Property A and best L^1-Approximation

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Abstract

An important property within the linear theory of best L^1-approximation, the so-called Property A, is discussed. Indeed, this property characterizes all finite-dimensional $L^1(\mu)$-unicity subspaces of continuous functions with respect to measures μ of a big class. Moreover, some extensions are considered: Property A^k, characterizing those subspaces where the sets of best $L^1(\mu)$-approximations have at most dimension k, and Property A_{per} characterizing $L^1(\mu)$-unicity subspaces of continuous periodic functions.

1 Introduction

In this paper we give a survey on relevant results during the period 1975–2012 concerning best L^1-approximation from finite-dimensional subspaces of real-valued continuous functions. The central role plays Property A characterizing all unicity subspaces. We also discuss some extensions, Property A^k characterizing all subspaces with Chebyshev rank k, and Property A_{per} characterizing all unicity subspaces of continuous periodic functions.

To introduce the approximation problem of interest, assume that K is a compact subset of $\mathbb{R}^d$ ($d \geq 1$) such that $K = \overline{\text{int } K}$ (the closure of its interior), and let G denote a finite-dimensional subspace of $C(K)$, the linear space of real-valued continuous functions defined on K. Moreover, let a set W of measures be given by

$$W = \{\mu : d\mu = wd\lambda, \ w \in L^{\infty}(K), \ \text{ess inf } w > 0 \text{ on } K\}$$

(λ denotes the Lebesgue measure on $\mathbb{R}^d$). For $\mu \in W$, let us define the *weighted $L^1(\mu)$-norm* $\|\cdot\|_{\mu}$ by

$$\|f\|_{\mu} = \int\limits_K |f| d\mu \quad (f \in C(K)).$$

Let $C_1(K, \mu)$ denote the linear space $C(K)$ endowed with norm $\|\cdot\|_{\mu}$. If G is a finite-dimensional subspace of $C_1(K, \mu)$, then $g_0 \in G$ is called a *best $L^1(\mu)$-approximation of $f \in C(K)$ from G* if $\|f - g_0\|_{\mu} \leq \|f - g\|_{\mu}$ for every $g \in G$. $P_G^{\mu}(f)$ denotes the set of all best $L^1(\mu)$-approximations of f from G.

It is well-known that, for each $f \in C(K)$, $P_G^{\mu}(f)$ is a non-empty, convex and compact subset of G. We say that $P_G^{\mu}(f)$ has *dimension k* ($0 \leq k \leq \dim G$), denoted by $\dim P_G^{\mu}(f) = k$, if there exist functions $g_0, \ldots, g_k$ in $P_G^{\mu}(f)$ such that $\{g_i - g_0\}_{i=1}^k$ are linearly independent and k is maximal under this property (this corresponds to the dimension of the smallest affine subspace of G containing $P_G^{\mu}(f)$). If $P_G^{\mu}(f)$ is a singleton, then $\dim P_G^{\mu}(f) = 0$.

Moreover, we say that G is *k-convex* (or G has *Chebyshev rank k*) *with respect to μ* ($0 \leq k \leq \dim G$), denoted by $\text{cr}(G, \mu) = k$, if $\dim P_G^{\mu}(f) \leq k$ for every $f \in C(K)$, and there exists an $\hat{f} \in C(K)$ such that $\dim P_G^{\mu}(\hat{f}) = k$. If $\text{cr}(G, \mu) = 0$, then every $f \in C(K)$ has a unique best $L^1(\mu)$-approximation from G, i.e., G is a *unicity subspace for $C_1(K, \mu)$*. Finally, we say that G has *Chebyshev rank k with respect to W* or, for brevity, G has *Chebyshev rank k*, if

$$\text{cr}(G) = \max_{\mu \in W} \text{cr}(G, \mu) = k.$$

Many of the results stated in this survey are due to András Kroó. His research was a strong motivation for the author of this paper to study problems of best L^1-approximation during a long period.

2 Property A

Property A plays a central role in the theory of best L^1-approximation.

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Definition 2.1. Let G be a finite-dimensional subspace of $C(K)$. Then G is said to satisfy *Property* A (or G is an A-*space*), if for every $g_0 \in G \setminus \{0\}$ and every function ψ such that $\psi = 0$ on $Z(g_0)$, $|\psi| = 1$ and ψ is continuous on $K \setminus Z(g_0)$, there exists a $g \in G(g_0) \setminus \{0\}$ satisfying $\psi g \geq 0$ on K.

Here $Z(g_0)$ denotes the set of zeros of g_0, and $G(g_0)$ is defined by $G(g_0) = \{g \in G : g = 0 \text{ a.e. on } Z(g_0)\}$.

The actual version of Property A was established by Kroó [3]. It is based on a Condition A given by Strauss [29] for $K = [a, b]$ and $\mu = \lambda$, the Lebesgue measure. Strauss proved the following statement.

Theorem 2.1. *Let G be a finite-dimensional subspace of $C[a, b]$ satisfying Condition A. Moreover assume that each $g \in G$ has only finitely many separated zeros. Then G is a unicity subspace for $C_1([a, b], \lambda)$.*

Using this condition Strauss verified that subspaces of polynomial splines with simple knots are unicity subspaces for $C_1([a, b], \lambda)$. In [18], [19] we showed that subspaces of generalized spline functions, including subspaces of polynomial and Chebyshevian splines and subspaces of piecing together Haar systems, also satisfy Condition A.

Thm. 2.1 was the key result for extensive studies on existence and characterization of L^1-unicity subspaces. The first relevant statement was an extension given by Kroó [3].

Theorem 2.2. *Let G be a finite-dimensional subspace of $C[a, b]$ satisfying Property A. Then G is a unicity subspace for $C_1([a, b], \mu)$ for all $\mu \in W$.*

He conjectured that the converse of Thm. 2.2 should also be true, and could verify it for $\dim G = 1$. Based on his result, in [20] we proved the converse for finite-dimensional subspaces G satisfying certain properties A1 and A2.

Finally, Kroó [4] verified his conjecture.

Theorem 2.3. *Let G be a finite-dimensional subspace of $C[a, b]$. The following statements are equivalent.*

(i) G is a unicity space for $C_1([a, b], \mu)$ for all $\mu \in W$.

(ii) G satisfies Property A.

Independently, Pinkus [13] also verified the converse for those subspaces of $C[a, b]$ satisfying $\lambda(Z(g)) = \lambda(\text{int } Z(g))$ for all $g \in G$, and all measures μ of the form $d\mu = wd\lambda$ where w is strictly positive and continuous.

How is the situation for the case when $K = \overline{\text{int } K} \subset \mathbb{R}^d$ $(d \geq 1)$, K compact? Independently, in Kroó [8] and our paper [21] the converse of Thm. 2.2 was verified for this general case. Since also the statement of Thm. 2.2 remains true, a characterization of $L^1(\mu)$-unicity subspaces of $C(K)$ was then completed.

Theorem 2.4. *Let $K = \overline{int\,K} \subset \mathbb{R}^d$ $(d \geq 1)$, K compact, and assume that G denotes a finite-dimensional subspace of $C(K)$. The following statements are equivalent.*

(i) G is a unicity space for $C_1(K, \mu)$ for all $\mu \in W$.

(ii) G satisfies Property A.

Remark 1. Kroó [8] extended this result to $L^1(\mu)$-unicity subspaces of $C(K, B)$ where B denotes a real Banach space. Schmidt [16] gave an analogue to Thm. 2.4 replacing the class W by a smaller subclass of measures in statement (i).

Of course, Thm. 2.4 states that G satisfies Property A if and only if $\text{cr}(G) = 0$ (the Chebyshev rank of G, see Section 1). In Section 6, a natural extension of Property A, Property A^k, is considered characterizing all the G with $\text{cr}(G) = k$ $(0 \leq k \leq \dim G)$.

3   Characterization of A-Subspaces of $C(K)$ where $K \subset \mathbb{R}$

As we now know is that Property A characterizes the unicity subspaces of $C_1(K, \mu)$ for all $\mu \in W$. But what does an A-space look like, at least in the case when $K = [a, b]$ or, more generally, K is compact and $K = \overline{\text{int } K} \subset \mathbb{R}$?

In the following certain classes of subspaces play an important role. An n-dimensional subspace G of $C(K)$, $K \subset \mathbb{R}$, is said to be a *Chebyshev space* or a *Haar system* if every $g \in G \setminus \{0\}$ has at most $n - 1$ zeros on K, whereas G is said to be *weak Chebyshev* if every $g \in G$ has at most $n - 1$ sign changes on K, i.e., there do not exist points $\{x_0, \ldots, x_n\} \subset K$ such that $x_0 < \cdots < x_n$ and $g(x_i)g(x_{i+1}) < 0$, $0 \leq i \leq n - 1$.

We have mentioned above that some classes of subspaces of generalized splines satisfy Condition A (which corresponds to Property A in the considered cases). All these subspaces of $C[a, b]$ have also the weak Chebyshev property. This observation is not surprising as we have shown in [20], [21].

Theorem 3.1. *Let $K = \overline{int\,K} \subset \mathbb{R}$, K compact, and let G denote a finite-dimensional A-subspace of $C(K)$. Then G is a weak Chebyshev space.*

To establish a complete characterization of A-subspaces of $C(K)$ let us go back to the general case when $K = \overline{\text{int } K} \subset \mathbb{R}^d$ $(d \geq 1)$, K compact. Moreover, let G be an n-dimensional subspace of $C(K)$. If $g \in G \setminus \{0\}$, then $K \setminus Z(g)$ is open with respect to K. As such it is an at most countable union of open (w.r.t. K) connected components. Denote the number of such components by $[K \setminus Z(g)]$. This number may be infinite. Let $Z(G) = \bigcap \{Z(g) : g \in G\}$.

The following statement due to Pinkus and Wajnryb [15] plays a crucial role.

Theorem 3.2. *Let K be given as above. Assume that G is an A-space. Then the following is true.*

(i) Let $g^* \in G \setminus \{0\}$. Then for every $g \in G(g^*)$,

$$[K \setminus Z(g)] \leq \dim G(g^*).$$

(ii) If $K \setminus Z(G)$ is not connected, then G decomposes, i.e., $K \setminus Z(G) = \bigcup_{i=1}^{k} A_i$, where A_i is open and connected in K, and if $\dim G|_{A_i} = m_i$ $(m_i \geq 1)$, $1 \leq i \leq k$, then $\sum_{i=1}^{k} m_i = n = \dim G$, and there exist functions $\{g_1^{(i)}, \ldots, g_{m_i}^{(i)}\}$ in G such that

$$G|_{A_i} = span\{g_1^{(i)}|_{A_i}, \ldots, g_{m_i}^{(i)}|_{A_i}\}$$

and $g_j^{(i)}$ vanishes identically off A_i, $1 \leq j \leq m_i$, $1 \leq i \leq k$.

Remark 2. (i) If $K \subset \mathbb{R}$, then statement (i) implies that $G(g^*)$ is a weak Chebyshev subspace of $C(K)$.
(ii) Using the notations of statement (ii), set

$$G_i := span\{g_1^{(i)}, \ldots, g_{m_i}^{(i)}\},$$

$1 \leq i \leq k$. Then by statement (ii),

$$G = G_1 \oplus \cdots \oplus G_k.$$

Moreover, it is easily verified that each G_i is an A-subspace of $C(K)$, $1 \leq i \leq k$.
(iii) Conversely, let us assume that $\widetilde{G}_i$ is an A-subspace of $C(K)$ such that all functions in $\widetilde{G}_i$ vanish identically off A_i, $1 \leq i \leq k$. Then the space $\widetilde{G}$ defined by

$$\widetilde{G} = \widetilde{G}_1 \oplus \cdots \oplus \widetilde{G}_k$$

is also an A-subspace of $C(K)$.

In the case when $K \subset \mathbb{R}$, the connected components A_i in K reduce to real bounded closed, open or half-open intervals. On the basis of Thm. 3.2 and Remark 2, one may therefore assume that $K = [a, b]$ and $Z(G) \cap (a, b) = \emptyset$, G an n-dimensional subspace of $C[a, b]$. Moreover define, for any subset M of $[a, b]$,

$$G(M) = \{g \in G : g = 0 \text{ on } M\}.$$

Under these assumptions Pinkus [13], [14] totally classified all A-spaces as follows.

Theorem 3.3. *Let G be given as above. Then G is an A-space if and only if the following conditions (i)–(iv) hold.*

(i) *G is weak Chebyshev;*

(ii) *There exist points $a = c_0 < c_1 < \cdots < c_s = b$, $s \leq 2n - 1$, such that $G|_{(c_{i-1}, c_i)}$ is a Chebyshev subspace, $1 \leq i \leq s$;*

(iii) *$G([c_i, c_j]) = G([c_0, c_j]) \oplus G([c_i, c_s])$, $0 \leq i < j \leq s$;*

(iv) *$G([c_0, c_i) \cup (c_j, c_s])$ is weak Chebyshev on $[a, b]$, $0 \leq i < j \leq s$.*

Li [12] showed that (i) implies (iv) in Thm. 3.3, and (i) and (iii) imply (ii). Hence he obtained the following simplified characterization of A-subspaces G of $C[a, b]$ with $Z(G) \cap (a, b) = \emptyset$.

Theorem 3.4. *G is an A-space if and only if G is weak Chebyshev and $G([c, d]) = G([a, d]) \oplus G([c, b])$ for any $a < c < d < b$.*

Remark 3. (i) The Thms. 3.2–3.4 together completely characterize all A-subspaces of $C(K)$ where $K = \overline{int\,K} \subset \mathbb{R}$, K compact. Especially, Thm. 3.3 states that an A-space is a space of generalized splines, because it has a "spline-like" structure. In [19] we already constructed a large class of such spaces with Property A.
(ii) Of course, many important classes of finite-dimensional subspaces of $C[a, b]$ possess Property A, including all Chebyshev spaces and all spaces of polynomial spline functions of finite degree to a fixed knot partition of $[a, b]$.

Using Thm. 3.3, in [23] we could verify that every A-space is a complete A-space in the following sense.

Theorem 3.5. *Let $K = \overline{int\,K} \subset \mathbb{R}$, K compact, and let G be an n-dimensional A-subspace of $C(K)$. Then G has a basis $\{g_1, \ldots, g_n\}$ such that $span\{g_1, \ldots, g_i\}$ is an A-space, $1 \leq i \leq n - 1$.*

The corresponding analogous property for weak Chebyshev spaces is known for a long time (see [17]). Indeed, every such subspace G of $C(K)$ is completely weak Chebyshev, i.e., there exists a basis $\{g_1, \ldots, g_n\}$ of G such that $span\{g_1, \ldots, g_i\}$ is weak Chebyshev on K, $1 \leq i \leq n - 1$. Moreover, as we showed in [23], such a G satisfies the property that $G|_{\widetilde{K}}$ is also weak Chebyshev for every $\widetilde{K} \subset K$.

Does an A-space satisfy corresponding properties? Using Thm. 3.2 we obtained the following result [23].

Theorem 3.6. *Let $K = \overline{int\,K} \subset \mathbb{R}$, K compact, and let G denote an n-dimensional A-subspace of $C(K)$. If I is a real bounded interval, then $\widetilde{G} = G|_{\widetilde{K}}$ is an A-subspace of $C(\widetilde{K})$ where $\widetilde{K} = \overline{int\,(I \cap K)}$.*

Remark 4. (i) We showed by an example that the above statement fails if the interval I is replaced by an arbitrary compact subset $\widetilde{K}$ of K with $\widetilde{K} = \overline{int\,\widetilde{K}}$. Hence in view of the above remarks, the situation is unlike the situation for weak Chebyshev subspaces.
(ii) Moreover, we showed by an example that there can exist an $(n - 1)$-dimensional weak Chebyshev subspace $\widetilde{G}$ of an A-space G such that $\widetilde{G}$ fails to be an A-space.

Thm. 3.2 implies that the existence of "non-trivial" A-subspaces of $C(K)$ is essentially restricted to the case when K is connected. In particular, this holds for an important class of subspaces (see e.g. [5]).

Corollary 3.7. *Let K be a compact and disconnected subset of $\mathbb{R}$ such that $K = \overline{int\,K}$. Then every n-dimensional Chebyshev subspace of $C(K)$ fails to satisfy Property A.*

4 A-Spaces and the Hobby-Rice Theorem

In two papers [10], [11] Kroó, Schmidt and Sommer studied Property A and its connection with the Hobby-Rice theorem. The first main result [10] is stated as follows .

Theorem 4.1. *Let G be an n-dimensional subspace of $C[a,b]$. Then G is an A-space if and only if the following conditions (i) and (ii) hold.*

(i) *G is weak Chebyshev.*

(ii) *If $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$ and $\{x_1, \ldots, x_n\} \subset Z(g)$ for some $g \in G \setminus \{0\}$, then there exists a $\tilde{g} \in G \setminus \{0\}$ such that $(-1)^i \tilde{g} \geq 0$ on $[x_i, x_{i+1}]$, $0 \leq i \leq n$.*

Remark 5. Condition (ii), stated with $n-1$ or fewer points in (a,b), is one of the many characterizations of weak Chebyshev spaces. As such, this condition is indeed a strengthened version of the weak Chebyshev property.

Of particular interest is that condition (ii) has arisen in connection with the Hobby-Rice theorem. This result plays a fundamental role in the theory of L^1-approximation. That is, if G is an n-dimensional subspace of $L^1([a,b], \nu)$ (the space of all real-valued, ν-measurable functions on $[a,b]$ where ν is a positive, non-atomic measure on $[a,b]$), then there exist points $a = x_0 < x_1 < \cdots < x_m < x_{m+1} = b$ with $0 \leq m \leq n$ such that

$$\sum_{i=0}^{m} (-1)^i \int_{x_i}^{x_{i+1}} g \, d\nu = 0$$

for all $g \in G$.

In [10], [11] our studies were concerned with subspaces G of $C[a,b]$ and measures ν of the form $d\nu = w d\lambda$, where w is in the class C^+ of positive, continuous functions on $[a,b]$ (hence $\nu \in W$). Then such a set $\{x_1, \ldots, x_m\}$ is called a *w-canonical set for G*. Kroó [9] showed that for an n-dimensional subspace G of $C[a,b]$, the w-canonical sets contain n points for all $w \in C^+$, if and only if G is a weak Chebyshev space. Moreover, in studying uniqueness of w-canonical sets, he verified that, for a weak Chebyshev space G, condition (ii) of Thm. 4.1 is equivalent to G having a unique w-canonical set $\{x_1, \ldots, x_n\}$ and $\dim G|_{\{x_1, \ldots, x_n\}} = \dim G = n$ for every $w \in W$. Let us call the dimension of $G|_{\{x_1, \ldots, x_n\}}$ the *rank* of the w-canonical set $\{x_1, \ldots, x_n\}$.

As an application of Thm. 4.1, in [10] the following corollary connecting Property A and the Hobby-Rice theorem was verified.

Corollary 4.2. *Let G be an n-dimensional subspace of $C[a,b]$. Then G is an A-space if and only if, for every $w \in C^+$, the w-canonical set is unique, contains n points $x_1 < \cdots < x_n$, and has rank n.*

To state the results given in [11], assume again that G is an n-dimensional subspace of $C[a,b]$. G is said to satisfy the *splitting property* provided that if $g \in G$ and $g = 0$ on $[c,d]$ where $a < c < d < b$, then $g\chi_{[a,c]}$, $g\chi_{[d,b]} \in G$ where χ_J denotes the characteristic function of $J \subset [a,b]$. Moreover, G is said to satisfy the *decomposition property* if $z \in Z(G) \cap (a,b)$ implies that $G = G([a,z]) \oplus G([z,b])$. Hence the Pinkus-Li characterization (Thms. 3.3 and 3.4) states that G is an A-space if and only if it satisfies the weak Chebyshev, splitting and decomposition properties.

The following characterization is the first major result in [11].

Theorem 4.3. *Let G be an n-dimensional subspace of $C[a,b]$. Then*

(i) *the w-canonical sets for G contain n points for all $w \in C^+$ if and only if G satisfies the weak Chebyshev property;*

(ii) *the w-canonical set for G contains n points and is unique for all $w \in C^+$ if and only if G satisfies the weak Chebyshev and splitting properties;*

(iii) *the w-canonical set for G contains n points, is unique, and has rank n for all $w \in C^+$ if and only if G satisfies the weak Chebyshev, splitting, and decomposition properties (that is, G is an A-space).*

Statement (ii) is a new result whereas (iii) corresponds to Cor. 4.2 and (i) was already known (see [11] for references). For the proof of (ii) and (iii) a result of Kroó [9] was used. He characterized there those weak Chebyshev spaces that have locally unique w-canonical sets.

The problem of extending spaces with a given property has been widely studied, especially extension of n-dimensional Chebyshev spaces to $(n+1)$-dimensional Chebyshev spaces. In [11] the extension of A-spaces was verified.

Theorem 4.4. *Let G be an n-dimensional A-space in $C[a,b]$. Then there exists an $f \in C[a,b]$ such that $G \oplus \text{span}\{f\}$ is an $(n+1)$-dimensional A-space.*

5 A-Subspaces of $C(K)$ where $K \subset \mathbb{R}^d$, $d > 1$

Up to now only few results on A-spaces have been stated for the case when K is a subset of $\mathbb{R}^d$ where $d > 1$.

The first result in this context is due to Kroó [2].

Theorem 5.1. *Let K be a compact convex subset of $\mathbb{R}^d$ satisfying $K = \overline{int\,K}$. Set*

$$G = \{g(x_1, \ldots, x_d) = \sum_{i=1}^{d} a_i x_i + a_{d+1} : a_i \in \mathbb{R}\},$$

the set of affine linear functions. Then G satisfies Property A.

The most important non-trivial example of an A-space is related to multivariate linear splines on regular partitions. To define the spaces of interest, let K be a bounded and connected polygonal domain in $\mathbb{R}^d$ $(d \geq 2)$ such that

$$K = \bigcup_{i=1}^{N} K_i \quad (N \in \mathbb{N})$$

where $\Delta = \{K_i\}_{i=1}^{N}$ is a collection of d-dimensional simplices. Moreover, assume that Δ is a *regular partition* of K, i.e., any pair of simplices in Δ intersect at most at an l-dimensional simplex where $0 \leq l < d$ (for details see [28]).

Let $\mathcal{P}_1$ denote the linear space of all d-variate polynomials of total degree one. Set

$$S_1^0(\Delta) = \{s \in C(K) : s|_{K_i} \in \mathcal{P}_1, \ 1 \leq i \leq N\},$$

the space of *d-variate linear splines over* Δ.

In [28] we verified the following statement.

Theorem 5.2. *Let* $G = S_1^0(\Delta)$. *Then* G *satisfies Property A.*
Especially, if $N = 1$, *i.e.,* $K = K_1$, *a d-dimensional simplex, then* $G = \mathcal{P}_1$ *is an A-space (see also Thm. 5.1).*

Remark 6. In [22] we have already obtained this result for so-called uni-diagonal triangulations in $\mathbb{R}^2$. Pinkus [14] generalized it for the case of bivariate regular triangulations.

The next result, given in [7], [10], shows that tensor product A-spaces can exist only under rather restrictive assumptions.

Theorem 5.3. *Let* G *and* $\widetilde{G}$ *be finite-dimensional subspaces of* $C(I)$ *and* $C(J)$, *respectively, where* I *and* J *are real intervals. The tensor product space* $G \otimes \widetilde{G}$ *is an A-space if and only if* G *and* $\widetilde{G}$ *are A-spaces and* G *or* $\widetilde{G}$ *is a direct sum of one-dimensional A-spaces having pairwise disjoint supports.*

Remark 7. In the literature some further examples of A-spaces in $\mathbb{R}^d$, $d > 1$, are presented (see Pinkus [14]).

Thm. 5.1 states that the space of d-variate polynomials of total degree one satisfies Property A. Unfortunately, in view of Thm. 5.3, this nice property fails already for the case when G forms the tensor product of univariate linear polynomials on a rectangular in $\mathbb{R}^2$. However, also for such spaces there are interesting results if the question of L^1-uniqueness is studied only for special measures, for instance $w \equiv 1$.

Assume that $K = [0, 1] \times [0, 1]$, and let, for $k, m \in \mathbb{N}$, $k \leq m$, $\widetilde{G}_k$ and $\widetilde{G}_m$ denote Chebyshev subspaces of $C[0,1]$ of dimension k and m, respectively. Moreover, assume that $\widetilde{G}_k \subset \widetilde{G}_m$. Let $\phi \in C[0,1]$, an increasing function, and define the subspace G of $C(K)$ by

$$G = G_{m+k} = \{g(x,y) = \phi(y)g_k(x) + g_m(x) : \ g_k \in \widetilde{G}_k, \ g_m \in \widetilde{G}_m\}.$$

Kroó [2] obtained the following result.

Theorem 5.4. *G is a unicity subspace for* $C_1(K, \mu)$ *for the special measure* $d\mu = d\lambda$, *i.e.,* $w \equiv 1$.

Following the lines of Kroó's proof, in [22] we extended the statement for all weight functions with separated variables as follows.

Theorem 5.5. *G is a unicity subspace for* $C_1(K, \mu)$ *for all weights* $\mu \in W$ *of the type* $d\mu = wd\lambda$ *satisfying* $w(x,y) = w_1(x)w_2(y)$, $(x,y) \in K$.

Remark 8. (i) The most important example of G is the space

$$G = \mathcal{P}_{m,1} = \{g(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{1} a_{ij} x^i y^j : a_{ij} \in \mathbb{R}\},$$

the tensor product of polynomials of degree m in the first variable and degree one in the second one (obviously, $\phi(y) = y$, $y \in [0,1]$). Therefore, $\mathcal{P}_{m,1}$ is a unicity subspace for $C_1(K, \mu)$ for all weights with separated variables.
(ii) On the other hand, Thm. 5.3 implies that $\mathcal{P}_{m,1}$ fails to be an A-space. However, as the next section will show, such spaces of bivariate polynomials satisfy a Property A^m, a natural extension of Property A.

6   Chebyshev Rank and Property A^k

Consider again K as a compact subset of $\mathbb{R}^d$ $(d \geq 1)$ such that $K = \overline{\text{int } K}$. To characterize subspaces G of $C(K)$ having Chebyshev rank k, $0 \leq k \leq \dim G$, Kroó [5] gave a natural extension of Property A, the so-called Property A^k. It is also an intrinsic property of the considered linear space. To define it, we first need some notations. For $f \in C(K)$ and $g_0, \ldots, g_r \in G \subset C(K)$ $(r \geq 0)$, set

$$Z(g_0, \ldots, g_r) = \bigcap_{i=0}^{r} Z(g_i), \quad G(g_0, \ldots, g_r) = \{h \in G : h = 0 \text{ a.e. on } Z(g_0, \ldots, g_r)\}.$$

Definition 6.1. Let G be a linear subspace of $C(K)$ with $\dim G = n$. G is said to satisfy *Property* A^k (or G is an A^k-*space*), $0 \le k \le n-1$, if for every choice of $k+1$ linearly independent functions $g_0, \ldots, g_k$ in G and every function ψ such that $\psi = 0$ on $Z(g_0, \ldots, g_k)$, $|\psi| = 1$ and ψ is continuous on $K \setminus Z(g_0, \ldots, g_k)$, there exists a $g \in G(g_0, \ldots, g_k) \setminus \{0\}$ satisfying $\psi g \ge 0$ on K.

Of course, Property A^0 corresponds to Property A.

Kroó [5] proved the following statement generalizing Thm. 2.4.

Theorem 6.1. Let G be an n-dimensional subspace of $C(K)$, and let $k \in \{0, \ldots, n-1\}$. Then $cr(G) \le k$ if and only if G satisfies Property A^k.

It is obvious that Property A^k implies Property A^{k+1}. This leads to the following corollary.

Corollary 6.2. Let G be given as above and let $k \in \{1, \ldots, n-1\}$. Then $cr(G) = k$ if and only if G satisfies Property A^k and does not satisfy Property A^{k-1}.

In the same paper [5] Kroó gave some applications and examples of A^k-spaces. The following results (including Thm. 6.7) are all taken from that paper.

Theorem 6.3. Let G be given as above and assume that G contains an $(n-k)$-dimensional A-space. Then G is an A^k-*space*.

The next statement follows immediately.

Corollary 6.4. If $G \subset C[a,b]$ contains an $(n-k)$-dimensional Chebyshev subspace, then G is an A^k-space $(0 \le k \le n-1)$.

To give a result for spaces defined on disjoint real intervals, set $K_m = \bigcup_{j=1}^{m}[\alpha_j, \beta_j]$, where $m \in \mathbb{N}$ and $\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_m < \beta_m$, $\text{int } K_m = \bigcup_{j=1}^{m}(\alpha_j, \beta_j)$.

Theorem 6.5. Let G be an n-dimensional subspace of $C(K_m)$ and assume that G is Chebyshev on $\text{int } K_m$, where $1 \le m \le n$. Then G is an A^{m-1}-*space*.

Consider now the special case of lacunary polynomials with k "gaps". Assume that $1 \le r_1 < \cdots < r_k \le n-1$ be arbitrary integers $(1 \le k \le n-1)$. Set

$$\mathcal{P}(n,k) = \text{span}\left\{x^j : 0 \le j \le n, j \ne r_i, 1 \le i \le k\right\}.$$

Theorem 6.6. Let $G = \mathcal{P}(n,k)$ such that $1 \le k \le [n/2]$. Then G is an A^k-space on $[-1,1]$ and, in general, it does not satisfy Property A^{k-1}, i.e., $cr(G) = k$.

Let us finish this part with the following statement.

Theorem 6.7. Let G be an n-dimensional Chebyshev subspace of $C[a,b]$ and assume that $\phi \in C[a,b]$ has k distinct zeros $(1 \le k \le n-1)$ in (a,b). Then $\widetilde{G} = \phi G = \{\phi g : g \in G\}$ is an A^k-space and does not satisfy Property A^{k-1}, i.e., $cr(\widetilde{G}) = k$.

Using the statement of Thm. 6.1 we studied the problem of determining the Chebyshev rank of $G = \mathcal{P}_m$, the linear space of multivariate polynomials of total degree at most m defined on a convex and compact subset K of $\mathbb{R}^d$ $(d \ge 2)$ such that $K = \overline{\text{int } K}$. In [27] we verified the following statement.

Theorem 6.8. Let $G = \mathcal{P}_m$. Then

$$\text{cr}(\mathcal{P}_m) = \binom{m-2+d}{d} = \dim \mathcal{P}_{m-2}, \ m \ge 2.$$

If $m = 1$, then $cr(\mathcal{P}_1) = 0$ (this follows from Thm. 5.2).

We were also interested in determining the Chebyshev rank of bivariate polynomials

$$G = \mathcal{P}_{k,m} = \{g(x,y) = \sum_{i=0}^{k}\sum_{j=0}^{m} a_{ij}x^i y^j : a_{ij} \in \mathbb{R}\} \quad k,m \ge 1,$$

defined on $K = [0,1] \times [0,1] \subset \mathbb{R}^2$. In [24] we obtained the following result.

Theorem 6.9. Let $G = \mathcal{P}_{k,m}$. Then G satisfies Property A^{km} and fails to satisfy Property A^{km-1}, i.e.,

$$cr(G) = km.$$

Remark 9. Together with Thm. 5.5 the following is true for $G = \mathcal{P}_{k,1}$:

$$\text{cr}(G) = k, \text{ but } \text{cr}(G, \mu) = 0$$

for all weights of the type $d\mu = wd\lambda$ satisfying $w(x,y) = w_1(x)w_2(y)$, $(x,y) \in K$.

Remark 10. (i) In [28] we have given lower and upper bounds for the Chebyshev rank of $G = S_m^r(\Delta)$, the linear space of r times continuously differentiable, d-variate splines of degree $m \ge 1$ over regular partitions Δ, for the cases $d \ge 3$ and $0 \le r \le m-1$ (recall that Thm. 5.2 is related to the special case when $m = 1$ and $r = 0$ stating Property A for linear d-variate splines). For $d = 3$ and simple partitions we determined the Chebyshev rank of the corresponding splines while for general partitions we were not able to close the gap between the given lower and upper bounds.

(ii) We wish to mention that results on the Chebyshev rank have been also obtained by other authors, for instance by Babenko et al [1] for best L^1-approximation by classes of functions having finitely many points of discontinuity.

7   Characterization of Periodic Unicity Subspaces in Best L^1-Approximation

Let C_{b-a} denote the subspace of all continuous, $(b-a)$-periodic functions $f : \mathbb{R} \to \mathbb{R}$ where $a < b$, i.e.,

$$C_{b-a} = \{f \in C(\mathbb{R}) : f(x) = f(x + (b-a)), x \in \mathbb{R}\}.$$

In analogy to the non-periodic case we define the following: If G is a finite-dimensional subspace of C_{b-a} and $\mu \in W$ (W defined as in Section 1), then G is said to be a *periodic unicity space for* $C_1([a,b],\mu)$, if to each $f \in C_{b-a}$ there exists a unique best approximation from G on $[a,b]$ in the norm $\|\cdot\|_\mu$.

   It turns out that for this important class C_{b-a} the general statements on L^1-uniqueness, represented in the above sections, may not be applied. Even though one can identify continuous periodic functions on $\mathbb{R}$ with continuous functions on the unit sphere S in $\mathbb{R}^2$, the set S however - as a subset of $\mathbb{R}^2$ - does not satisfy the condition $S = \overline{\text{int } S}$.

   In two papers [25], [26] we have therefore studied the special problem of characterizing unicity subspaces of C_{b-a} in best L^1-approximation.

   As a first result, we observed that the statement of Thm. 2.4 holds analogously.

Theorem 7.1. *A finite-dimensional subspace G of C_{b-a} is a periodic unicity space for $C_1([a,b],\mu)$ for all $\mu \in W$ if and only if G (as a subspace of $C(S)$) satisfies Property A on S.*

   To illustrate the difference between G satisfying Property A on $[a,b]$ (which corresponds to the non-periodic case), and Property A on S (G considered as subspace of $C(S)$), we gave the following definition.

Definition 7.1. We say that the subspace G of C_{b-a} satisfies *Property* A_{per} if for every $g \in G \setminus \{0\}$ and every $g^* \in C_{b-a}$ such that $|g^*| = |g|$ on $[a,b]$, there exists a $\tilde{g} \in G(g) \setminus \{0\}$ for which $\tilde{g}g^* \geq 0$ on $[a,b]$.

   Property A_{per} is actually weaker than Property A on $K = [a,b]$ (see [25]).

   Thm. 7.1 can now be reformulated as follows.

Theorem 7.2. *A finite-dimensional subspace G of C_{b-a} is a periodic unicity space for $C_1([a,b],\mu)$ for all $\mu \in W$ if and only if G satisfies Property A_{per}.*

   Following the lines of the proof of Thm. 3.2 we verified an analogous statement in [25].

Theorem 7.3. *Suppose that the subspace G of C_{b-a} satisfies Property A_{per}. Then*

   (i) $[S \setminus Z(g)] \leq \dim G(g)$ for every $g \in G$;

   (ii) G decomposes, if $[S \setminus Z(G)] \geq 2$.

Remark 11. To characterize the subspaces G of C_{b-a} which satisfy Property A_{per}, on the basis of Thm. 7.3 one has only to treat the cases $Z(G) = \emptyset$ and $Z(G) \cap [a,b) = \{\tilde{x}\}$ (see [25]). Since in the latter case G obviously satisfies Property A on $\widetilde{K} = [\tilde{x}, \tilde{x} + b - a]$ for any $\tilde{x} \in Z(G)$, the general non-periodic case is given. Hence the statements of Section 3 (Thms. 3.3 and 3.4) may be applied for characterizing the subspace G of $C(\widetilde{K})$ with Property A. In particular, it follows that such a G has to have a spline-like structure.

   Thus, there still remains the most interesting case of the considered periodic approximation problem when $Z(G) = \emptyset$. In [25] we characterized the corresponding subspaces G of C_{b-a} which satisfy Property A_{per}.

Theorem 7.4. *Assume that G is an n-dimensional subspace of C_{b-a} satisfying $Z(G) = \emptyset$. The following statements (i) and (ii) are equivalent.*

   (i) G satisfies Property A_{per}.

   (ii)   (a) $[S \setminus Z(g)] \leq \dim G(g) = d(g)$ for every $g \in G$.
   (b) For every $g \in G \setminus \{0\}$ and every set $\{x_i\}_{i=1}^{m+1}$ of separated zeros of g satisfying

$$a \leq x_1 < \cdots < x_m \leq b \leq x_{m+1} = x_1 + b - a$$

   and $x_m - x_1 < b - a$ where $1 \leq m \leq d(g)$, there exists a $\tilde{g} \in G(g) \setminus \{0\}$ such that

$$(-1)^i \tilde{g}(x) \geq 0, \ x \in [x_i, x_{i+1}], \ 1 \leq i \leq m$$

   (the zeros are said to be separated if there exist points $y_i \in (x_i, x_{i+1})$, $1 \leq i \leq m$, for which $g(y_i) \neq 0$).

   The above stated conditions cannot be easily verified, in general. Therefore, the main objective of our paper [26] was to establish some simpler characterizations. Recall that in the non-periodic case the dimension of G (odd or even) does not play any role in characterizing Property A. The situation is completely different in the periodic case. In fact, we verified that if $\dim G = 2k + 1$ ($k \geq 0$) and G satisfies Property A_{per}, then G is necessarily a weak Chebyshev space. On the other hand, it is well-known that every subspace G of C_{b-a} of even dimension such that $Z(G) = \emptyset$ fails to have this property. To establish simpler intrinsic characterizations of Property A_{per}, in [26] we therefore considered the cases odd and even separately and obtained the following results.

   Assume in both cases that G is an n-dimensional subspace of C_{b-a} satisfying $Z(G) = \emptyset$.

Theorem 7.5. *Assume that $\dim G = n = 2k + 1$. The following conditions (i) and (ii) are equivalent.*

   (i) G satisfies Property A_{per}.

   (ii)   (a) G is a weak Chebyshev space on $[a,b]$.

(b) *If $g \in G$ and g has zero intervals $J_i = [c_i, d_i]$, $i = 1, 2$, such that $c_1 < d_1 < c_2 < d_2 < c_1 + b - a$, then*

$$G(J_1 \cup J_2) = G([c_1, d_2]) \oplus G([c_2, d_1 + b - a]).$$

Corollary 7.6. *Let G be given as above. Moreover, suppose that every $g \in G$ has at most one zero interval on S (of course, g could have two zero intervals $[a, d], [c, b]$ on $[a, b]$ with $d < c$ which can be identified with one zero interval $[c, d + b - a]$). Then G satisfies Property A_{per} if and only if G is weak Chebyshev on $[a, b]$.*

Theorem 7.7. *Assume that $\dim G = n = 2k$. The following conditions (i) and (ii) are equivalent.*

(i) *G satisfies Property A_{per}.*

(ii) (a) *Every $g \in G$ has at most n separated zeros on $[a, b)$.*
(b) *If $g \in G$ and g has a zero interval $[c, d]$ with $c < d$, then*

$$[[c, c + b - a] \setminus Z(g)] \leq \dim G([c, d]).$$

(c) *If $g \in G$ and g has zero intervals $J_i = [c_i, d_i]$, $i = 1, 2$, such that $c_1 < d_1 < c_2 < d_2 < c_1 + b - a$, then*

$$G(J_1 \cup J_2) = G([c_1, d_2]) \oplus G([c_2, d_1 + b - a]).$$

(d) *For every $g \in G \setminus \{0\}$ and every set $\{x_i\}_{i=1}^{m+1}$ of separated zeros of g satisfying*

$$a \leq x_1 < \cdots < x_m \leq b \leq x_{m+1} = x_1 + b - a$$

and $x_m - x_1 < b - a$ where $1 \leq m \leq n$, there exists a $\tilde{g} \in G \setminus \{0\}$ such that

$$(-1)^i \tilde{g}(x) \geq 0, \ x \in [x_i, x_{i+1}], \ 1 \leq i \leq m.$$

Remark 12. In [26] we showed by examples that the conditions (ii) (b) of Thm. 7.5 and Thm. 7.7 and condition (ii) (d) of Thm. 7.7 may not be omitted.

Moreover, in [25] we verified Property A_{per} for some important subspaces of periodic functions.

Example 7.1. (Trigonometric polynomials) Let $K = [0, 2\pi]$ and assume that G denotes the $(2n+1)$-dimensional subspace of all trigonometric polynomials g of order n, i.e.,

$$g(x) = a_0 + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx), \ x \in [0, 2\pi].$$

Then G satisfies Property A_{per}.

Example 7.2. (Piecing together Haar systems) Let $a = e_0 < e_1 < \cdots < e_{k+1} = b$. On each interval $I_i = [e_{i-1}, e_i]$, let G_i be a Haar system of real-valued continuous functions with dimension $n_i \geq 1$, $1 \leq i \leq k + 1$. For convenience, we especially assume that $n_1 \geq 2$ and $n_{k+1} \geq 2$. Define the subspace G of piecing together Haar systems $C[a, b]$ by

$$G = \{g \in C[a, b] : g|_{I_i} \in G_i, \ 1 \leq i \leq k + 1\}.$$

It is well-known (see [14]) that $\dim G = \sum_{i=1}^{k+1} n_i - k$ and G satisfies Property A on $[a, b]$.
To investigate its periodic analogue we consider the subspace $\widetilde{G}$ of C_{b-a} defined by

$$\widetilde{G} = \{g \in C_{b-a} : g|_{I_i} \in G_i, \ 1 \leq i \leq k + 1\}.$$

Thus, $\widetilde{G}$ is the space of all periodic extensions of functions $g \in G$ such that $g(a) = g(b)$. It follows that $\dim \widetilde{G} = \dim G - 1$. Moreover, we verified that $\widetilde{G}$ satisfies Property A_{per}.

Example 7.3. (Periodic splines) Given $k \geq 0$ and $l \geq 1$, let $a = e_0 < e_1 < \cdots < e_{k+1} = b$. Extend this knot vector to a knot sequence on $\mathbb{R}$ by

$$e_{i+j(k+1)} = e_i + j(b - a), \ 0 \leq i \leq k + 1, \ j \in \mathbb{Z} \setminus \{0\}.$$

Set $\Delta = \{e_i\}_{i \in \mathbb{Z}}$ and $I_i = [e_{i-1}, e_i]$, $i \in \mathbb{Z}$. Let $\mathcal{P}_l$ denote the linear space of all polynomials of degree at most l. For any $q \in \{1, \ldots, l\}$ consider the linear space $S_l^{l-q}(\Delta)$ defined by

$$S_l^{l-q}(\Delta) = \{s \in C^{l-q}(\mathbb{R}) : s|_{I_i} \in \mathcal{P}_l, i \in \mathbb{Z}\},$$

the subspace of polynomial spline functions of degree l with the fixed knots $\{e_i\}_{i \in \mathbb{Z}}$ of multiplicity q. It is well-known (see [17]) that $\dim S_l^{l-q}(\Delta)|_{[a,b]} = l + 1 + qk$. Moreover, this space satisfies Property A (see [14]).
We are now interested in the subspace

$$G = S_l^{l-q}(\Delta) \cap C_{b-a},$$

the subspace of periodic splines of degree l with the fixed knots $\{e_i\}_{i \in \mathbb{Z}}$ of multiplicity q. It is easily verified that

$$\dim G = l + 1 + qk - (l - q + 1) = q(k + 1).$$

Moreover, we could show that G satisfies Property A_{per}.

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