PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 19 · 2026 · Pages 146–155

Algebraic semigroups, C_0-semigroups and generalized convexity

Ana-Maria Acu [a] · Georgian Chivu [a] · Ioan Raşa [b]

Communicated by Wolfgang Erb

Abstract

An Altomare projection gives rise to a C_0-semigroup and to generalized convexity. Besides other properties, it has one which can be described in geometric terms as well as in terms of algebraic semigroups. We study operators having only this last property and the types of convexity induced by them. In this framework we consider also a conjecture formulated by F. Altomare and I. Rasa in 1999 and present related results.

1 Introduction

Certain C_0-semigroups can be approximated by suitable iterates of positive linear operators. This area of research was intensively investigated by Francesco Altomare and his school (see [3]-[7], [10], [11], [18] and the references therein). The approximation operators, called *Bernstein-Schnabl operators*, are constructed in terms of special projections T called *Altomare projections*; see, e.g., [15]-[17]. An Altomare projection is positive, its range contains the affine functions, and it has a property with an algebraic/geometric flavor (see (i) from Theorem 4.3) intertwined with the convexity theory. The aim of this paper is twofold. On one hand we consider operators having the third property mentioned above and study them. On the other hand we return to a conjecture formulated by F. Altomare and I. Raşa in [8]. It involves the infinitesimal generator A of a C_0-semigroup and the convexity properties of a function f with $A(f)\geq 0$.

In Section 2 we consider an algebraic semigroup of linear operators acting on $C(X)$, the space of continuous functions defined on a compact convex set X. These operators are instrumental in Section 3 where we describe the Bernstein-Schnabl operators B_n associated with an arbitrary bounded linear operator T acting on $C(X)$. The construction of B_n is based on two de Casteljau type algorithms. Section 4 introduces the T-convex and T-affine functions. An example illustrates these notions when T is not positive.

In Theorem 4.3 we consider an operator T, not necessarily positive, but having the algebraic properties needed in Altomare's theory.

In Section 5 we combine these properties with the positivity of T and present the associated C_0-semigroup together with the corresponding notions of T-convex functions. An example illustrates the general theory. Section 6 is devoted to a conjecture relating a notion of subharmonicity with notions of convexity. The conjecture was formulated in [8] and a related result was obtained in [1]. Here we present another related result, Theorem 6.2. In Section 7 we consider the canonical projection on the standard simplex in $\mathbb{R}^2$ and illustrate all the notions and results presented in the previous sections.

2 A semigroup of linear operators

Let X be a compact convex subset of a Hausdorff locally convex real space. By $C(X)$ we denote the space of all continuous real-valued functions defined on X. $B(C(X))$ will stand for the space of all bounded linear operators on $C(X)$ and $I_{C(X)}$ for the identity operator on $C(X)$.

Let $z\in X$, $a\in[0,1]$. Consider the operator $K_{z,a}\in B(C(X))$ defined by

$$K_{z,a}f(x):=f(ax+(1-a)z),\ f\in C(X),\ x\in X. \tag{2.1}$$

Set

$$\mathcal{K}:=\left\{K_{z,a}\,|z\in X,\ a\in[0,1]\right\}.$$

---

[a] Department of Mathematics and Informatics, Lucian Blaga University of Sibiu, Romania
[b] Department of Mathematics, Technical University of Cluj-Napoca, Romania

Proposition 2.1. $(\mathcal{K},\circ)$ is a subsemigroup of $(B(C(X)),\circ)$ and $I_{C(X)}$ is the unit element.

Proof. For $z \in X$ we have $K_{z,1}f(x) = f(x)$, $f \in C(X)$, $x \in X$, and so $I_{C(X)} = K_{z,1} \in \mathcal{K}$. Moreover, let $a, b \in [0,1]$, $y, z \in X$. Then

$$K_{y,b} \circ K_{z,a} = \begin{cases} K_{\frac{(1-a)z+a(1-b)y}{1-ab},ab}, \text{ if } ab < 1, \\ I_{C(X)}, \text{ if } a = b = 1. \end{cases} \tag{2.2}$$

Therefore $K_{y,b} \circ K_{z,a} \in \mathcal{K}$ and this completes the proof. □

The algebraic structure of the semigroup $(\mathcal{K},\circ)$, interesting in itself, will be investigated in a forthcoming paper in relation with several results from [13], [14]. Here we collect a list of preliminary properties.

Proposition 2.2. Let $x, y, z \in X$, $a, b, c \in [0,1]$. Then

i) If $K_{z,a} = K_{y,b}$, then $a = b$.

ii) If $K_{z,a} = K_{y,b}$ and $a = b < 1$, then $z = y$.

iii) $K_{x,a} \circ K_{x,b} = K_{x,b} \circ K_{x,a} = K_{x,ab}$.

iv) If $K_{x,a} \circ K_{y,b} = K_{y,b} \circ K_{x,a}$ and $a < 1$, $b < 1$, then $x = y$.

v) If $K_{y,b} \circ K_{z,a} = K_{y,b} \circ K_{x,c}$ and $b > 0$, then $K_{z,a} = K_{x,c}$.

vi) If $K_{z,a} \circ K_{y,b} = K_{x,c} \circ K_{y,b}$ and $b > 0$, then $K_{x,a} = K_{x,c}$.

The proof is straightforward and we omit it.

3 Bernstein-Schnabl type operators and de Casteljau type algorithms

Let $T \in B(C(X))$. For $x \in X$ consider the Radon measure μ_x defined by

$$\mu_x(f) := Tf(x),\ f \in C(X). \tag{3.1}$$

We shall use also the notation

$$Tf(x) = \int_X f(t)d\mu_x(t). \tag{3.2}$$

The Bernstein-Schnabl operators associated with T are defined by

$$B_nf(x) := \int_{X^n} f\left(\frac{t_1 + \cdots + t_n}{n}\right)d\mu_x(t_1)\ldots d\mu_x(t_n), \tag{3.3}$$

where $n \in \mathbb{N}$, $f \in C(X)$, $x \in X$. See [3], [5] and the references therein.

The results presented in the next three theorems were obtained in [17] for a more specialized family of operators T. For the sake of completeness we give here simplified, more detalied proofs for arbitrary operators $T \in B(C(X))$.

Theorem 3.1. *(First de Casteljau type algorithm). Let $n \geq 1$, $f \in C(X)$, $x \in X$ be given. Define the functions $f_0^x, f_1^x, \ldots, f_n^x$, of the variable $t \in X$, by $f_0^x := f$,*

$$f_i^x(t) := \left(TK_{t,1/(n-i+1)}f_{i-1}^x\right)(x),\ i = 1, \ldots, n. \tag{3.4}$$

Then the function f_n^x is constant, namely

$$f_n^x(t) = B_nf(x),\ t \in X. \tag{3.5}$$

Proof. For $i = 1, \ldots, n-1$ we have

$$\begin{aligned}\int_X f_{i-1}^x\left(\frac{t_1 + \cdots + t_{n-i} + t}{n-i+1}\right)d\mu_x(t) &= \int_X \left(K_{\frac{t_1+\cdots+t_{n-i}}{n-i},\frac{1}{n-i+1}}f_{i-1}^x\right)(t)d\mu_x(t) \\ &= T\left(K_{\frac{t_1+\cdots+t_{n-i}}{n-i},\frac{1}{n-i+1}}f_{i-1}^x\right)(x) = f_i^x\left(\frac{t_1 + \cdots + t_{n-i}}{n-i}\right).\end{aligned}$$

Together with (3.4), this leads to

$$\begin{aligned}B_nf(x) &= \int_{X^{n-1}}\left[\int_X f_0^x\left(\frac{t_1 + \cdots + t_{n-1} + t}{n}\right)d\mu_x(t)\right]d\mu_x(t_1)\ldots d\mu_x(t_{n-1}) \\ &= \int_{X^{n-1}} f_1^x\left(\frac{t_1 + \cdots + t_{n-1}}{n-1}\right)d\mu_x(t_1)\ldots d\mu_x(t_{n-1}) = \cdots \\ &= \int_X f_{n-1}^x(t_1)d\mu_x(t_1) = Tf_{n-1}^x(x) = TK_{t,1}f_{n-1}^x(x) = f_n^x(t).\end{aligned}$$

This proves (3.5). □

For $n \geq 1$, $i = 0, 1, \ldots, n$ and $x \in X$ set

$$D_{n,i,x} := \left\{ \frac{t_1 + \cdots + t_{n-i} + ix}{n} \,\middle|\, t_1, \ldots, t_{n-i} \in X \right\}. \tag{3.6}$$

Let $f \in C(X)$. Define the functions $p_i^x : D_{n,i,x} \to \mathbb{R}$ by $p_0^x = f$,

$$p_i^x(t) = \left(TK_{\frac{nt-x}{n-1},\frac{1}{n}} p_{i-1}^x \right)(x), \; i = 1, \ldots, n, \; t \in D_{n,i,x}. \tag{3.7}$$

Theorem 3.2. *(Second de Casteljau type algorithm) With the above notation,*

$$p_n^x(x) = B_n f(x). \tag{3.8}$$

Proof. First, we have for $i = 1, \ldots, n$ and $t_1, \ldots, t_{n-i+1} \in X$,

$$\int_X p_{i-1}^x \left(\frac{t_1 + \cdots + t_{n-i+1} + (i-1)x}{n} \right) d\mu_x(t_{n-i+1}) = p_i^x \left(\frac{t_1 + \cdots + t_{n-i} + ix}{n} \right). \tag{3.9}$$

Indeed, set $t := \dfrac{t_1 + \cdots + t_{n-1} + ix}{n}$. Then

$$\begin{aligned} p_i^x \left(\frac{t_1 + \cdots + t_{n-i} + ix}{n} \right) &= p_i^x(t) = \left(TK_{\frac{nt-x}{n-1},\frac{1}{n}} p_{i-1}^x \right)(x) \\ &= \int_X K_{\frac{nt-x}{n-1},\frac{1}{n}} p_{i-1}^x(t_{n-i+1}) d\mu_x(t_{n-i+1}) = \int_X p_{i-1}^x \left(\frac{nt - x + t_{n-i+1}}{n} \right) d\mu_x(t_{n-i+1}) \\ &= \int_X p_{i-1}^x \left(\frac{t_1 + \cdots + t_{n-i+1} + (i-1)x}{n} \right) d\mu_x(t_{n-i+1}), \end{aligned}$$

and this proves (3.9). Furthermore, using (3.9),

$$\begin{aligned} B_n f(x) &= \int_{X^{n-1}} \left[\int_X p_0^x \left(\frac{t_1 + \cdots + t_n}{n} \right) d\mu_n(t_n) \right] d\mu_x(t_1) \ldots\ldots d\mu_x(t_{n-1}) \\ &= \int_{X^{n-1}} p_1^x \left(\frac{t_1 + \cdots + t_{n-1} + x}{n} \right) d\mu_x(t_1) \ldots d\mu_x(t_{n-1}) \\ &= \int_{X^{n-2}} p_2^x \left(\frac{t_1 + \cdots + t_{n-2} + 2x}{n} \right) d\mu_x(t_1) \ldots d\mu_x(t_{n-2}) = \cdots \\ &= \int_X p_{n-1}^x \left(\frac{t_1 + (n-1)x}{n} \right) d\mu_x(t_1) = p_n^x(x), \end{aligned}$$

and this completes the proof. □

Remark 1. If $X = [0,1]$ and $Tf(x) = (1-x)f(0) + xf(1)$, $f \in C[0,1]$, $x \in [0,1]$, the second algorithm is the classical de Casteljau algorithm for the computation of the Bernstein polynomials (see, e.g., [9], [12]).

Theorem 3.3. *For each $x, t \in X$ and $f \in C(X)$ one has $f_n^x(t) = p_n^x(x)$ and*

$$f_i^x(t) = K_{x,\frac{n-i}{n}} p_i^x(t), \; i = 0, 1, \ldots, n-1. \tag{3.10}$$

Proof. We use induction on i. For $i = 0$, (3.10) reduces to $f_0^x = p_0^x = f$. Suppose that

$$f_{i-1}^x(t) = K_{x,\frac{n-i+1}{n}} p_{i-1}^x(t). \tag{3.11}$$

Using (3.4) it follows

$$\begin{aligned} f_i^x(t) &= \left(TK_{t,\frac{1}{n-i+1}} f_{i-1}^x \right)(x) \\ &= \left(TK_{t,\frac{1}{n-i+1}} K_{x,\frac{n-i+1}{n}} p_{i-1}^x \right)(x). \end{aligned}$$

From relation (2.2) we get

$$f_i^x(t) = \left(TK_{\frac{(i-1)x+(n-i)t}{n-1},\frac{1}{n}} p_{i-1}^x \right)(x). \tag{3.12}$$

On the other hand,

$$K_{x,\frac{n-i}{n}} p_i^x(t) = p_i^x \left(\frac{n-i}{n} t + \frac{i}{n} x \right).$$

Using (3.7), one has

$$K_{x,\frac{n-i}{n}} p_i^x(t) = \left(TK_{\frac{(n-i)t+(i-1)x}{n-1},\frac{1}{n}} p_{i-1}^x \right)(x).$$

This shows that (3.10) is valid also for i. Moreover, remark that according to (3.5) and (3.8)

$$f_n^x(t) = p_n^x(x) = B_n f(x).$$

Thus the proof is complete. □

4 T-convex and T-affine functions

Let $T \in B(C(X))$ and $f \in C(X)$.

Definition 4.1. We say that f is T-convex if

$$TKf \geq Kf \text{ for all } K \in \mathcal{K}. \tag{4.1}$$

As usual, T is called a positive linear operator if $Tg \geq 0$ for all $g \in C(X)$, $g \geq 0$.

Theorem 4.1. *If T is positive and f is T-convex, then*

$$B_n f(x) \geq f(x),\ n \geq 1,\ x \in X. \tag{4.2}$$

Proof. First, we prove that

$$f_i^x \geq K_{x,\frac{n-i}{n}} f,\ i = 0, 1, \ldots, n. \tag{4.3}$$

Clearly, (4.3) is true for $i = 0$. Suppose that it is true for a given $i \in \{0, \ldots, n-1\}$. Then

$$f_{i+1}^x(t) = \left(TK_{t,\frac{1}{n-i}} f_i^x\right)(x) \geq \left(TK_{t,\frac{1}{n-i}} K_{x,\frac{n-i}{n}} f\right)(x).$$

Using (2.2) and (4.1) we get

$$\left(TK_{t,\frac{1}{n-i}} K_{x,\frac{n-i}{n}} f\right)(x) = T\left(K_{\frac{ix+(n-i-1)t}{n-1},\frac{1}{n}} f\right)(x) \geq \left(K_{\frac{ix+(n-i-1)t}{n-1},\frac{1}{n}} f\right)(x) = \left(K_{x,\frac{n-i-1}{n}} f\right)(t),$$

and consequently (4.3) is proved. For $i = n$, using (3.5), one has

$$B_n f(x) = f_n^x(x) \geq K_{x,0} f(x) = f(x)$$

and the proof of (4.2) is completed. ☐

Definition 4.2. We say that $f \in C(X)$ is T-affine if

$$TKf = Kf \text{ for all } K \in \mathcal{K}. \tag{4.4}$$

According to Theorem 4.1, if T is positive and f is T-affine, then $B_n f = f$, $n \geq 1$. In fact, the condition "T positive" is not necessary, as we will see in the next theorem.

Theorem 4.2. *Let $T \in B(C(X))$ and $f \in C(X)$ T-affine. Then*

$$B_n f = f,\ n \geq 1. \tag{4.5}$$

Proof. Consider the functions $p_i^x : D_{n,i,x} \to \mathbb{R}$, $i = 0, 1, \ldots, n$ (see (3.6)). We will prove that

$$p_i^x(t) = f(t),\ t \in D_{n,i,x}. \tag{4.6}$$

For $i = 0$, (4.6) is obviously true. Suppose that it is true for a given $i \in \{0, 1, \ldots, n-1\}$.

Let $t \in D_{n,i+1,x}$ and $s \in X$. Then $\dfrac{s+nt-x}{n} \in D_{n,i,x}$, hence

$$p_i^x\left(\frac{s+nt-x}{n}\right) = f\left(\frac{s+nt-x}{n}\right).$$

This entails

$$K_{\frac{nt-x}{n-1},\frac{1}{n}} p_i^x(s) = K_{\frac{nt-x}{n-1},\frac{1}{n}} f(s),\ s \in X.$$

Therefore,

$$K_{\frac{nt-x}{n-1},\frac{1}{n}} p_i^x = K_{\frac{nt-x}{n-1},\frac{1}{n}} f.$$

Now, using (3.7) and (4.4), we can write

$$p_{i+1}^x(t) = \left(TK_{\frac{nt-x}{n-1},\frac{1}{n}} p_i^x\right)(x) = \left(TK_{\frac{nt-x}{n-1},\frac{1}{n}} f\right)(x) = f(t).$$

This completes the proof of (4.6). In particular, for $i = n$, (4.6) yields $p_n^x(x) = f(x)$, hence $B_n f(x) = f(x)$ follows from Theorem 3.2. ☐

Example 4.1. Let $X = [0,1]$ and

$$Tf(x) = (1-x)f(0) - xf(1),\ f \in C[0,1],\ x \in [0,1]. \tag{4.7}$$

Then

$$TK_{z,a}f(x) = (1-x)f((1-a)z) - xf(a+(1-a)z),$$

and consequently the following are equivalent

$$f \text{ is } T-\text{convex}, \tag{4.8}$$

$$(1-x)f((1-a)z) - xf(a+(1-a)z) \geq f(ax+(1-a)z),\ a,x,z \in [0,1]. \tag{4.9}$$

I) If f is Jensen convex and $f \leq 0$, then f is T-convex. Indeed, in this case we can write

$$\begin{aligned}(1-x)f((1-a)z) - xf(a+(1-a)z) &\geq (1-x)f((1-a)z) + xf(a+(1-a)z)\\ &\geq f(ax+(1-a)z),\end{aligned}$$

i.e., f is T-convex.

II) If f is T-convex and $f \geq 0$, then f is Jensen convex. Indeed, we have

$$(1-x)f((1-a)z) + xf(a+(1-a)z) \geq (1-x)f((1-a)z) - xf(a+(1-a)z) \geq f(ax+(1-a)z).$$

Let $0 \leq u \leq v \leq 1$. It is easy to see that there exist $a,z \in [0,1]$ such that

$$(1-a)z = u,\ a+(1-a)z = v.$$

It follows that

$$(1-x)f(u) + xf(v) \geq f((1-x)u+xv)$$

for all $x \in [0,1]$ and $0 \leq u \leq v \leq 1$, i.e., f is Jensen convex.

III) Let $f \in C[0,1]$ be T-affine. Then

$$(1-x)f((1-a)z) - xf(a+(1-a)z) = f(ax+(1-a)z),\ a,x,z \in [0,1].$$

Setting $a = 0$ we get $xf(z) = 0$ for all $x,z \in [0,1]$. So, f is T-affine if and only if $f = 0$.

Returning to the general case, let $T \in B(C(X))$ and set

$$H := T(C(X)) = \{Tg \mid g \in C(X)\}.$$

Denote by $\mathbf{1}$ the constant function on x having the value 1.

Theorem 4.3. *Consider the following statements.*

(i) $TKT = KT$ *for all* $K \in \mathcal{K}$*,*

(ii) H *is the space of all* T*-affine functions,*

(iii) T *is a projection, i.e.,* $T^2 = T$*, and so* $H = \{f \in C(X) \mid Tf = f\}$*,*

(iv) $B_n h = h$ *for all* $h \in H$ *and* $n \geq 1$*,*

(v) If $T \neq 0$*, then* $T\mathbf{1} = \mathbf{1}$*.*

Then (i) $\Longleftrightarrow$ *(ii). Moreover, (i) implies (iii), (iv), (v).*

Proof. (i) $\Longrightarrow$ (ii). Let $f \in H$, i.e., $f = Tg$ for some $g \in C(X)$. According to (i), for all $K \in \mathcal{K}$ we have $TKTg = KTg$, i.e., $TKf = Kf$, which means that f is T-affine. Conversely, if $f \in C(X)$ is T-affine, then $TI_{C(X)}f = I_{C(X)}f$, i.e. $Tf = f$ and so $f \in H$.

(ii) $\Longrightarrow$ (i). Let $g \in C(X)$, $K \in \mathcal{K}$, $f = Tg$. Then $f \in H$ and according to (ii) we have $TKf = Kf$, i.e., $TKTg = KTg$, and so (i) is proved.

(i) $\Longrightarrow$ (iii). It suffices to take $K = I_{C(X)}$ in (i).

(i) $\Longrightarrow$ (iv). This is a consequence of (ii) and Theorem 4.2.

(i) $\Longrightarrow$ (v). Suppose that $T \neq 0$. Then there exist $f \in C(X)$ and $z \in X$ such that $Tf(z) \neq 0$. Using (i) we have $TK_{z,0}Tf = K_{z,0}Tf$. But $K_{z,0}Tf = Tf(z)\mathbf{1}$, so that

$$T(Tf(z)\mathbf{1}) = Tf(z)\mathbf{1},$$

which implies $T\mathbf{1} = \mathbf{1}$. ☐

Remark 2. 1) Suppose that T_1 and T_2 have the property (i) from Theorem 4.3 and $T_1T_2 = T_2T_1$. Then the operators T_1T_2 and $T_1 \bigoplus T_2 := T_1 + T_2 - T_1T_2$ have the same property.

2) Let $X = [0,1]$ and consider the Lagrange interpolation operator L_n associated to the knots $0 \leq t_0 < t_1 < \cdots < t_n \leq 1$. Then L_n has the property (i) and generally it is not positive.

5 A C_0-semigroup

Let E be a Banach space. Denote by $B(E)$ the space of all bounded linear operators on E.

A semigroup of bounded linear operators on E is a family $(T(t))_{t\geq 0}$ of elements of $B(E)$ such that

$$T(0) = I_E,$$

$$T(s+t) = T(s)T(t) \text{ for every } s,t \geq 0,$$

where I_E denotes the identity operator on E, i.e., $I_E(f) := f$ for every $f \in E$.

A semigroup $(T(t))_{t\geq 0}$ on E is a C_0-semigroup if

$$\lim_{t\to 0^+} \|T(t)f - f\| = 0, \text{ for all } f \in E.$$

The generator of the C_0-semigroup $(T(t))_{t\geq 0}$ is the operator $A : D(A) \to E$ defined on the linear subspace

$$D(A) := \left\{ f \in E \,|\, \text{There exists } \lim_{t\to 0^+} \frac{T(t)f - f}{t} \in E \right\},$$

by

$$Af := \lim_{t\to 0^+} \frac{T(t)f - f}{t} \text{ for every } f \in D(A).$$

Let X be a convex compact subset of $\mathbb{R}^p$, $p \geq 1$, having nonempty interior. Denote by $C(X)$ the Banach lattice of all real-valued, continuous functions on X equipped with the supremum norm and the usual ordering. $C^2(X)$ stands for the subspace of all functions $f \in C(X)$ which are two times continuously differentiable on the interior $intX$ and whose partial derivatives of order ≤ 2 can be continuously extended to X.

Denote by $pr_j \in C(X)$, $j = 1, \ldots, p$, the function defined by

$$pr_j(x) = x_j, \; x = (x_1, \ldots, x_p) \in X.$$

Let $T \in B(C(X))$, $T \neq 0$, $T \neq I_{C(X)}$. Suppose that

$$T(pr_j) = pr_j, \; j = 1, \ldots, p, \tag{5.1}$$

$$TKT = KT \text{ for all } K \in \mathcal{K}. \tag{5.2}$$

According to Theorem 4.3, we have $T^2 = T$, $T\mathbf{1} = \mathbf{1}$,

$$H := T(C(X)) = \{h \in C(X) \,|\, Th = h\}.$$

For each $i, j = 1, \ldots, p$, denote

$$a_{ij} := T(pr_i pr_j) - pr_i pr_j.$$

Consider the differential operator $W_T : C^2(X) \to C(X)$ defined by

$$W_T u(x) := \frac{1}{2} \sum_{i,j=1}^{p} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j},$$

for every $u \in C^2(X)$ and $x \in X$.

Let $A_m(X)$, $m \geq 1$, denote the subspace of the restrictions to X of all polynomial functions of degree $\leq m$. Then, $A_1(X) = A(X)$ is the subspace of all continuous affine functions on X.

Theorem 5.1. *([2]) Suppose that T is positive and $T(A_2(X)) \subset A(X)$. Then the operator $(W_T, C^2(X))$ is closable and its closure $(A, D(A))$ generates a C_0-semigroup on $C(X)$.*

Other properties of this C_0-semigroup can be found in [3], [5] and the references therein.

For the sake of simplicity, in what follows we use also the notation

$$f_{z,a}(x) := K_{z,a} f(x) = f(ax + (1-a)z).$$

In agreement with Definition 4.1, we will use

Definition 5.1. ([15]) A function $f \in C(X)$ is said to be T-convex if

$$f_{z,a} \leq T(f_{z,a}) \text{ for each } z \in X, \; a \in [0,1].$$

Moreover, we need

Definition 5.2. ([8]) A function $f \in C(X)$ is said to be weakly T-convex if

$$f(z) \leq T(f_{z,a})(z) \text{ for each } z \in X, \; a \in [0,1].$$

Proposition 5.1. ([8]) A convex function is T-convex. A T-convex function is weakly-T convex.

Definition 5.3. Let X be a simplex. The function $f \in C(X)$ is called axially convex if f is convex on each segment parallel to an edge of X (see, e.g, [15], [19]-[21]).

Example 5.1. ([8], [22, Ex.18.15]) Let $X = [0,1]$ and T be the canonical projection associated with X, described by

$$Tf(x) = (1-x)f(0) + xf(1),\ f \in C[0,1],\ x \in [0,1].$$

(1) f is T-convex if and only if

$$(1-x)f((1-a)z) + xf(a+(1-a)z) \geq f(ax+(1-a)z),$$

for all $a, x, z \in [0,1]$.

(2) f is weakly-T-convex if and only if

$$(1-z)f((1-a)z) + zf(a+(1-a)z) \geq f(z),\ \text{for all } a, z \in [0,1].$$

(3) The following statements about $f \in C[0,1]$ are equivalent.

(a) f is convex;

(b) f is axially convex;

(c) f is T-convex;

(d) f is weakly-T-convex.

Similar results for operators and functions on the standard simplex in $\mathbb{R}^2$ will be presented in Section 7.

6 A conjecture and related results

Definition 6.1. ([8]) Under the same hypotheses as in Theorem 5.1, $f \in D(A)$ is called a generalized A-subharmonic function if $A(f) \geq 0$.

The following result was obtained in [8, Prop. 6.2].

Proposition 6.1. Assume that $f \in C^2(X)$. If f is weakly T-convex, then f is a generalized A-subharmonic function.

Conjecture 6.3 from the paper [8] reads as follows.

Conjecture 6.1. If $f \in D(A)$ and $Af \geq 0$, then f is weakly T-convex.

Examples and results supporting this conjecture can be found in [8] but as far as we know the problem is still open.

A result related to Conjecture 6.1 was obtained in [1]. Briefly speaking, it was proved that a stronger hypothesis entails a stronger conclusion.

Theorem 6.1. *([1]) Given $f \in C(X)$, let $F : X \times [0,1] \to \mathbb{R}$,*

$$F(z,a) := Tf_{z,a}(z) - f(z),\ z \in X,\ a \in [0,1]. \tag{6.1}$$

Then,

(i) $F(z,0) = 0,\ z \in X$.

(ii) If f is convex, then $F(z,\cdot)$ is increasing on $[0,1]$ for each $z \in X$. In particular, $F \geq 0$ on $X \times [0,1]$.

Remark 3. According to Proposition 6.1, the hypothesis "f is convex" in Theorem 6.1 is stronger than the hypothesis "$A(f) \geq 0$" in Conjecture 6.1. On the other hand, the conclusion of Theorem 6.1, namely "$F(z,\cdot)$ is increasing on $[0,1]$ for each $z \in X$ and $F(z,0) = 0,\ z \in X$" implies $F \geq 0$ and so it is stronger that the conclusion of Conjecture 6.1. The relationship between these hypotheses, conclusions, and other conditions/properties of f deserves to be further investigated.

Here is a result in this sense.

Theorem 6.2. *Let $f \in C(X)$ be a convex function and $z \in X$. Then $F(z,\cdot)$ is convex on $[0,1]$.*

Proof. Let $a, b \in [0,1]$ and $z \in X$. Then

$$f_{z,\frac{a+b}{2}}(t) = f\left(\frac{a+b}{2}t + \left(1 - \frac{a+b}{2}z\right)\right) = f\left(\frac{at+(1-a)z+bt+(1-b)z}{2}\right)$$
$$\leq \frac{f(at+(1-a)z)+f(bt+(1-b)z)}{2} = \frac{f_{z,a}(t)+f_{z,b}(t)}{2}.$$

Thus $f_{z,\frac{a+b}{2}} \leq \frac{1}{2} f_{z,a} + \frac{1}{2} f_{z,b}$, which yields

$$T f_{z,\frac{a+b}{2}} \leq \frac{1}{2} T f_{z,a} + \frac{1}{2} T f_{z,b}.$$

In particular,

$$T f_{z,\frac{a+b}{2}}(z) \leq \frac{1}{2} T f_{z,a}(z) + \frac{1}{2} T f_{z,b}(z),$$

i.e.

$$T f_{z,\frac{a+b}{2}}(z) - f(x) \leq \frac{1}{2}\left(T f_{z,a}(z) - f(z)\right) + \frac{1}{2}\left(T f_{z,b}(z) - f(z)\right).$$

This entails

$$F\left(z, \frac{a+b}{2}\right) \leq \frac{1}{2} F(z,a) + \frac{1}{2} F(z,b),$$

which means that $F(z,\cdot)$ is convex on $[0,1]$. □

7 The canonical projection on the standard simplex in $\mathbb{R}^2$

Let $X = \left\{(x,y) \in \mathbb{R}^2 \,|\, x,y \geq 0,\ x+y \leq 1\right\}$ be the standard simplex in $\mathbb{R}^2$ and T be the canonical projection, namely

$$T f(x,y) = (1-x-y) f(0,0) + x f(1,0) + y f(0,1),$$

for all $f \in C(X)$, $(x,y) \in X$.

Example 7.1. According to Definition 4.1, f is T-convex if and only if

$$(1-x-y) f((1-a)u,(1-a)v) + x f(a+(1-a)u,(1-a)v)$$
$$+ y f((1-a)u, a+(1-a)v) \geq f(ax+(1-a)u, ay+(1-a)v),$$

for all $a \in [0,1]$ and $(x,y),(u,v) \in X$.

According to Definition 5.1, f is weakly-T-convex if and only if

$$(1-u-v) f((1-a)u,(1-a)v) + u f(a+(1-a)u,(1-a)v)$$
$$+ v f((1-a)u, a+(1-a)v) \geq f(u,v),$$

for all $a \in [0,1]$ and $(u,v) \in X$.

Example 7.2. ([8], [22, Ex. 18.16, p.101]) A function f is T-convex if and only if f is axially convex. There exist functions T-convex that are not convex and there exist functions weakly-T-convex that are not T-convex.

If we refer to the statements

(a) f is convex;

(b) f is axially convex;

(c) f is T-convex;

(d) f is weakly-T-convex.

formulated for a function $f \in C(X)$, we have the following implications:

$$(a) \longrightarrow (b) \longleftrightarrow (c) \longrightarrow (d).$$

Remark 4. ([22, p.102]) It follows that $f \in C(X)$ is T-convex if and only if it is convex on every segment within the triangle X that is parallel to the Ox-axis, the Oy-axis, or the line $x+y=1$

If $f \in C^2(X)$, the T-convexity is characterized by

$$f''_{xx} \geq 0,\ f''_{yy} \geq 0,\ f''_{xx} + f''_{yy} - 2 f''_{xy} \geq 0. \tag{7.1}$$

If $f \in C^2(X)$, the convexity is expressed as

$$f''_{xx} \geq 0,\ f''_{yy} \geq 0,\ f''_{xx} f''_{yy} - (f''_{xy})^2 \geq 0. \tag{7.2}$$

Now we can better appreciate the difference between convexity and T-convexity on the standard simplex; in particular, it is clear that (7.2) implies (7.1).

Example 7.3. Let $f \in C^2(X)$, $z=(x,y)\in X$, $a\in[0,1]$. Consider the points $P(x-ax,y-ay)$, $Q(x+a(1-x),y-ay)$, $R(x-ax,y+a(1-y)$. Then, according to (6.1),

$$F(z,a)=(1-x-y)f(P)+xf(Q)+yf(R)-f(x,y)$$

and

$$F'_a(z,a)=(1-x-y)\left[f'_x(P)(-x)+f'_y(P)(-y)\right]$$
$$+x\left[f'_x(Q)(1-x)+f'_y(Q)(-y)\right]+y\left[f'_x(R)(-x)+f'_y(R)(1-y)\right].$$

$$\begin{aligned}F''_{a^2}(z,a)&=(1-x-y)\left[x^2f''_{x^2}(P)+2xyf''_{xy}(P)+y^2f''_{y^2}(P)\right]\\&+x\left[(1-x)^2f''_{x^2}(B)-2(1-x)yf''_{xy}(Q)+y^2f''_{y^2}(Q)\right]\\&+y\left[x^2f''_{x^2}(R)-2x(1-y)f''_{xy}(R)+(1-y)^2f''_{y^2}(R)\right].\end{aligned}$$

Suppose that $f\in C^2(X)$ is convex. Then $F''_{a^2}(z,a)\geq 0$, so that $F(z,\cdot)$ is convex on $[0,1]$. Moreover, if $a=0$, then the points P, Q, R are reduced to (x,y). It follows that $F'_a(z,0)=0$, and consequently $F(z,\cdot)\geq 0$. This Example illustrates Theorem 6.1.

Example 7.4. It is easy to verify that the infinitesimal generator corresponding to the canonical projection T acts on $C^2(X)$ as follows

$$A(f)(x,y)=\frac{x(1-x)}{2}f''_{x^2}(x,y)-xyf''_{xy}(x,y)+\frac{y(1-y)}{2}f''_{y^2}(x,y).$$

In this case the Conjecture 6.1 asks to prove that if $f\in C^2(X)$ and

$$\frac{x(1-x)}{2}f''_{x^2}(x,y)-xyf''_{xy}(x,y)+\frac{y(1-y)}{2}f''_{y^2}(x,y)\geq 0,\ x,y\in X,$$

then

$$(1-u-v)f((1-a)u,(1-a)v)+uf(a+(1-a)u,(1-a)v)$$
$$+vf((1-a)u,a+(1-a)v)\geq f(u,v),$$

for all $a\in[0,1]$ and $(u,v)\in X$.

Acknowledgement. The work was supported by the project "Mathematical Methods and Models for Biomedical Applications" financed by National Recovery and Resilience Plan PNRR-III-C9-2022-I8.

References

[1] A.M. Acu, I. RaÅa, Generalized subharmonic and weakly convex functions. In: A.M. Candela, M. Cappelletti Montano, E. Mangino (eds) Recent Advances in Mathematical Analysis, Trends in Mathematics, BirkhÃ¤user, Cham, 2023.

[2] F. Altomare, Lototsky-Schnabl operators on compact convex sets and their associated semigroups, Mh. Math. 114 (1992), 1-13.

[3] F. Altomare, M. Campiti, Korovkin-type Approximation Theory and its Applications, Series: De Gruyter Studies in Mathematics, 17, 1994.

[4] F. Altomare, M. Cappelletti Montano, and S. Diomede, Degenerate elliptic operators, Feller semigroups and modified Bernstein-Schnabl operators, Math. Nachr. 284 (2011), no. 5-6, 587â607.

[5] F. Altomare, M. Cappelletti Montano, V. Leonessa, I. Raşa, Markov Operators, Positive Semigroups and Approximation Processes, De Gruyter, 2014.

[6] F. Altomare and S. Milella, On the C_0-semigroups generated by second order differential operators on the real line, Taiwanese J. Math. 13 (2009), no. 1, 25â46.

[7] F. Altomare, S. Milella, and G. Musceo, On a class of positive C_0-semigroups of operators on weighted continuous function spaces, Note Mat. 31 (2011), no. 1, 15â27.

[8] F. Altomare, I. Raşa, Feller semigroups, Bernstein type operators and generalized convexity associated with positive projections, New Developments in Approximation Theory (Dortmund, 1998), Internat. Ser. Numer. Math., vol. 132, Birkhäuser, Basel, 1999, pp. 9â32.

[9] W. Boehm, A. Müller, On de Casteljau's algorithm, Computer Aided Geometric Design Volume 16, Issue 7, 1999, 587-605

[10] M. Cappelletti Montano, S. Diomede, Modified Bernstein-Schnabl operators on convex compact subsets of locally convex spaces and their limit semigroups, Commun. Appl. Anal. 13 (2009), no. 4, 609â632.

[11] M. Cappelletti Montano, V. Leonessa, A modification of Bernstein-Durrmeyer operators with Jacobi weights on the unit interval. In: A.M. Candela, M. Cappelletti Montano, E. Mangino (eds) Recent Advances in Mathematical Analysis, Trends in Mathematics, BirkhÃ¤user, Cham, 2023.

[12] G.M. Phillips, A de Casteljau algorithm for generalized Bernstein polynomials, BIT 36:1 (1996), 232-236.

[13] A.H. Clifford, G. B. Preston, The Algebraic Theory of Semigroups, Vol. I, American Mathematical Society, Providence, Rhode Island, 1961.

[14] A.H. Clifford, G. B. Preston, The Algebraic Theory of Semigroups, Vol. II, American Mathematical Society, Providence, Rhode Island, 1967.

[15] I. Raşa, Altomare projections and Lototsky-Schnabl operators, Proceedings of the Second International Conference in Functional Analysis and Approximation Theory (Acquafredda di Maratea, 1992), no. 33, 1993, pp. 439â451.

[16] I. Raşa, On some properties of Altomare projections, Confer. Sem. Mat. Univ. Bari (1993), no. 253, ii+19 pp. (1994).

[17] I. Raşa, Feller semigroups, elliptic operators and Altomare projections, Proceedings of the Fourth International Conference on Functional Analysis and Approximation Theory, Vol. I (Potenza, 2000), no. 68, part I, 2002, pp. 133â155.

[18] M. Romito, Lototsky-Schnabl operators associated with a strictly elliptic differential operator and their corresponding Feller semigroup, Monatsh. Math. 126 (1998), no. 4, 329â352.

[19] T. Sauer, Multivariate Bernstein polynomials and convexity, Comput. Aided Geom. Design 8 (1991), no. 6, 465â478.

[20] T. Sauer, Ein Bernstein-Durrmeyer Operator auf dem Simplex, Ph.D. thesis, Dissertation, Universität Erlangen-Nürnberg, 1992.

[21] T. Sauer, Axially parallel subsimplices and convexity, Comput. Aided Geom. Design 12 (1995), no. 5, 491â505.

[22] T. Vladislav, I. Raşa, Analiză Numerică. Aproximare, Problema lui Cauchy Abstractă, Proiectori Altomare, Editura Tehnică, Bucureşti, 1999. (in Romanian)