+1}{w}]} f\left(\frac{\mathtt{k}'_j}{w}, u\right) du \right] \prod_{i \neq j} \bar{\chi}_{i,m}(wt_i - k_i) \chi_j\left(wt_j - k_j + \frac{m}{2}\right) + \\ & - \frac{w^2}{m} \sum_{\tilde{\mathtt{k}} \in \mathbb{Z}^N} \left[\int_{[\frac{\tilde{k}_j - m}{w}, \frac{\tilde{k}_j - m+1}{w}]} f\left(\frac{\tilde{\mathtt{k}}'_j}{w}, u\right) du \right] \prod_{i \neq j} \bar{\chi}_{i,m}(wt_i - \tilde{k}_i) \chi_j\left(wt_j - \tilde{k}_j + \frac{m}{2}\right). \end{aligned} \tag{14}$$

If we now put, in the second integral, $v = u + \frac{m}{w}$, then

$$\frac{\partial \bar{K}_{w,j}^m f}{\partial t_j}(\mathtt{t}) = \frac{w^2}{m} \sum_{\mathtt{k} \in \mathbb{Z}^N} \int_{[\frac{k_j}{w}, \frac{k_j+1}{w}]} \left[f\left(\frac{\mathtt{k}'_j}{w}, u\right) - f\left(\frac{\mathtt{k}'_j}{w}, u - \frac{m}{w}\right) \right] du \prod_{i \neq j} \bar{\chi}_{i,m}(wt_i - k_i) \chi_j\left(wt_j - k_j + \frac{m}{2}\right),$$

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and therefore

$$\left|\frac{\partial \bar{K}^m_{w,j} f}{\partial t_j}(\mathtt{t})\right| \leq \frac{w}{m} \sum_{\mathtt{k}\in\mathbb{Z}^N} V_{[\frac{k_j-m}{w},\frac{k_j+1}{w}]}\left[f\left(\frac{\mathtt{k}'_j}{w},\cdot\right)\right] \prod_{i\neq j} |\bar{\chi}_{i,m}(wt_i-k_i)| \left|\chi_j\left(wt_j-k_j+\frac{m}{2}\right)\right|$$

$$\leq w\frac{m+1}{m} \sum_{\mathtt{k}'_j\in\mathbb{Z}^{N-1}} V_{\mathbb{R}}\left[f\left(\frac{\mathtt{k}'_j}{w},\cdot\right)\right] \prod_{i\neq j} |\bar{\chi}_{i,m}(wt_i-k_i)| \left|\chi_j\left(wt_j-k_j+\frac{m}{2}\right)\right|.$$

Then, by (13),

$$\|V^j[\bar{K}^m_{w,j}f]\|_{L^1(\mathbb{R}^{N-1})} = \left\|\frac{\partial \bar{K}^m_{w,j} f}{\partial t_j}\right\|_{L^1(\mathbb{R}^N)} \leq w\frac{m+1}{m}\int_{\mathbb{R}^N} \sum_{\mathtt{k}'_j\in\mathbb{Z}^{N-1}} V_{\mathbb{R}}\left[f\left(\frac{\mathtt{k}'_j}{w},\cdot\right)\right] \prod_{i\neq j} |\bar{\chi}_{i,m}(wt_i-k_i)| \left|\chi_j\left(wt_j-k_j+\frac{m}{2}\right)\right| d\mathtt{t}$$

$$= \frac{m+1}{m}\sum_{\mathtt{k}'_j\in\mathbb{Z}^{N-1}} \frac{1}{w^{N-1}} V^j[f]\left(\frac{\mathtt{k}'_j}{w}\right) \int_{\mathbb{R}^N} w^N \prod_{i\neq j}|\bar{\chi}_{i,m}(wt_i-k_i)| \left|\chi_j\left(wt_j-k_j+\frac{m}{2}\right)\right| d\mathtt{t} \leq \frac{m+1}{m}\|V^j[f]\|_{\ell^1(\Sigma^{N-1}_w)} \prod_{i=1}^N \|\chi_i\|_{L^1(\mathbb{R})}.$$

Let us now take $i\neq j$. First notice that, similarly to (11) and (12), for $i\neq j$ there holds $\displaystyle\int_{\mathbb{R}} \left|\frac{\partial \bar{K}^m_{w,j} f}{\partial t_i}(\mathtt{x}'_i,u)\right| du = V^i[\bar{K}^m_{w,j}f](\mathtt{x}'_i)$, a.e. $\mathtt{x}'_i\in\mathbb{R}^{N-1}$. Moreover, similarly to (14),

$$\frac{\partial \bar{K}^m_{w,j} f}{\partial t_i}(\mathtt{t}) = \frac{w^2}{m}\sum_{\mathtt{k}\in\mathbb{Z}^N}\left[\int_{[\frac{k_j}{w},\frac{k_j+1}{w}]} f\left(\frac{\mathtt{k}'_j}{w},u\right)du\right]\cdot \prod_{l\neq i}\bar{\chi}_{l,m}(wt_l-k_l)\left[\chi_i\left(wt_i-k_i+\frac{m}{2}\right)-\chi_i\left(wt_i-k_i+\frac{m}{2}\right)\right].$$

Putting in the second series $\tilde{k}_i = k_i$ for $i\neq j$ and $\tilde{k}_j = k_j+m$,

$$\frac{\partial \bar{K}^m_{w,j} f}{\partial t_i}(\mathtt{t}) = \frac{w^2}{m}\sum_{\mathtt{k}\in\mathbb{Z}^N}\int_{[\frac{k_j}{w},\frac{k_j+1}{w}]}\left[f\left(\frac{\mathtt{k}'_j}{w},u_j\right)-f\left(\frac{\mathtt{k}''_{j,i}}{w},\frac{k_i-m}{w},u_j\right)\right]du_j \prod_{l\neq i}\bar{\chi}_{l,m}(wt_l-k_l)\chi_i\left(wt_i-k_i+\frac{m}{2}\right)$$

$$\leq \frac{w^2}{m}\sum_{\mathtt{k}\in\mathbb{Z}^N}\int_{[\frac{k_j}{w},\frac{k_j+1}{w}]} V_{[\frac{k_i-m}{w},\frac{k_i}{w}]}\left[f\left(\frac{\mathtt{k}''_{j,i}}{w},u_j,\cdot\right)\right]du_j \prod_{l\neq i}\bar{\chi}_{l,m}(wt_l-k_l)\chi_i\left(wt_i-k_i+\frac{m}{2}\right)$$

$$\leq \frac{w^2}{m}\sum_{\mathtt{k}''_{j,i}\in\mathbb{Z}^{N-2}}\sum_{k_j\in\mathbb{Z}}\int_{[\frac{k_j}{w},\frac{k_j+1}{w}]}\sum_{k_i\in\mathbb{Z}}\left\{V_{[\frac{k_i-m}{w},\frac{k_i-m+1}{w}]}\left[f\left(\frac{\mathtt{k}''_{j,i}}{w},u_j,\cdot\right)\right]+\right.$$

$$\left.+\ldots+V_{[\frac{k_i-1}{w},\frac{k_i}{w}]}\left[f\left(\frac{\mathtt{k}''_{j,i}}{w},u_j,\cdot\right)\right]\right\}du_j\prod_{l\neq i}\bar{\chi}_{l,m}(wt_l-k_l)\chi_i\left(wt_i-k_i+\frac{m}{2}\right)$$

$$= w^2\sum_{\mathtt{k}''_{j,i}\in\mathbb{Z}^{N-2}}\sum_{k_j\in\mathbb{Z}}\int_{[\frac{k_j}{w},\frac{k_j+1}{w}]} V^i[f]\left(\frac{\mathtt{k}''_{j,i}}{w},u_j\right)du_j\prod_{l\neq i}\bar{\chi}_{l,m}(wt_l-k_l)\chi_i\left(wt_i-k_i+\frac{m}{2}\right).$$

Therefore there holds

$$\|V^i[\bar{K}^m_{w,j}f]\|_{L^1(\mathbb{R}^{N-1})} = \int_{\mathbb{R}^N}\left|\frac{\partial \bar{K}^m_{w,j} f}{\partial t_i}(\mathtt{t})\right| d\mathtt{t}$$

$$\leq w^2\int_{\mathbb{R}^N}\sum_{\mathtt{k}''_{j,i}\in\mathbb{Z}^{N-2}}\sum_{k_j\in\mathbb{Z}}\int_{[\frac{k_j}{w},\frac{k_j+1}{w}]} V^i[f]\left(\frac{\mathtt{k}''_{j,i}}{w},u_j\right)du_j\prod_{l\neq i}|\bar{\chi}_{l,m}(wt_l-k_l)|\left|\chi_i\left(wt_i-k_i+\frac{m}{2}\right)\right| d\mathtt{t}$$

$$= w^2\sum_{\mathtt{k}''_{j,i}\in\mathbb{Z}^{N-2}}\sum_{k_j\in\mathbb{Z}}\int_{[\frac{k_j}{w},\frac{k_j+1}{w}]} V^i[f]\left(\frac{\mathtt{k}''_{j,i}}{w},u_j\right)du_j\int_{\mathbb{R}^N}\prod_{l\neq i}|\bar{\chi}_{l,m}(wt_l-k_l)|\left|\chi_i\left(wt_i-k_i+\frac{m}{2}\right)\right| d\mathtt{t}$$

$$= \sum_{\mathtt{k}''_{j,i}\in\mathbb{Z}^{N-2}}\frac{1}{w^{N-2}}\sum_{k_j\in\mathbb{Z}}\int_{[\frac{k_j}{w},\frac{k_j+1}{w}]} V^i[f]\left(\frac{\mathtt{k}''_{j,i}}{w},u_j\right)du_j\prod_{l\neq i}\|\bar{\chi}_{l,m}\|_{L^1(\mathbb{R})}\|\chi_i\|_{L^1(\mathbb{R})}$$

$$\leq \sum_{\mathtt{k}''_{j,i}\in\mathbb{Z}^{N-2}}\frac{1}{w^{N-1}}\sum_{k_j\in\mathbb{Z}}\sup_{u_j\in[\frac{k_j}{w},\frac{k_j+1}{w}]} V^i[f]\left(\frac{\mathtt{k}''_{j,i}}{w},u_j\right)\prod_{l\neq i}\|\bar{\chi}_{l,m}\|_{L^1(\mathbb{R})}\|\chi_i\|_{L^1(\mathbb{R})} \leq \|V^i[f]\|_{\ell^1(\Sigma^{N-1}_w)}\prod_{l=1}^N\|\chi_l\|_{L^1(\mathbb{R})},$$

and hence inequality (10) follows.

The second estimate follows again taking into account that $V[\bar{K}^m_{w,j}f] \leq \sum_{i=1}^N \|V^i[\bar{K}^m_{w,j}f]\|_{L^1(\mathbb{R}^{N-1})}$. □

Remark 1. Throughout the paper we have assumed that $N > 1$. Similar estimates in variation in the one-dimensional case have already been obtained: we refer to [6] for case of the generalized sampling series and to [8] for the case of the sampling-Kantorovich operators (notice that, for $N = 1$ the mixed sampling-Kantorovich operators obviously coincide with the classical sampling-Kantorovich operators).

Remark 2. a) We point out that Theorem 3.1 can be viewed as a generalization of Proposition 5 of [7]. Indeed, let us consider a step function f with compact support $[\mathtt{a},\mathtt{b}] \subset \mathbb{R}^N$, $[\mathtt{a},\mathtt{b}] \subset \mathbb{R}^N$, where $\mathtt{a},\mathtt{b} \in \mathbb{Z}^N$, $a_i < b_i$, $i = 1,...,N$, and $f(\mathtt{x}) = f(\mathtt{i})$ for every $\mathtt{x} \in [\mathtt{i},\mathtt{j}[$, i.e., f is constant on each interval of a grid of multi-dimensional intervals of the form $[\mathtt{i},\mathtt{j}] \subset [\mathtt{a},\mathtt{b}]$, with $\mathtt{i},\mathtt{j} \in \mathbb{Z}^N$, and $|j_\nu - i_\nu| = 1$, $\nu = 1,...,N$, that form a partition of $[\mathtt{a},\mathtt{b}[$. Then it is not difficult to see that, for such function, $V^j[f]$ is also a step type function on the same grid and therefore the $\ell^1(\Sigma_w^{N-1})$ and the L^1-norm of $V^j[f]$ actually coincide. This implies that, in this case, the thesis of Theorem 3.1 says that the Tonelli integrals of $\bar{S}_w^m f$ are smaller than the Tonelli integrals of f multiplied by $\prod_{i=1}^N \|\chi_i\|_{L^1(\mathbb{R})}$, and therefore, passing to the Euclidean norm, we obtain the thesis of Proposition 5 of [7].

b) We finally point out that, in case of the same step-type function of a), the mixed sampling-Kantorovich operators coincide with the classical Kantorovich operators (therefore Theorem 3.2 becomes Theorem 1 of [8]) and also with the generalized sampling series: hence, as pointed out in Corollary 1 of [8], the sharper estimate

$$V^i[\bar{K}_{w,j}^m f] = V^i[\bar{K}_w^m f] = V^i[\bar{S}_w^m f] \le \prod_{l=1}^N \|\chi_l\|_{L^1(\mathbb{R})} V^i[f]$$

actually holds for every $i, j = 1, \ldots, N$. As a consequence, we also have

$$V[\bar{K}_{w,j}^m f] = V[\bar{K}_w^m f] = V[\bar{S}_w^m f] \le \prod_{l=1}^N \|\chi_l\|_{L^1(\mathbb{R})} \sum_{i=1}^N V^i[f].$$

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