Dolomites Research Notes on Approximation
Special issue dedicated to Andràs Kroó on the occasion of his 65th birthday, Volume 12 · 2019 · Pages 10–15

A Note on Orthogonal Dirichlet Polynomials with Rational Weight[1]

S. Doron Lubinsky[a]

Communicated by S. De Marchi

Abstract

Let $\{\lambda_j\}_{j=1}^{\infty}$ be a strictly increasing sequence of positive numbers with $\lambda_1 > 0$. We find an explicit formula for the orthogonal Dirichlet polynomials $\{\phi_n\}$ formed from linear combinations of $\left\{\lambda_j^{-it}\right\}_{j=1}^{n}$, associated with rational weights

$$w(t) = \sum_{j=1}^{L} \frac{c_j}{\pi\left(1+\left(b_j t\right)^2\right)},$$

where $0 < b_1 < b_2 < \ldots$, and the $\{c_j\}$ are appropriately chosen. Only $\left\{\lambda_j^{-it}\right\}_{j=n-L}^{n}$ appear in the formula. In the case $L = 2$, we show that the weight can always be taken positive in $\mathbb{R}$.

Keywords: Dirichlet polynomials, orthogonal polynomials.
AMS Subject Class 2010: 42C05

1 Introduction

Throughout, let

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots . \tag{1}$$

Let $\mathcal{L}_n$ denote the set of Dirichlet polynomials

$$\sum_{j=1}^{n} c_j \lambda_j^{-it}$$

with complex coefficients $\{c_j\}$.

In a 2014 paper [5], we showed that

$$\phi_n(t) = \frac{\lambda_n^{1-it} - \lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} = \frac{-1}{\sqrt{\lambda_{n-1}^{-2} - \lambda_n^{-2}}} \det\begin{bmatrix} \lambda_{n-1}^{-it} & \lambda_n^{-it} \\ \lambda_{n-1}^{-1} & \lambda_n^{-1} \end{bmatrix}$$

is the nth orthogonal Dirichlet polynomial for the arctan density, that is

$$\int_{-\infty}^{\infty} \phi_n(t)\overline{\phi_m(t)} \frac{dt}{\pi(1+t^2)} = \delta_{mn}, \; m, n \geq 1. \tag{2}$$

We also estimated the Christoffel functions, convergence of associated orthonormal expansions, and universality limits. These orthonormal polynomials have been applied and provided in a variety of questions by Weber and Dimitrov as well as the author [4], [6], [8], [10], [11], [12]. In a follow up paper [7], the author considered orthogonal Dirichlet polynomials for the Laguerre weight, though it turned out that much of the material there was already subsumed by Műntz orthogonal polynomials [3].

In this note, we consider rational densities

$$w(t) = \sum_{m=1}^{L} \frac{c_m}{\pi\left(1+\left(b_m t\right)^2\right)} \tag{3}$$

with appropriately chosen $\{c_j\}$. Here $L \geq 1$, and

$$1 = b_1 < b_2 < \ldots < b_L . \tag{4}$$

[1]Research supported by NSF grant DMS1800251
[a]School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160 (USA). email: lubinsky@math.gatech.edu

Define, for $n \geq L$,

$$\psi_n(t) = \det \begin{bmatrix} \lambda_{n-L}^{-it} & \lambda_{n-L+1}^{-it} & \cdots & \lambda_{n-1}^{-it} & \lambda_n^{-it} \\ \lambda_{n-L}^{-1/b_1} & \lambda_{n-L+1}^{-1/b_1} & \cdots & \lambda_{n-1}^{-1/b_1} & \lambda_n^{-1/b_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-L}^{-1/b_{L-1}} & \lambda_{n-L+1}^{-1/b_{L-1}} & \cdots & \lambda_{n-1}^{-1/b_{L-1}} & \lambda_n^{-1/b_{L-1}} \\ \lambda_{n-L}^{-1/b_L} & \lambda_{n-L+1}^{-1/b_L} & \cdots & \lambda_{n-1}^{-1/b_L} & \lambda_n^{-1/b_L} \end{bmatrix}. \tag{5}$$

Observe that $\psi_n(t)$ is a linear combination of only $\left\{\lambda_j^{-it}\right\}_{n-L \leq j \leq n}$. Also define for a given fixed n, and $j \geq 1$, $1 \leq m \leq L$,

$$d_{jm} = \int_{-\infty}^{\infty} \psi_n(t) \frac{\lambda_j^{it}}{\pi\left(1+(b_m t)^2\right)}\, dt \tag{6}$$

and let B be the $(L-1) \times L$ matrix

$$B = \begin{bmatrix} d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\ d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \end{bmatrix} \tag{7}$$

and

$$D = \det \begin{bmatrix} d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\ d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \\ d_{n,1} & d_{n,2} & \cdots & d_{n,L} \end{bmatrix}. \tag{8}$$

Theorem 1
Let $n \geq L \geq 1$. Let $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$ and ψ_n be given by (5).
(a) Let $\mathbf{c} = [c_1\ c_2 \ldots c_L]^T$ be taken as any non-trivial solution of $B\mathbf{c} = \mathbf{0}$. Let

$$w(t) = \sum_{m=1}^{L} \frac{c_m}{\pi\left(1+(b_m t)^2\right)}. \tag{9}$$

Then for $1 \leq j \leq n-1$,

$$\int_{-\infty}^{\infty} \psi_n(t)\, \lambda_j^{it} w(t)\, dt = 0. \tag{10}$$

(b) If D defined by (8) is non-0, then we can take

$$w(t) = A \det \begin{bmatrix} d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\ d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \\ \frac{1}{\pi\left(1+(b_1 t)^2\right)} & \frac{1}{\pi\left(1+(b_2 t)^2\right)} & \cdots & \frac{1}{\pi\left(1+(b_L t)^2\right)} \end{bmatrix}, \tag{11}$$

for any $A \neq 0$, while

$$\int_{-\infty}^{\infty} \psi_n(t)\, \lambda_n^{it} w(t)\, dt = AD. \tag{12}$$

(c)

$$\psi_n(t) = \sum_{j=n-L}^{n} \alpha_j \lambda_j^{-it} \tag{13}$$

where for $n-L \leq j \leq n$,

$$\alpha_j (-1)^{j-n+L} > 0. \tag{14}$$

Remarks
(a) Note that as $\left\{\frac{1}{\pi\left(1+(b_m t)^2\right)}\right\}_{m=1}^{L}$ are linearly independent, w above is not identically 0. As an even rational function with numerator degree at most $2L-2$ and denominator degree $2L$, w has at most $L-1$ sign changes in $(0,\infty)$. It seems to be an interesting problem to investigate the positivity of w.
(b) In addition to the orthogonality relation above, we note that for any $1 \leq m \leq L$, and $0 < \lambda \leq \lambda_{n-L}$,

$$\int_{-\infty}^{\infty} \psi_n(t) \frac{\lambda^{it}}{\pi\left(1+(b_m t)^2\right)}\, dt = 0.$$

This does not require anything of the $\{c_j\}$ above.

In the case $L=2$, we can prove positivity of the weight:

Theorem 2
Assume the notation of Theorem 1 with $L=2$. Then we can choose $c_1<0<c_2$ such that if

$$w(t)=\sum_{k=1}^{2}\frac{c_k}{\pi\left(1+(b_kt)^2\right)}$$

then

$$w(t)>0,\ t\in\mathbb{R},$$

and w is given by the determinant (11), with

$$A=\frac{c_2}{d_{n-1,1}}<0.$$

Remark
In the proof of Theorem 2, we show that one can take

$$c_1=-c_2\frac{g\left(\frac{1}{b_2}\right)}{g\left(\frac{1}{b_1}\right)}$$

where

$$g(s)=s\left[\left(\frac{\lambda_{n-2}}{\lambda_{n-1}}\right)^{s}-\left(\frac{\lambda_{n-2}}{\lambda_{n-1}}\right)^{-s}\right].$$

We prove the theorems in the next section.

2 Proofs

Proof of Theorem 1
(a) We use the following simple consequence of the residue theorem: for real μ,

$$\int_{-\infty}^{\infty}\frac{e^{i\mu t}}{\pi(1+t^2)}dt=e^{-|\mu|}. \tag{15}$$

Then if $0<\lambda\le\lambda_{n-L}$, and $n-L\le k\le n$,

$$\int_{-\infty}^{\infty}\frac{(\lambda/\lambda_k)^{it}}{\pi\left(1+(b_mt)^2\right)}dt=\frac{1}{b_m}\int_{-\infty}^{\infty}\frac{e^{isb_m^{-1}\log(\lambda/\lambda_k)}}{\pi(1+s^2)}ds=\frac{1}{b_m}\left(\frac{\lambda}{\lambda_k}\right)^{1/b_m}.$$

Then for such λ,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\psi_n(t)\frac{\lambda^{it}}{\pi\left(1+(b_mt)^2\right)}dt\\
=&\det\begin{bmatrix}\int_{-\infty}^{\infty}\frac{(\lambda/\lambda_{n-L})^{it}}{\pi(1+(b_mt)^2)}dt & \cdots & \int_{-\infty}^{\infty}\frac{(\lambda/\lambda_{n-1})^{it}}{\pi(1+(b_mt)^2)}dt & \int_{-\infty}^{\infty}\frac{(\lambda/\lambda_{n})^{it}}{\pi(1+(b_mt)^2)}dt\\ \lambda_{n-L}^{-1/b_1} & \cdots & \lambda_{n-1}^{-1/b_1} & \lambda_n^{-1/b_1}\\ \vdots & \ddots & \vdots & \vdots\\ \lambda_{n-L}^{-1/b_L} & \cdots & \lambda_{n-1}^{-1/b_L} & \lambda_n^{-1/b_L}\end{bmatrix}\\
=&\det\begin{bmatrix}\frac{1}{b_m}\left(\frac{\lambda}{\lambda_{n-L}}\right)^{1/b_m} & \cdots & \frac{1}{b_m}\left(\frac{\lambda}{\lambda_{n-1}}\right)^{1/b_m} & \frac{1}{b_m}\left(\frac{\lambda}{\lambda_{n}}\right)^{1/b_m}\\ \lambda_{n-L}^{-1/b_1} & \cdots & \lambda_{n-1}^{-1/b_1} & \lambda_n^{-1/b_1}\\ \vdots & \ddots & \vdots & \vdots\\ \lambda_{n-L}^{-1/b_L} & \cdots & \lambda_{n-1}^{-1/b_L} & \lambda_n^{-1/b_L}\end{bmatrix}=0,
\end{aligned}$$

by taking $\frac{1}{b_m}\lambda^{1/b_m}$ times row $m+1$ from the first row. So we have the orthogonality relation (10) for $\lambda=\lambda_j$, all $j\le n-L$. Next, the equations

$$\int_{-\infty}^{\infty}\psi_n(t)\lambda_{n-L+j}^{it}w(t)dt=0,\ 1\le j\le L-1$$

are equivalent to (recall (3) and (6))

$$\sum_{m=1}^{L}c_md_{n-L+j,m}=\sum_{m=1}^{L}c_m\int_{-\infty}^{\infty}\psi_n(t)\frac{\lambda_{n-L+j}^{it}}{\pi\left(1+(b_mt)^2\right)}dt=0,\ 1\le j\le L-1$$

which in turn is equivalent to $B\mathbf{c} = \mathbf{0}$, recall (7). This is a system of $L-1$ homogeneous linear equations in L variables, so there is a non-trivial solution for $\mathbf{c}$.
(b) First observe that w defined by (11) is indeed a linear combination of $\left\{\frac{1}{\pi\left(1+(b_m t)^2\right)}\right\}_{m=1}^{L}$. Next, we see from (11) that

$$\int_{-\infty}^{\infty} \psi_n(t)\lambda_k^{it} w(t)\,dt = A\det\begin{bmatrix} d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\ d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \\ d_{k,1} & d_{k,2} & \cdots & d_{k,L} \end{bmatrix} = 0,$$

if $n-L+1 \le k \le n-1$. If $k = n$, we instead obtain the non-0 number AD. It also then follows that w cannot be the zero function.
(c) Let E be the $L \times (L+1)$ matrix

$$E = \begin{bmatrix} \lambda_{n-L}^{-1/b_1} & \lambda_{n-L+1}^{-1/b_1} & \cdots & \lambda_{n-1}^{-1/b_1} & \lambda_n^{-1/b_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-L}^{-1/b_{L-1}} & \lambda_{n-L+1}^{-1/b_{L-1}} & \cdots & \lambda_{n-1}^{-1/b_{L-1}} & \lambda_n^{-1/b_{L-1}} \\ \lambda_{n-L}^{-1/b_L} & \lambda_{n-L+1}^{-1/b_L} & \cdots & \lambda_{n-1}^{-1/b_L} & \lambda_n^{-1/b_L} \end{bmatrix}.$$

Thus E consists of the last L rows of the matrix used to define ψ_n. For $1 \le k \le L+1$, let $E(k)$ denote the $L \times L$ matrix obtained from E by deleting its kth column. Then with the notation (13), we see that

$$\alpha_j = (-1)^{j-n+L}\det(E(j-n+L+1)).$$

To show that each $\det(E(k)) > 0$, we use the fact that the kernel $K(s,t) = e^{st}$ is totally positive for $s,t \in \mathbb{R}$ [1, p. 212] or [9]. If we set $s_j = -\frac{1}{b_j}$, while $t_i = \log\lambda_{n-L+i-1}$, then $s_1 < s_2 < \ldots s_L$ and $t_1 < t_2 < \ldots < t_L$, then

$$\det(E(k)) = \det\begin{bmatrix} K(s_1,t_1) & \ldots & K(s_1,t_{k-1}) & K(s_1,t_{k+1}) & \ldots & K(s_1,t_{L+1}) \\ K(s_2,t_1) & \ldots & K(s_2,t_{k-1}) & K(s_2,t_{k+1}) & \ldots & K(s_2,t_{L+1}) \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ K(s_L,t_1) & \ldots & K(s_L,t_{k-1}) & K(s_L,t_{k+1}) & \ldots & K(s_L,t_{L+1}) \end{bmatrix} > 0.$$

■

Proof of Theorem 2
From (5) for $L = 2$,

$$\psi_n(t) = \det\begin{bmatrix} \lambda_{n-2}^{-it} & \lambda_{n-1}^{-it} & \lambda_n^{-it} \\ \lambda_{n-2}^{-1/b_1} & \lambda_{n-1}^{-1/b_1} & \lambda_n^{-1/b_1} \\ \lambda_{n-2}^{-1/b_2} & \lambda_{n-1}^{-1/b_2} & \lambda_n^{-1/b_2} \end{bmatrix}. \tag{16}$$

Let

$$w(t) = \sum_{k=1}^{2} \frac{c_k}{\pi\left(1+(b_k t)^2\right)}$$

where for the moment we do not specify the choice of c_1, c_2. Then we already have for $k = 1, 2, \ldots, n-2$,

$$\int_{-\infty}^{\infty} \psi_n(t)\lambda_k^{it} w(t)\,dt = 0$$

no matter what is the choice of c_1, c_2 - as follows from the proof of Theorem 1(a). So let us investigate the remaining condition in (10), namely

$$\int_{-\infty}^{\infty} \psi_n(t)\lambda_{n-1}^{-it} w(t)\,dt = 0.$$

This is equivalent to

$$0 = \sum_{k=1}^{2} c_k \int_{-\infty}^{\infty} \psi_n(t)\lambda_{n-1}^{it} \frac{dt}{\pi\left(1+(b_k t)^2\right)} = c_1 d_{n-1,1} + c_2 d_{n-1,2}. \tag{17}$$

Now for $k=1,2$, we see from the determinant expression (16) and then from (15) that

$$\begin{aligned}
d_{n-1,k} &= \frac{1}{b_k}\det\begin{bmatrix} \int_{-\infty}^{\infty}\left(\frac{\lambda_{n-1}}{\lambda_{n-2}}\right)^{is/b_k}\frac{ds}{\pi\left(1+s^2\right)} & 1 & \int_{-\infty}^{\infty}\left(\frac{\lambda_{n-1}}{\lambda_{n}}\right)^{is/b_k}\frac{ds}{\pi\left(1+s^2\right)} \\ \lambda_{n-2}^{-1/b_1} & \lambda_{n-1}^{-1/b_1} & \lambda_{n}^{-1/b_1} \\ \lambda_{n-2}^{-1/b_2} & \lambda_{n-1}^{-1/b_2} & \lambda_{n}^{-1/b_2} \end{bmatrix} \\
&= \frac{1}{b_k}\det\begin{bmatrix} \left(\frac{\lambda_{n-2}}{\lambda_{n-1}}\right)^{1/b_k} & 1 & \left(\frac{\lambda_{n-1}}{\lambda_{n}}\right)^{1/b_k} \\ \lambda_{n-2}^{-1/b_1} & \lambda_{n-1}^{-1/b_1} & \lambda_{n}^{-1/b_1} \\ \lambda_{n-2}^{-1/b_2} & \lambda_{n-1}^{-1/b_2} & \lambda_{n}^{-1/b_2} \end{bmatrix} \\
&= \frac{1}{b_k}\lambda_{n-1}^{1/b_k}\det\begin{bmatrix} \left(\frac{\lambda_{n-2}}{\lambda_{n-1}^2}\right)^{1/b_k} & \lambda_{n-1}^{-1/b_k} & \lambda_{n}^{-1/b_k} \\ \lambda_{n-2}^{-1/b_1} & \lambda_{n-1}^{-1/b_1} & \lambda_{n}^{-1/b_1} \\ \lambda_{n-2}^{-1/b_2} & \lambda_{n-1}^{-1/b_2} & \lambda_{n}^{-1/b_2} \end{bmatrix} \\
&= \frac{1}{b_k}\lambda_{n-1}^{1/b_k}\det\begin{bmatrix} \left(\frac{\lambda_{n-2}}{\lambda_{n-1}^2}\right)^{1/b_k}-\lambda_{n-2}^{-1/b_k} & 0 & 0 \\ \lambda_{n-2}^{-1/b_1} & \lambda_{n-1}^{-1/b_1} & \lambda_{n}^{-1/b_1} \\ \lambda_{n-2}^{-1/b_2} & \lambda_{n-1}^{-1/b_2} & \lambda_{n}^{-1/b_2} \end{bmatrix} \\
&= \frac{1}{b_k}\left[\left(\frac{\lambda_{n-2}}{\lambda_{n-1}}\right)^{1/b_k}-\left(\frac{\lambda_{n-1}}{\lambda_{n-2}}\right)^{1/b_k}\right]\left[\lambda_{n-1}^{-1/b_1}\lambda_{n}^{-1/b_2}-\lambda_{n}^{-1/b_1}\lambda_{n-1}^{-1/b_2}\right]<0,
\end{aligned} \tag{18}$$

as $\frac{\lambda_{n-2}}{\lambda_{n-1}}\in(0,1)$, $\frac{1}{b_1}-\frac{1}{b_2}>0$, and

$$\begin{aligned}
&\lambda_{n-1}^{-1/b_1}\lambda_{n}^{-1/b_2}-\lambda_{n}^{-1/b_1}\lambda_{n-1}^{-1/b_2} \\
=\;&\lambda_{n-1}^{-1/b_1}\lambda_{n}^{-1/b_2}\left[1-\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{\frac{1}{b_1}-\frac{1}{b_2}}\right]>0.
\end{aligned}$$

In summary,

$$d_{n-1,k}<0,\ k=1,2. \tag{19}$$

Next, let $r=\frac{\lambda_{n-2}}{\lambda_{n-1}}\in(0,1)$, and

$$g(s)=s\left[r^s-r^{-s}\right].$$

From (18) and (17) and cancelling a common factor of $\lambda_{n-1}^{-1/b_1}\lambda_{n}^{-1/b_2}-\lambda_{n}^{-1/b_1}\lambda_{n-1}^{-1/b_2}$, we have

$$c_1 g\left(\frac{1}{b_1}\right)+c_2 g\left(\frac{1}{b_2}\right)=0. \tag{20}$$

Here

$$g'(s)=\left(r^s-r^{-s}\right)+(s\ln r)\left(r^s+r^{-s}\right)<0,$$

as $r=\frac{\lambda_{n-2}}{\lambda_{n-1}}<1$ so $\ln r<0$. Then g is decreasing and negative, and

$$0>g\left(\frac{1}{b_2}\right)>g\left(\frac{1}{b_1}\right)$$

so (20) gives

$$c_1=-c_2\frac{g\left(\frac{1}{b_2}\right)}{g\left(\frac{1}{b_1}\right)} \text{ and } |c_1|<|c_2|. \tag{21}$$

To ensure that $w(0)=\frac{1}{\pi}(c_1+c_2)>0$, we then need to choose $c_1<0<c_2$. To ensure that $w(t)>0$ for all t, we need for all such t.

$$|c_1|\le c_2\frac{1+(b_1t)^2}{1+(b_2t)^2}.$$

As

$$\min_{t\in\mathbb{R}}\frac{1+(b_1t)^2}{1+(b_2t)^2}=\left(\frac{b_1}{b_2}\right)^2,$$

this is equivalent to

$$\frac{g\left(\frac{1}{b_2}\right)}{g\left(\frac{1}{b_1}\right)}\le\left(\frac{b_1}{b_2}\right)^2,$$

that is, (recall $g<0$),

$$b_2\left[r^{-1/b_2}-r^{1/b_2}\right]\leq b_1\left[r^{-1/b_1}-r^{1/b_1}\right].$$

Now let

$$h(s)=\frac{1}{s}\left[r^{-s}-r^{s}\right],$$

so that we want

$$h\left(\frac{1}{b_2}\right)\leq h\left(\frac{1}{b_1}\right). \tag{22}$$

This would be true if h is increasing over the range $\left[\frac{1}{b_2},\frac{1}{b_1}\right]$. Now

$$\begin{aligned} h'(s) &= -\frac{1}{s^2}\left[r^{-s}-r^{s}\right]-\frac{1}{s}(\ln r)\left[r^{-s}+r^{s}\right] \qquad (23)\\ &= -\frac{r^{-s}}{s^2}\left[1-r^{2s}+\frac{1}{2}\left(\ln r^{2s}\right)\left[1+r^{2s}\right]\right]=-\frac{r^{-s}}{s^2}G(x)\end{aligned}$$

where

$$x(s)=r^{2s}\in(0,1) \text{ decreases as } s \text{ increases}$$

and

$$G(x)=1-x+\frac{1}{2}(\ln x)(1+x).$$

Here $G(0+)=-\infty$ and $G(1)=0$ while for $x\in(0,1)$,

$$\begin{aligned} G'(x) &= -\frac{1}{2}+\frac{1}{2x}+\frac{1}{2}\ln x\\ &\Rightarrow\quad G''(x)=\frac{1}{2x}\left(1-\frac{1}{x}\right)<0.\end{aligned}$$

Thus G is concave in $(0,1)$ and G' is a decreasing function of x with $G'(0+)=\infty$ and $G'(1)=0=G(1)$. It follows that $G'(x)>0$ for $x\in(0,1)$, so

$$G(x)<G(1)=0 \text{ for } x\in(0,1).$$

So, indeed,

$$h'(s)=-\frac{r^{-s}}{s^2}G(x)>0 \text{ for } s>0,$$

and as desired, we have (22). Then with c_1 and c_2 given by (21), and $c_2>0$, we do have

$$w(t)>0,\ t\in(-\infty,\infty).$$

It remains to show that this w is also given by (11) with $L=2$. We know that c_1,c_2 are non-0 so

$$\begin{aligned} &\det\begin{bmatrix} d_{n-1,1} & d_{n-1,2}\\ \frac{1}{\pi\left(1+(b_1t)^2\right)} & \frac{1}{\pi\left(1+(b_2t)^2\right)}\end{bmatrix}\\ =\ &\det\begin{bmatrix} d_{n-1,1} & d_{n-1,2}+\frac{c_1}{c_2}d_{n-1,1}\\ \frac{1}{\pi\left(1+(b_1t)^2\right)} & \frac{1}{\pi\left(1+(b_2t)^2\right)}+\frac{c_1}{c_2}\frac{1}{\pi\left(1+(b_1t)^2\right)}\end{bmatrix}\\ =\ &\det\begin{bmatrix} d_{n-1,1} & 0\\ \frac{1}{\pi\left(1+(b_1t)^2\right)} & \frac{1}{c_2}w(t)\end{bmatrix}\\ =\ &\frac{d_{n-1,1}}{c_2}w(t).\end{aligned}$$

Thus the determinant is of one sign. Choosing $A=\frac{c_2}{d_{n-1,1}}<0$ gives the result. ■

References

[1] T. Ando, *Totally Positive Matrices*, Linear Algebra and its Applications, 90(1987), 165-219.

[2] H. Bohr, *Almost Periodic Functions*, Chelsea, New York, 1947.

[3] P. Borwein and T. Erdelyi, *Polynomials and Polynomial Inequalities*, Springer, New York, 1995.

[4] D.K. Dimitrov and W.D. Oliveira, *An Extremal Problem Related to Generalizations of the Nyman-Beurling and Baez-Duarte Criteria*, manuscript.

[5] D.S. Lubinsky, *Orthogonal Dirichlet Polynomials with Arctan Density*, J. Approx. Theory, 177(2014), 43-56.

[6] D.S. Lubinsky, *Uniform Mean Value Estimates and Discrete Hilbert Inequalties via Orthogonal Dirichlet Series*, Acta Math Hungarica, 143(2014), 422-438.

[7] D.S. Lubinsky, *Orthogonal Dirichlet Polynomials with Laguerre Weight*, J. Approx. Theory, 194(2015), 146-156.

[8] W.D. Oliveira, *Zeros of Dirichlet Polynomials via a Density Criterion*, Journal of Number Theory, to appear.

[9] A. Pinkus, *Totally Positive Matrices*, Cambridge Tracts in Mathematics, Vol. 181, Cambridge University Press, Cambridge, 2009.

[10] M. Weber, *Dirichlet polynomials: some old and recent results, and their interplay in number theory*, (in) Dependence in probability, analysis and number theory, (2010), Kendrick Press, Heber City, UT, pp. 323–353.

[11] M. Weber, *On mean values of Dirichlet polynomials*, Math. Inequal. Appl. 14 (2011), 529–534.

[12] M. Weber, *Cauchy Means of Dirichlet Polynomials*, J. Approx. Theory, 204(2016), 61-79.