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Lebesgue constants and convergence of barycentric rational interpolation on arbitrary nodes

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Abstract

We investigate the order of magnitude of the Lebesgue constant of barycentric interpolation on arbitrary nodes, and explore its role in the order of approximation.

Key words: barycentric rational interpolation, Lebesgue constant.
MSC: 40A15

1 Introduction

For each $n = 1, 2, \dots$ let $X_n := \{0 = x_{0n} < x_{1n} < \dots < x_{nn} = 1\}$ be an arbitrary partition of the interval $I := [0, 1]$, and let $f(x)$ be a function defined on I. The so-called *barycentric interpolation operator*

$$B_{n,0}(f,x) := \frac{\sum_{k=0}^{n} \frac{(-1)^k f(x_k)}{x-x_k}}{\sum_{k=0}^{n} \frac{(-1)^k}{x-x_k}}, \qquad n = 1, 2, \dots \tag{1}$$

has been introduced in [1], and extensively investigated by several authors (for a comprehensive survey on the subject, see J.-P. Berrut and G. Klein [2]). Note that each x_k may depend on n; we use this short notation instead of x_{kn} for simplicity of the formulas.

The most important properties of this linear operator are the order of magnitude of its norm (the Lebesgue constant), and the convergence-divergence behavior as means of approximation. Of course, everything depends on the choice of the nodes x_k.

The error of uniform approximation by this operator is usually measured as a function of the quantity

$$h_n := \max_{0 \le k \le n-1} (x_{k+1} - x_k) \ge \frac{1}{n}. \tag{2}$$

The best known error estimate is $O(h_n)$, for twice differentiable functions (see Floater and Hormann [6], Theorem 3), which yields $O(1/n)$ for equidistant nodes. It is conjectured that the latter is the saturation order of the operator $B_{n,0}(f,x)$ even for equidistant nodes (cf. Mastroianni and Szabados [8], Conjecture 2).

In order to increase the rate of approximation, Floater and Hormann [6] generalized the operator (1) in the following way: let d be a fixed nonnegative integer, and let

$$B_{n,d}(f,x) := \frac{\sum_{k=0}^{n-d} \lambda_i(x) p_i(f,x)}{D(x)}, \qquad n \ge d \tag{3}$$

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where

$$D(x) := \sum_{i=0}^{n-d} \lambda_i(x), \qquad \lambda_i(x) := (-1)^i \prod_{k=i}^{i+d} \frac{1}{x - x_k}, \qquad i = 0, \ldots, n-d\,, \tag{4}$$

and

$$p_i(f, x) := \sum_{k=i}^{i+d} f(x_k)\ell_{i,k}(x), \qquad i = 0, \ldots, n-d\,, \tag{5}$$

is the dth degree Lagrange interpolating polynomial of $f(x)$ based on the nodes $\{x_k\}_{k=i}^{i+d}$ with the fundamental polynomials

$$\ell_{i,k}(x) := \prod_{\substack{s=i \\ s\neq k}}^{i+d} \frac{x - x_s}{x_k - x_s}, \qquad k = i, \ldots, i+d;\ i = 0, \ldots, n-d\,. \tag{6}$$

Obviously, (1) is the special case $d = 0$ of (3).

An easy calculation yields that this operator can be written in the form

$$B_{n,d}(f, x) = \frac{\sum_{j=0}^{n} \frac{(-1)^{d+j} f(x_j)}{x - x_j} \sum_{i=\max(0,j-d)}^{\min(j,n-d)} \prod_{\substack{s=i \\ s\neq j}}^{i+d} \frac{1}{|x_j - x_s|}}{D(x)}, \qquad n \geq d\,.$$

When we know some continuity properties of the function or its derivative, the so-called Lebesgue constant, i.e. the operator norm, plays an important role. For the operators (1) and (3) it is readily seen to be

$$\Lambda_d(X_n) := \sup_{x\in I} \frac{\sum_{j=0}^{n} \frac{1}{|x - x_j|} \sum_{i=\max(0,j-d)}^{\min(j,n-d)} \prod_{\substack{s=i \\ s\neq j}}^{i+d} \frac{1}{|x_j - x_s|}}{|D(x)|}, \qquad n \geq d\,. \tag{7}$$

(Here the empty product in case $d = 0$ is defined as 1.)

The purpose of this paper is to give estimates for the Lebesgue constants, as well as error estimates of the approximation by these barycentric operators.

2 Estimates for the Lebesgue constant

For equidistant nodes $X_n = \{x_k = k/n,\ k = 0, 1, \ldots, n\}$ and $d = 0$, Len Bos et al. ([3], Theorems 1 and 2), proved the estimates

$$\frac{2n}{4 + n\pi} \log(n + 1) \leq \Lambda_0(X_n) \leq \log n + 2\,.$$

G. Halász proved that *for any system of nodes* X_n

$$\Lambda_0(X_n) \geq \frac{1}{8} \log n, \qquad n = 2, 3, \ldots$$

holds (see Vértesi [9], Theorem 3.1).

Using a similar but more involved method, we prove the following generalization of this result.

Theorem 2.1. *For any $d \geq 0$ and any system of nodes X_n we have*

$$\Lambda_d(X_n) \geq \frac{1}{2(d + 1)!} \log \frac{n}{4}, \qquad n = 5, 6, \ldots\,. \tag{8}$$

Apart from d, this lower estimate is sharp in n, since similar upper estimates are shown for some special nodes (like quasi-equidistant, extended Chebyshev and Gauss–Lobato nodes); see e.g. [7] and [5].

Proof. Let k be such that

$$h_n = x_{k+1} - x_k$$

and let $y_k = \frac{x_k + x_{k+1}}{2}$. Without loss of generality we may assume that $n/2 \leq k \leq n$.

First we give an upper bound for the denominator $D(x)$ in (4) at $x = y_k$:

$$
\begin{aligned}
|D(y_k)| &= \left|\sum_{i=0}^{n-d} \lambda_i(y_k)\right| \\
&= \left|\sum_{i=0}^{k-d-1} \lambda_i(y_k)\right| + \sum_{i=\max(0,k-d)}^{\min(k+1,n-d)} |\lambda_i(y_k)| + \left|\sum_{i=\min(k+2,n-d+1)}^{n-d} \lambda_i(y_k)\right| \\
&=: D_1(y_k) + D_2(y_k) + D_3(y_k),
\end{aligned}
$$

with the understanding that $\sum_a^b = 0$ whenever $a < b$. Using the fact that in $D_1(y_k)$ the $\lambda_i(y_k)$'s form an alternating sequence increasing in absolute value when i increases, we get

$$
D_1(y_k) \le |\lambda_{k-d-1}(y_k)| \le \frac{1}{(y_k - x_{k-1})^{d+1}} \le \left(\frac{2}{h_n}\right)^{d+1} \quad (\text{if } d+1 \le k \le n-1), \tag{9}
$$

while $D_1(y_k) = 0$ when $0 \le k \le d$. Similarly, in $D_3(y_k)$ the $\lambda_i(y_k)$'s form an alternating sequence decreasing in absolute value when i increases, whence

$$
D_3(y_k) \le |\lambda_{k+2}(y_k)| \le \frac{1}{(x_{k+2} - y_k)^{d+1}} \le \left(\frac{2}{h_n}\right)^{d+1} \qquad (\text{if} \quad 0 \le k \le n-d-1), \tag{10}
$$

while $D_3(y_k) = 0$ when $n-d \le k \le n-1$. In $D_2(y_k)$, each $\lambda_i(y_k)$ contains either the factor $y_k - x_k = h_n/2$ or $x_{k+1} - y_k = h/2$, thus

$$
D_2(y_k) \le (d+2)\left(\frac{2}{h_n}\right)^{d+1}. \tag{11}
$$

Summarizing the estimates we obtain

$$
D(y_k) \le (d+4)\left(\frac{2}{h_n}\right)^{d+1}. \tag{12}
$$

Next, we estimate the numerator

$$
N(x) := \sum_{j=0}^{n} \frac{M_j}{|x - x_j|}, \quad \text{where} \quad M_j := \sum_{i=\max(0,j-d)}^{\min(j,n-d)} \prod_{\substack{s=i \\ s\ne j}}^{i+d} \frac{1}{|x_j - x_s|}. \tag{13}
$$

We have

$$
\prod_{\substack{s=i \\ s\ne j}}^{i+d} \frac{1}{|x_j - x_s|} \ge \frac{\binom{d}{j-i}}{d!h_n^d}, \qquad 0 \le i \le j, \quad d \le 0 \le k-1. \tag{14}
$$

Using the inequality

$$
|y_k - x_j| \le (k-j)h_n \qquad (d \le j \le k-1)
$$

we obtain from (13) and (14),

$$
N(y_k) \ge \frac{1}{d!h_n^{d+1}} \sum_{0 \le j \le k-1} \frac{1}{k-j} \sum_{i=0}^{j} \binom{d}{j-i} \ge \frac{2^d}{d!h_n^{d+1}} \log\frac{k}{2} \ge \frac{2^d}{d!h_n^{d+1}} \log\frac{n}{4}.
$$

This together with (12) yields the statement of the theorem, since by (2), $h_n \ge 1/n$. □

It is well-known that the $O(\log n)$ behavior for the Lebesgue constant in case of equidistant nodes is attained (for $d = 0$ see L. Bos et al. [3], Theorem 2, and for $d \ge 1$, L. Bos et al. [4], Theorem 1).

From a numerical point of view, the most important case is the equidistant nodes. However, theoretically it is equally interesting to investigate how large the Lebesgue constant can be for some other systems of nodes. The analogous situation for Lagrange interpolation is clear: if we move two adjacent nodes arbitrarily close to each other, then the Lebesgue constant can be arbitrarily large. The following example shows that in case of barycentric interpolation the situation is similar, although the construction of a system of nodes with arbitrarily large Lebesgue constant is not that simple.

Theorem 2.2. *Let a_n, $n = 1, 2, \ldots$, be an arbitrary sequence of positive numbers. Then there exists a sequence of nodes X_n, $n = 1, 2, \ldots$, such that*

$$
\Lambda_d(X_n) \ge c_d a_n, \qquad n = 1, 2, \ldots
$$

where $c_d > 0$ depends only on $d \ge 0$.

Proof. Evidently, we may assume that $a_n \geq (2n)^{\frac{d+1}{2}}$, $n = 1, 2, \ldots$. Let

$$x_k = \frac{k}{a_n^{\frac{2}{d+1}}}, \quad k = 0, 1, \ldots, n-2; \quad x_{n-1} = 1 - \frac{1}{a_n^{\frac{2}{d+1}}}$$

when $n-d$ is odd, and

$$x_k = \frac{k}{a_n^{\frac{2}{d+1}}}, \quad k = 0, 1, \ldots, n-1$$

when $n-d$ is even, and of course $x_n = 1$ in both cases. For the denominator in (7) we obtain $\left(\text{with } y = \frac{n-1}{a_n^{\frac{2}{d+1}}} + \frac{1}{a_n^{\frac{1}{d+1}}}\right)$

$$|D(y)| \leq \sum_{k=0}^{\left[\frac{n-d}{2}\right]-1} |\lambda_{2k}(y) + \lambda_{2k+1}(y)| + \sum_{i=2\left[\frac{n-d}{2}\right]}^{n-d} |\lambda_i(y)|$$

$$\leq \sum_{k=0}^{\left[\frac{n-d}{2}\right]-1} \frac{x_{2k+d+1} - x_{2k}}{\prod_{s=2k}^{2k+d+1}(y - x_s)} + c_d a_n^{\frac{d}{d}d+1} \leq \frac{n a_n^{\frac{d+2}{d+1}}}{a_n^{\frac{2}{d+1}}} + c_d a_n^{\frac{d}{d+1}} \leq c_d n a_n^{\frac{d}{d+1}}.$$

As for the numerator, we obtain

$$N(y) = \sum_{j=d}^{n-d} \frac{1}{|y - x_j|} \sum_{i=j-d}^{d} \prod_{\substack{s=i \\ s \neq j}}^{i+d} \frac{a_n^{\frac{2}{d+1}}}{|j-s|} \geq c_d \sum_{j=d}^{n-d} a_n^{\frac{1}{d+1}} a_n^{\frac{2d}{d+1}} \leq c_d n a_n^{\frac{2d+1}{d+1}}$$

whence

$$\Lambda_d(X_n) \geq \frac{N(y)}{D(y)} \geq c_d a_n. \qquad \square$$

3 Order of approximation

Concerning the order of approximation, Floater and Hormann ([6], Theorems 2 and 3) proved the following. Let

$$\beta_n = \begin{cases} 1 + \max\limits_{1 \leq i \leq n-2} \min\left(\frac{x_{i+1}-x_i}{x_i - x_{i-1}}, \frac{x_{i+1}-x_i}{x_{i+2}-x_{i+1}}\right) & \text{if} \quad d = 0, \\ 1 & \text{if} \quad d \geq 1. \end{cases}$$

Then for all $f^{(d+2)} \in C[0,1]$ it holds

$$\|f - B_{n,d}(f)\| \leq \beta_n h_n^{d+1}\left(\frac{\|f^{(d+2)}\|}{d+2} + [1 + (-1)^{n-d}]\frac{\|f^{(d+1)}\|}{d+1}\right). \tag{15}$$

We intend to give an error estimate where the Lebesgue constant $\Lambda_d(X_n)$ of the operator $B_{n,d}$ appears, and the class of functions is wider than in the above estimate (15). In fact, in the next theorem we assume only the continuity of $f^{(d)}$ instead of the boundedness of $f^{(d+2)}$ as above.

Theorem 3.1. *For any system of nodes X_n we have*

$$\|f - B_{n,d}(f)\| \leq h_n^d \omega(f^{(d)}, h_n)\left(\frac{(d+1)(d+2)}{2^{d-1}}\Lambda_d(X_n) + [1 + (-1)^{n-d}]\beta_n\right)$$

where $\omega(f^{(d)}, \cdot)$ is the modulus of continuity of $f^{(d)} \in C[0,1]$.

Proof. Using the formula (14) from [6], as well as the notation (4), we have

$$E(x) := D(x)|f(x) - B_{n,d}(f,x)| = \left|\sum_{i=0}^{n-d}(-1)^i f[x_i, \ldots, x_{i+d}, x]\right| \tag{16}$$

$$\leq \sum_{i=0}^{\left[\frac{n-d-1}{2}\right]} |f[x_{2i}, \ldots, x_{2i+d}, x] - f[x_{2i+1}, \ldots, x_{2i+d+1}, x]|$$

$$+ \frac{1 + (-1)^{n-d}}{2} \cdot |f[x_{n-d}, \ldots, x_n, x]|$$

where $f[\dots]$ is the divided difference of order $d+1$ of f with the corresponding nodes. Here, using the definition of divided differences we can reduce the order of divided differences to d:

$$f[x_{2i},\dots,x_{2i+d},x] = \frac{f[x_{2i+1},\dots,x_{2i+d},x] - f[x_{2i},\dots,x_{2i+d}]}{x - x_{2i}},$$

$$f[x_{2i+1},\dots,x_{2i+d+1},x] = \frac{f[x_{2i+1},\dots,x_{2i+d},x] - f[x_{2i+1},\dots,x_{2i+d+1}]}{x - x_{2i+d+1}},$$

and

$$f[x_{n-d},\dots,x_n,x] = \frac{f[x_{n-d+1},\dots,x_n,x] - f[x_{n-d},\dots,x_n]}{x - x_{n-d}}.$$

In the second formula we also used the symmetry of the divided differences. Thus

$$\begin{aligned}
E(x) \le & \sum_{i=0}^{\left[\frac{n-d-1}{2}\right]} \frac{|f[x_{2i+1},\dots,x_{2i+d},x] - f[x_{2i},\dots,x_{2i+d}]|(x_{2i+d+1} - x_{2i})}{|(x - x_{2i})(x - x_{2i+d+1})|} \qquad (17) \\
& + \sum_{i=0}^{\left[\frac{n-d-1}{2}\right]} \frac{|f[x_{2i},\dots,x_{2i+d}] - f[x_{2i+1},\dots,x_{2i+d+1}]|}{|x - x_{2i+d+1}|} \\
& + \frac{1+(-1)^{n-d}}{2} \cdot \frac{|f[x_{n-d+1},\dots,x_n,x] - f[x_{n-d},\dots,x_n]|}{|x - x_{n-d}|} \\
\le & \frac{1}{d!} \sum_{i=0}^{\left[\frac{n-d-1}{2}\right]} \left\{ \frac{|f^{(d)}(\xi_i) - f^{(d)}(\eta_i)|(d+1)h_n}{|(x - x_{2i})(x - x_{2i+d+1})|} + \frac{|f^{(d)}(\eta_i) - f^{(d)}(\zeta_i)|}{|x - x_{2i+d+1}|} \right\} \\
& + \frac{1+(-1)^{n-d}}{2} \cdot \frac{|f^{(d)}(\alpha) - f^{(d)}(\beta)|}{d!|x - x_{n-d}|} \\
\le & \frac{1}{d!} \sum_{i=0}^{\left[\frac{n-d-1}{2}\right]} \left\{ \frac{(d+1)h_n \omega(f^{(d)}, |\xi_i - \eta_i|)}{|(x - x_{2i})(x - x_{2i+d+1})|} + \frac{\omega(f^{(d)}, |\eta_i - \zeta_i|)}{|x - x_{2i+d+1}|} \right\} \\
& + \frac{1+(-1)^{n-d}}{2} \cdot \frac{\omega(f^{(d)}, |\alpha - \beta|)}{d!|x - x_{n-d}|},
\end{aligned}$$

where the η_i, ζ_i, α, β are intermediate values in the corresponding intervals,

$$\begin{aligned}
|\xi_i - \eta_i| &\le \max(x_{2i},\dots,x_{2i+d+1},x) - \min(x_{2i},\dots,x_{2i+d+1},x) \\
&= \max((d+1)h_n, |x - x_{2i}|, |x - x_{2i+d+1}|) =: M, \\
|\eta_i - \zeta_i| &\le x_{2i+d+1} - x_{2i} \le (d+1)h_n
\end{aligned}$$

and

$$|\alpha - \beta| \le 1 - \min(x, x_{n-d}).$$

Now if $M = (d+1)h_n$, then

$$\frac{(d+1)h_n \omega(f^{(d)}, |\xi_i - \eta_i|)}{|(x - x_{2i})(x - x_{2i+d+1})|} \le (d+1)^2 \omega(f^{(d)}, h_n) \left\{ \frac{1}{|x - x_{2i}|} + \frac{1}{|x - x_{2i+d+1}|} \right\},$$

while if $M = |x - x_{2i}| \ge (d+1)h_n$ then using the inequality

$$\omega(f^{(d)}, T) \le 2T\omega(f^{(d)}, t)/t \quad (0 < t \le T) \qquad (18)$$

we get

$$\frac{(d+1)h_n \omega(f^{(d)}, |\xi_i - \eta_i|)}{|(x - x_{2i})(x - x_{2i+d+1})|} \le \frac{2(d+1)\omega(f^{(d)}, h_n)}{|x - x_{2i+d+1}|}.$$

Similarly, if $M = |x - x_{2i+d+1}| \ge (d+1)h_n$, then

$$\frac{(d+1)h_n \omega(f^{(d)}, |\xi_i - \eta_i|)}{|(x - x_{2i})(x - x_{2i+d+1})|} \le \frac{2(d+1)\omega(f^{(d)}, h_n)}{|x - x_{2i}|}.$$

Also

$$\frac{\omega(f^{(d)}, |\eta_i - \zeta_i|)}{|x - x_{2i+d+1}|} \le \frac{(d+1)\omega(f^{(d)}, h_n)}{|x - x_{2i}|}.$$

Next, if $0 \le x \le x_{n-d}$, then $|\alpha - \beta| \le 1 - x$ and we obtain

$$\frac{|f^{(d)}(\alpha) - f^{(d)}\beta)|}{|x - x_{n-d}|} \le \frac{\omega(f^{(d)}, 1 - x_{n-d})}{x_{n-d} - x} + \frac{\omega(f^{(d)}, x_{n-d} - x)}{x_{n-d} - x}$$
$$\le \frac{(d+1)\omega(f^{(d)}, h_n)}{x_{n-d} - x} + \frac{\omega(f^{(d)}, x_{n-d} - x)}{x_{n-d} - x}.$$

Here we estimate the second term as

$$\frac{\omega(f^{(d)}, x_{n-d} - x)}{x_{n-d} - x} \le \begin{cases} \frac{\omega(f^{(d)}, h_n)}{x_{n-d} - x} & \text{if} \quad x_{n-d} - x \le h_n, \\ \frac{2\omega(f^{(d)}, h_n)}{h_n} & \text{if} \quad x_{n-d} - x \ge h_n, \end{cases}$$

where in the last line we have used the inequality (18) again.

Finally, if $0 \le x_{n-d} \le x < 1$, then $|\alpha - \beta| \le 1 - x_{n-d} \le (d+1)h_n$, and hence

$$\frac{|f^{(d)}(\alpha) - f^{(d)}\beta)|}{|x - x_{n-d}|} \le \frac{(d+1)\omega(f^{(d)}, h_n)}{x - x_{n-d}}.$$

Collecting these estimates, we obtain in all cases

$$\frac{|f^{(d)}(\alpha) - f^{(d)}\beta)|}{|x - x_{n-d}|} \le \frac{(d+2)\omega(f^{(d)}, h_n)}{|x - x_{n-d}|} + \frac{2\omega(f^{(d)}, h_n)}{h_n}.$$

Now substituting the obtained estimates in (17) we get

$$E(x) \le \frac{(d+1)(d+2)}{d!}\omega(f^{(d)}, h_n) \sum_{i=0}^{[\frac{n-d-1}{2}]} \left(\frac{1}{|x - x_{2i}|} + \frac{1}{|x - x_{2i+d+1}|} \right) \tag{19}$$
$$+[1 + (-1)^{n-d}] \left(\frac{(d+2)\omega(f^{(d)}, h_n)}{d!|x - x_{n-d}|} + \frac{\omega(f^{(d)}, h_n)}{d! h_n} \right)$$
$$\le \frac{2(d+1)(d+2)}{d!}\omega(f^{(d)}, h_n) \sum_{j=0}^{n} \frac{1}{|x - x_j|} + [1 + (-1)^{n-d}] \frac{\omega(f^{(d)}, h_n)}{d! h_n}.$$

To estimate M_j in (13) we use (2) to get

$$M_j \ge \sum_{i=\max(0,j-d)}^{\min(j,n-d)} \frac{1}{(j-i)!(i+d-j)! h_n^d} \ge \frac{2^d}{d! h_n^d}.$$

Thus (19) yields

$$E(x) \le \frac{(d+1)(d+2)}{2^{d-1}} h_n^d \omega(f^{(d)}, h_n) \sum_{j=0}^{n} \frac{M_j}{|x - x_j|} + [1 + (-1)^{n-d}] \frac{\omega(f^{(d)}, h_n)}{d! h_n}.$$

Using the estimate

$$|D(x)| \ge \frac{1}{d! h_n^{d+1}(1 + \beta_n)}$$

(see [6], p. 323 and the inequality (17) there) we finally obtain

$$|f(x) - B_{n,d}(f, x)| \le \frac{(d+1)(d+2)}{2^{d-1}} h_n^d \omega(f^{(d)}, h_n) \frac{E(x)}{D(x)}$$
$$+[1 + (-1)^{n-d}] \beta_n h_n^d \omega(f^{(d)}, h_n)$$
$$\le h_n^d \omega(f^{(d)}, h_n) \left(\frac{(d+1)(d+2)}{2^{d-1}} \Lambda_d(X_n) + [1 + (-1)^{n-d}] \beta_n \right).$$

□

4 Approximation of piecewise convex/concave functions for equidistant nodes

The presence of the Lebesgue constant in the previous theorem causes an an extra factor in the error estimate which is at least $O(\log n)$. Under some mild restriction on the structure of a continuous function, in case of $d = 0$ we can eliminate the effect of the Lebesgue constant as shown in the next theorem.

Theorem 4.1. *Let $0 = a_0 < a_1 < \cdots < a_s = 1$ be a fixed partition of the interval $[-1, 1]$, and assume that $f \in C[0, 1]$ is convex or concave in each of the intervals $I_j := [a_{j-1}, a_j]$, $j = 1, 2, \ldots, s$. Then for the equidistant nodes $E_n = \{k/n,\ k = 0, 1, \ldots, n\}$ we have*

$$\|f - B_{n,0}(f)\| \le c_s \omega\left(f, \frac{1}{n}\right)$$

where the constant $c_s > 0$ depends only on s.

Remark. This theorem has been stated in [8], but in the case $s \ge 2$ the proof contained an error.[1]

Lemma 4.2. *Let $f(x)$ be a convex or concave function in the finite interval $[a, b]$, and let $X_m = \{z_0 < \cdots < z_m\} \subset [a, b]$ be an arbitrary system of nodes. Then we have*

$$\left|\sum_{k=0}^{m}(-1)^k f[z_k, x]\right| \le |f[z_0, x]| + |f[z_m, x]|, \qquad x \in [a, b] \setminus X_m \,. \tag{20}$$

Proof. Denoting $b_k := f[z_k, x]$, $k = 0, \ldots, m$, we have

$$\left|\sum_{k=0}^{m}(-1)^k f[z_k, x]\right| = \left|\sum_{k=0}^{m}(-1)^k b_k\right| = \frac{1}{2}\left|b_0 + \sum_{k=0}^{m-1}(-1)^k(b_k - b_{k+1}) + (-1)^m b_m\right|$$
$$\le \frac{1}{2}\left(|b_0| + \sum_{k=0}^{m-1}|b_k - b_{k+1}| + |b_m|\right).$$

By assumption, $f(x)$ is convex (or concave) on $[a, b]$, therefore the sign of

$$b_k - b_{k+1} = f[z_k, x] - f[z_{k+1}, x] = -f[z_k, z_{k+1}, x](z_{k+1} - z_k)$$

is constant, whence the telescoping sum yields the statement of the lemma. □

Proof of Theorem 4. Using (16) with $d = 0$ we obtain

$$|f(x) - B_{n,0}(f, x)| = \frac{1}{D(x)}\left|\sum_{k=0}^{n}(-1)^k f[x_k, x]\right| \tag{21}$$

where D in (4) takes the form

$$D(x) := \left|\sum_{k=0}^{n}\frac{(-1)^k}{x - x_k}\right|. \tag{22}$$

Fix an $x \in [a_{j-1}, a_j)$, and let $x_i = i/n$ be a nearest node to x. If $x < x_i$ then

$$D(x) > \left|\frac{1}{x - x_i} - \frac{1}{x - x_{i+1}}\right| = \frac{1}{n(x_i - x)(x_i - x + 1/n)} \ge \frac{2}{3(x_i - x)}.$$

Evidently, a similar inequality holds if $x > x_i$, i.e. we have

$$D(x) \ge \frac{2}{3|x - x_i|}. \tag{23}$$

Next we give an upper estimate for the sum on the right hand side of (21). Individual terms of this sum can be easily estimated, since using (18) we get

$$|f[x_k, x]| \le \frac{\omega(f, |x - x_k|)}{|x - x_k|} \le \frac{2\omega(f, 1/n)}{|x - x_i|}, \qquad k = 0, \ldots, n\,.$$

[1] The author is grateful to Professor Walter F. Mascarenhas (Sao Paolo) for pointing out this mistake.

Based on this estimate, separating the nodes in the $1/n$-neighborhood of the a_j's, we partition the rest of the sum into several parts according to the position of the nodes:

$$\left|\sum_{k=0}^{n}(-1)^k f[x_k,x]\right| \tag{24}$$

$$\leq \left|\left(\sum_{\ell=1}^{j-1}\sum_{a_{\ell-1}+1/n\leq x_k\leq a_\ell-1/n}+\sum_{x_k\in I_j}+\sum_{\ell=j+1}^{s}\sum_{a_{\ell-1}+1/n\leq x_k\leq a_\ell-1/n}\right)(-1)^k f[x_k,x]\right|$$

$$+\frac{4s\omega(f,1/n)}{|x-x_i|}.$$

To estimate the middle sum here, we use the Lemma on the interval I_j (with equidistant nodes). Denoting the smallest and largest node in I_j by x_μ and x_ν, respectively, we get

$$\left|\sum_{x_k\in I_j}(-1)^k f[x_k,x]\right| \leq |f[x_\mu,x]|+|f[x_\nu,x]| \tag{25}$$

$$\leq 2\max_{0\leq k\leq n}\frac{\omega(f,|x-x_k|)}{|x-x_k|}\leq\frac{4\omega(f,|x-x_i|)}{|x-x_i|}\leq\frac{4\omega(f,1/n)}{|x-x_i|},$$

where we used inequality (18) with $T=|x-x_k|\geq t=|x-x_i|$.

Finally, we estimate the rest of the sums in (24). Since $f(x)$ is convex (or concave) in any interval $[a_{\ell-1}+1/n,a_\ell-1/n]$, $1\leq \ell\leq j-1$, choosing the largest node x_u in this interval and writing $y_\ell=x_u-1/(2n)$ we obtain again by the Lemma

$$\left|\sum_{a_{\ell-1}+1/n\leq x_k\leq a_\ell-1/n}(-1)^k f[x_k,y_\ell]\right|\leq c_s+\frac{\omega(f,x_u-y_\ell)}{y_\ell-x_u}\leq c_s n\omega(f,1/n),$$

$1\leq \ell\leq j-1$. Thus

$$\left|\sum_{a_{\ell-1}+1/n\leq x_k<a_\ell-1/n}(-1)^k f[x_k,x]\right|$$

$$\leq\left|\sum_{a_{\ell-1}+1/n\leq x_k\leq a_\ell-1/n}(-1)^k(f[x_k,x]-f[x_k,y_\ell])\right|+2c_s n\omega(f,1/n).$$

Now

$$\left|\sum_{a_{\ell-1}+1/n\leq x_k\leq a_\ell-1/n}(-1)^k(f[x_k,x]-f[x_k,y_\ell])\right|$$

$$\leq |f(x)-f(y_\ell)|\cdot\left|\sum_{a_{\ell-1}+1/n\leq x_k\leq a_\ell-1/n}\frac{(-1)^k}{x-x_k}\right|$$

$$+(x-y_\ell)\left|\sum_{a_{\ell-1}+1/n\leq x_k\leq a_\ell-1/n}\frac{(-1)^k f[x_k,y_\ell]}{x-x_k}\right|.$$

Here

$$|f(x)-f(y_\ell)|\cdot\left|\sum_{a_{\ell-1}+1/n\leq x_k\leq a_\ell-1/n}\frac{(-1)^k}{x-x_k}\right|\leq\frac{|f(x)-f(y_\ell)|}{x-x_u}$$

$$\leq\frac{|f(x)-f(x_u)|}{x-x_u}+\frac{|f(x_u)-f(y_\ell)|}{x_u-y_\ell}\leq\frac{2\omega(f,1/n)}{|x-x_i|}+2n\omega(f,1/n).$$

Denoting the smallest node greater than or equal to $a_{\ell-1}+1/n$ by x_v, we get by Abel summation

$$(x-y_\ell)\left|\sum_{a_{\ell-1}+1/n\leq x_k\leq a_\ell-1/n}\frac{(-1)^k f[x_k,y_\ell]}{x-x_k}\right|\leq\frac{x-y_\ell}{x-x_u}\max_{v\leq t\leq u}\left|\sum_{k=v}^{t}(-1)^k f[x_k,y_\ell]\right|$$

$$\leq|f[x_v,y_\ell]|+\max_{v\leq t\leq u}|f[x_t,y_\ell]|\leq 2c_s+2\max_{v\leq t\leq u}\frac{\omega(f,y_\ell-x_t)}{y_\ell-x_t}\leq 2c_s n\omega(f,1/n).$$

Collecting the above estimates we get

$$\left|\sum_{a_{\ell-1}\le x_k\le a_\ell}(-1)^k f[x_k,x]\right|\le c_s\left(n\omega(f,1/n)+\frac{\omega(f,1/n)}{|x-x_i|}\right),\qquad \ell=1,2,\dots,j\,.$$

Evidently, the same estimates hold for $j+1\le\ell\le s$.

Thus we obtain from (21) and (23),

$$|f(x)-B_{n,0}(f,x)|\le\frac{c_s}{|D(x)|}\left(n\omega(f,1/n)+\frac{\omega(f,1/n)}{|x-x_i|}\right)\le c_s\omega(f,1/n),\quad x\in I\,.$$

□

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