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# Analytic properties of telescoping series derived from the zeros of the polynomial components

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**Abstract**

Telescoping polynomial series with specified restrictions on the zeros of the polynomial components turn out to be entire functions. Applied to polynomial $L_p$-approximation, $1 < p \leq \infty$, on a compact set $E$, we obtain a converse theorem based only on the location of the zeros of the difference of consecutive polynomials and the asymptotic behavior of the zeros of the polynomials. In contrast to the Bernstein-Walsh theorem, no information about the asymptotic behavior of the error of approximation is needed.

**Key words**: polynomial approximation, telescoping series, analytic continuation, Bernstein-Walsh theorem
**MSC**: 41A10, 30B40, 30E10

## 1 Introduction

Let $E$ be a compact set in $\mathbb{C}$ with connected complement $\Omega := \overline{\mathbb{C}} \backslash E$ and positive capacity and let us denote by $\mathcal{P}_n$ the collection of algebraic polynomials of degree at most $n$.

If $f$ is an entire function, then there exist polynomials $p_n \in \mathcal{P}_n$ such that

$$\limsup_{n\to\infty} ||f - p_n||_E^{1/n} = 0, \tag{1}$$

where $||.||_E$ denotes the maximum norm on $E$. For example, we can choose $p_n$ as the best polynomial approximation to $f$ on $E$ with respect to $\mathcal{P}_n$, $n \in \mathbb{N}$. Or we can construct interpolating polynomials to $f$ on a scheme

$$Z_n : z_{n,0}, z_{n,1}, \ldots, z_{n,n} \in \partial E$$

such that the normalized counting measures $\nu_n$ of $Z_n$ converge weakly to the equilibrium measure $\mu$ of $E$. In order to achieve this, Leja [4] proposed an iterative scheme

$$Z_n = \{z_0, z_1, \ldots, z_n\}, n \in \mathbb{N},$$

where

$$Z_{n+1} = Z_n \cup \{z_{n+1}\}$$

and the point $z_{n+1}$ is chosen in such a way that

$$\prod_{i=0}^{n} |z_{n+1} - z_i| = \max_{z\in E} \prod_{i=0}^{n} |z - z_i|.$$

(The starting point $z_0$ is an arbitrary point of $E$).

In the case that $\{p_n\}_{n\in\mathbb{N}}$ is the sequence of best polynomial approximations of a real-valued function $f$ with respect to the maximum norm on a compact set $E \subset \mathbb{R}$, $p_{n-1}$ is characterized by $n+1$ points of alternation of the error function $f - p_{n-1}$. Hence,

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the difference $p_n - p_{n-1}$ has exactly $n$ different zeros, not necessarily in $E$ itself, but in the convex hull of $E$ if $p_n \neq p_{n-1}$. If $p_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, are the Leja interpolation polynomials on $Z_n = \{z_0, z_1, ..., z_n\}$ to $f$ again the difference $p_n - p_{n-1}$ has exactly $n$ different zeros on $E$, namely

$$(p_n - p_{n-1})(z_j) = 0 \text{ for } z_j \in Z_{n-1}.$$

If $\{p_n\}_{n\in\mathbb{N}}$ is a sequence of polynomials such that the asymptotic behavior (1) holds, then $f$ can be continued to an entire function. This converse result was proved by Bernstein and Walsh via the telescoping series

$$p_0 + \sum_{n=1}^{\infty}(p_n - p_{n-1})$$

and their Bernstein-Walsh lemma.

In this paper we want to obtain a converse theorem by using

(a) the location of the zeros of $p_n - p_{n-1}$, and
(b) a "concentration point" of the zeros of $p_n$.

The approach is stimulated by papers of Lungu ([5],[6]), who proved converse results for diagonal rational Chebyshev approximation on the interval $[-1, 1]$ and Padé approximation using restrictions for the location of the poles of the rational approximants. Generalisations of the Lungu results to the non-diagonal case were investigated by E. Nguyen in her dissertation [7].

## 2 A converse theorem

Let $E$ be a compact set in $\mathbb{C}$ with connected complement $\Omega = \overline{\mathbb{C}} \setminus E$ and positive capacity cap $E$, and let

$$f = p_0 + \sum_{n=1}^{\infty}(p_n - p_{n-1}) \tag{2}$$

be a telescoping polynomial series, $p_n \in \mathcal{P}_n$.

**Notations**: Let $\mathcal{N}_n$ be the set of zeros of $p_n$, the zeros listed according to their multiplicity, then we denote by $\mathcal{N}$ the set of limit points of $\{\mathcal{N}_n\}_{n=1}^{\infty}$, i.e.,

$$\mathcal{N} = \left\{\zeta \in \overline{\mathbb{C}} : \text{ there exists a subsequence } \Lambda \subset \mathbb{N} \text{ and } \zeta_n \in \mathcal{N}_n, n \in \Lambda, \text{ such that } \lim_{n\in\Lambda, n\to\infty} \zeta_n = \zeta\right\}.$$

For a fixed $a \in \overline{\mathbb{C}}$ and $\varepsilon > 0$, let us define the disks

$$K(a,\varepsilon) := \begin{cases} \{z \in \mathbb{C} : \quad |z - a| \leq \varepsilon\}, & a \in \mathbb{C}, \\ \{z \in \mathbb{C} : \quad |z| \geq 1/\varepsilon\}, & a = \infty. \end{cases}$$

We denote by $\psi_n(a,\varepsilon)$ the number of points of $\mathcal{N}_n$ in $K(a,\varepsilon)$ and set

$$\psi(a) := \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{\psi_n(a,\varepsilon)}{\deg(p_n)},$$

where $\deg(p_n)$ is the degree of $p_n$. The limit in $\lim_{\varepsilon\to 0}$ exists, since $\psi_n(a,\varepsilon)/\deg(p_n)$ is bounded by 1 and

$$\liminf_{n\to\infty} \frac{\psi_n(a,\varepsilon)}{\deg(p_n)}$$

is monotonically decreasing with $\varepsilon \to 0$.

**Theorem.** *Let $f$ be a telescoping polynomial series (2) with the following properties:*

*(i) There exist infinitely many numbers $n \in \mathbb{N}$ with $p_n \neq p_{n-1}$; if $p_n \neq p_{n-1}$ then $p_n - p_{n-1}$ has $n$ zeros in $E$.*

*(ii)* *There exists a point $\zeta \in \mathbb{C}\backslash E$ such that $\zeta \notin \mathcal{N}$.*

*(iii)* *There exists $a \in \overline{\mathbb{C}}\backslash E$ with $\psi(a) > 0$.*

*(iv)* $\limsup_{n\to\infty} ||p_n||_E^{1/n} = c < \infty$.

*Then $f$ is an entire function and, in the case that $a \neq \infty$, $f$ is identically* 0.

**Remarks and Examples**

1. Let $f$ be a power series with center at 0, but not a polynomial, and with positive radius $\rho$ of convergence. Let $E$ be compact in $D(\rho) = \{z \in \mathbb{C} : |z| < \rho\}$ and let us denote by $p_n$ the n-th partial sum of $f$. Then $p_n - p_{n-1}$ has $n$ zeros at 0 if $p_n \neq p_{n-1}$. Hence, $(i)$ is satisfied as well as $(iv)$ with $c = 1$. Moreover, we can choose a point $\xi \in D(\rho)\backslash E$ such that $(ii)$ holds. If we can find a point $a \in \overline{\mathbb{C}}\backslash E$ with $\psi(a) > 0$, then the theorem tells us that $f$ is an entire function:

   In the case that $\rho$ is finite there exists, due to a result of Szegő [11], a subsequence $\Lambda \subset \mathbb{N}$ such that the normalized zero counting measures $\nu_n$ of $p_n$ converge weakly to the equilibrium measure of the circle $\{z \in \mathbb{C} : |z| = \rho\}$ as $n \to \infty$, $n \in \Lambda$. Consequently, for any point $a \in \overline{\mathbb{C}}$ we obtain $\psi(a) = 0$ and condition $(iii)$ is never satisfied.

   In the case $\rho = \infty$, the uniform convergence of the power series in $\mathbb{C}$ leads to $\psi(a) = 0$ for all $a \in \mathbb{C}$ and $\psi(\infty) = 1$. Hence, the theorem shows that $f$ is an entire function, not at all surprising since the starting point was $\rho = \infty$.

   So we can summarize that we can prove the holomorphy of the power series $f$ in $\mathbb{C}$, based only on $\psi(\infty) = 1$, without the help of Szegő's theorem.

2. If $f$ is holomorphic on $E$, but not a polynomial, then the conditions $(i)$ and $(ii)$ are satisfied for Leja polynomials and for the best approximating polynomials of a real-valued function $f$ on an interval $E \subset \mathbb{R}$, because of the characterization by alternation points. In both examples, the condition $(iii)$ cannot hold for $a \in \mathbb{C}$. Namely, if $f$ is an entire function then the polynomials converge uniformly to $f$ in $\mathbb{C}$. Hence, again $\psi(a) = 0$ for $a \in \mathbb{C}$ and $\psi(\infty) = 1$. In the case that $f$ is holomorphic on $E$, but not entire, then best approximating or Leja polynomials to $f$ on $E$ converge maximally to $f$ (Walsh [13]) and, due a generalization of Szegő's result, the normalized zero counting measures of these polynomials converge weakly to the equilibrium of the maximal Green domain of holomorphy of $f$, at least for a subsequence $\Lambda \subset \mathbb{N}$ ([1],[2]). Hence, we have again $\psi(a) = 0$ for any $a \in \mathbb{C}$, and as in the case of a power series $f$, we obtain the holomorphy of $f$ in $\mathbb{C}$, based only on $\psi(\infty) = 1$, without the help of generalizations of Szegő's theorem.

3. Let $E$ be compact and $f$ holomorphic on $E$. We consider the Hermite-Lagrange interpolating polynomials $p_n$ on $\{z_0, \ldots, z_n\} \subset E$, $n \in \mathbb{N}$. Then $(i)$ holds and if additionally $(ii)-(iv)$ are satisfied, then the theorem can be applied.

4. In all previous examples the constant $c$ in $(iv)$ was 1. We want to discuss an example where $c > 1$: Let $E \subset \mathbb{R}$ be compact, but not an interval, and let $f$ be a continuous, real-valued function on $E$ which is not a polynomial. Let us consider the polynomials $p_n$ of the best approximation to $f$, then the characterization by alternation points leads to $n$ zeros of $p_n - p_{n-1}$ on the convex hull conv$(E)$ of $E$ if $p_n \neq p_{n-1}$. Then

$$\limsup_{n\to\infty} ||p_n||_E^{1/n} = 1$$

   and the Bernstein-Walsh lemma yields

$$\limsup_{n\to\infty} ||p_n||_{\mathrm{conv}(E)}^{1/n} \leq c = \max_{z\in\mathrm{conv}(E)} g_\Omega(z, \infty),$$

   where $g_\Omega(z, \infty)$ is Green's function of $\Omega$ with pole at $\infty$. Hence, replacing $E$ by conv$(E)$, the condition $(i)$ holds and $(iv)$ is true with a constant $c > 1$.

## 3 Proof

Using a translation of $E$, we may assume in $(ii)$ that $\zeta = 0$. Then we use the linear transformation

$$z = h(w) := \frac{1}{w} \tag{3}$$

and obtain the following properties:

(a) $h(0) = \infty$.

(b) Let $\widetilde{E} := h(E)$, then $\widetilde{E}$ is compact in $\mathbb{C}$ since $\zeta = 0$ is mapped to $\infty$ and $0 \notin E$.

(c) Since $\Omega = \overline{\mathbb{C}}\backslash E$ is connected, then the set $\widetilde{\Omega} := \overline{\mathbb{C}}\backslash\widetilde{E}$ is also connected.

(d) If $K$ is compact in $\mathbb{C}$, then $\widetilde{K} = h(K)$ is compact in $\overline{\mathbb{C}}$. But we remark that $\infty \in \widetilde{K}$ is possible.

Because of (3), we obtain

$$p_n(z) = (p_n \circ h)(w) = \frac{\widetilde{p}_n(w)}{w^n} \tag{4}$$

with a polynomial $\widetilde{p}_n \in \mathcal{P}_n$.

Let us define $D_n(z) := p_n(z) - p_{n-1}(z)$, then

$$\widetilde{D}_n(w) := D_n(h(w)) = (p_n \circ h)(w) - (p_{n-1} \circ h)(w) = \tag{5}$$

$$= \frac{\widetilde{p}_n(w) - w\widetilde{p}_{n-1}(w)}{w^n} \tag{6}$$

$$= \frac{\widetilde{q}_n(w)}{w^n} \tag{7}$$

with $\widetilde{q}_n \in \mathcal{P}_n$.

Now we assume in the following that $n$ is an index with $p_n \neq p_{n-1}$. Then $(ii)$ implies that there exist $n$ points

$$\zeta_{n,i} \in E, 1 \leq i \leq n,$$

(counted with their multiplicity) such that

$$\widetilde{D}_n(1/\zeta_{n,i}) = 0.$$

Therefore,

$$\widetilde{q}_n(w) = A_n \widetilde{W}_n(w),\ A_n \neq 0, \tag{8}$$

where

$$\widetilde{W}_n(w) = \prod_{i=1}^{n} (w - \frac{1}{\zeta_{n,i}}).$$

(7) and (8) lead to

$$A_n = \frac{\widetilde{q}_n(w_0)}{\widetilde{W}_n(w_0)}, \tag{9}$$

inserting any point $w_0$ with $\widetilde{W}_n(w_0) \neq 0$. Define

$$\widetilde{a} := h(a),\ \widetilde{\mathcal{N}}_n := h(\mathcal{N}_n)$$

and

$$\widetilde{\psi}(\widetilde{a}) := \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{\widetilde{\psi}_n(\widetilde{a}, \varepsilon)}{\deg(\widetilde{p}_n)},$$

where $\widetilde{\psi}_n(a, \varepsilon)$ denotes the number of points of $\widetilde{\mathcal{N}}_n$ in $K(\widetilde{a}, \varepsilon)$. Then $\widetilde{a} \in \mathbb{C}$ and $\widetilde{\psi}(\widetilde{a}) = \psi(a)$.

Since $\widetilde{a} \notin \widetilde{E}$, we can fix $\varepsilon$, $0 < \varepsilon < 1$, such that

$$0 < \varepsilon < \frac{\text{dist}\,(\widetilde{a}, \widetilde{E})}{2}.$$

Then there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $K(\widetilde{a}, \varepsilon)$ contains at least $\deg(\widetilde{p}_n)\widetilde{\psi}(\widetilde{a})/2$ points of $\widetilde{\mathcal{N}}_n$ for all $n \geq n_0(\varepsilon)$, whether $p_n = p_{n-1}$ or $p_n \neq p_{n-1}$.

We obtain with (6) - (8):

$$\widetilde{D}_n(w) = \frac{\widetilde{q}_n(w)}{w^n} = \frac{A_n \widetilde{W}_n(w)}{w^n}$$

or

$$A_n = \frac{\widetilde{q}_n(w_0)}{\widetilde{W}_n(w_0)} = \frac{\widetilde{p}_n(w_0)}{\widetilde{W}_n(w_0)},$$

inserting any point $w_0$ with $\widetilde{W}_n(w_0) \neq 0$ and $\widetilde{p}_{n-1}(w_0) = 0$.

So let us consider any index $n > n_0(\varepsilon)$ with $p_n \neq p_{n-1}$. We fix a zero $\eta_{n-1}$ of $\widetilde{p}_{n-1}$ in $K(\widetilde{a}, \varepsilon)$ and the special choice $w_0 = \eta_{n-1}$ leads to

$$A_n = \frac{\widetilde{p}_n(\eta_{n-1})}{\widetilde{W}_n(\eta_{n-1})}. \tag{10}$$

We estimate the numerator and the denominator in (10) separately.

**Upper bounds for $\widetilde{p}_n(\eta_{n-1})$**

Let $\widetilde{p}_n(w) = a_n w^n + \ldots$, then $a_n \neq 0$ since we consider only indices $n$ with $p_n \neq p_{n-1}$. Consequently

$$\widetilde{p}_n(w) = a_n \prod_{\eta \in \widetilde{\mathcal{N}}_n} (w - \eta), \tag{11}$$

where $\eta$ runs through the points of $\widetilde{\mathcal{N}}_n$ and

$$||\widetilde{p}_n||_{\widetilde{E}} \geq |a_n| (\text{cap } \widetilde{E})^n. \tag{12}$$

We have used for the last inequality the well-known bound

$$||q_n||_{\widetilde{E}} \geq (\text{cap } \widetilde{E})^n$$

for all monic polynomials $q_n \in \mathcal{P}_n$ (cf. Pommerenke [8], Lemma 11.2). Moreover,

$$||\widetilde{p}_n||_{\widetilde{E}} = ||w^n p_n(\frac{1}{w})||_{\widetilde{E}} \leq ||p_n||_E \max_{w \in \widetilde{E}} |w^n|.$$

Set

$$c_1 := \max_{w \in \widetilde{E}} |w|,$$

then $c_1 > 0$ and (12) leads to

$$|a_n| \leq \frac{c_1^{\,n}}{(\text{cap } \widetilde{E})^n} ||p_n||_E \tag{13}$$

(We keep in mind that cap $\widetilde{E} > 0$ because of cap $E > 0$). Using property $(iv)$ we obtain a constant $c_2 > 0$ such that

$$||p_n||_E^{1/n} \leq c_2 \text{ for all } n \in \mathbb{N}.$$

Let

$$c_3 := (\text{cap } \widetilde{E})^{-1} c_1 c_2,$$

then

$$|a_n| \leq c_3^n, \; n \in \mathbb{N}, \tag{14}$$

and

$$|\widetilde{p}_n(\eta_{n-1})| \leq c_3^n \prod_{\eta \in \widetilde{\mathcal{N}}_n} |\eta_{n-1} - \eta|.$$

We subdivide the product into two factors

$$\prod_{\eta \in \widetilde{\mathcal{N}}_n} = \prod_{\eta \in \widetilde{\mathcal{N}}_n, |\widetilde{a} - \eta| \leq \varepsilon} \prod_{\eta \in \widetilde{\mathcal{N}}_n, |\widetilde{a} - \eta| > \varepsilon}.$$

Since $0 \notin \mathcal{N}$, there exists a neighborhood $U$ of 0 and $n_1(\varepsilon) \geq n_0(\varepsilon)$ such that $\mathcal{N}_{n-1} \cap U = \emptyset$ for all $n \geq n_1(\varepsilon)$. Because the linear transformation $h$ maps $U$ to a neighborhood of $\infty$, there exists a constant $c_4 > 1$ such that

$$|\eta_{n-1} - \eta| \leq c_4 \text{ for any } \eta \in \widetilde{\mathcal{N}}_n, \; n \geq n_1(\varepsilon).$$

Hence,

$$\prod_{\eta \in \widetilde{\mathcal{N}}_n, |\widetilde{a} - \eta| > \varepsilon} |\eta_{n-1} - \eta| \leq c_4^n.$$

On the other hand

$$\prod_{\eta \in \widetilde{\mathcal{N}}_n, |\widetilde{a} - \eta| \leq \varepsilon} |\eta_{n-1} - \eta| \leq \varepsilon^{n \widetilde{\psi}(\widetilde{a})/2}, \; n \geq n_1(\varepsilon),$$

and therefore we get

$$|\widetilde{p}_n(\eta_{n-1})| \leq (c_3 c_4)^n \varepsilon^{n\psi(a)/2}, \; n \geq n_1(\varepsilon). \tag{15}$$

**Lower bounds for $\widetilde{W}_n(\eta_{n-1})$**

Since

$$\widetilde{W}_n(\eta_{n-1}) = \prod_{i=1}^{n} (\eta_{n-1} - \frac{1}{\zeta_{n,i}})$$

and $1/\zeta_{n,i} \in \widetilde{E}$, we have for $n \geq n_1(\varepsilon)$

$$|\eta_{n-1} - \frac{1}{\zeta_{n,i}}| \geq \operatorname{dist}(\eta_{n-1}, \widetilde{E}) \geq \frac{\operatorname{dist}(\widetilde{a}, \widetilde{E})}{2} =: 1/c_5$$

or

$$|\widetilde{W}_n(\eta_{n-1})| \geq (1/c_5)^n. \tag{16}$$

Together with (10), we finally obtain from and (15) and (16)

$$|A_n| \leq (c_3 c_4 c_5)^n \varepsilon^{n\psi(a)/2} \text{ for } n \geq n_1(\varepsilon). \tag{17}$$

Summarizing, we have got for $w \in \overline{\mathbb{C}} \backslash \widetilde{E}$, $w \neq 0$ :

$$\widetilde{D}_n(w) = A_n \frac{1}{w^n} \prod_{i=1}^{n} (w - \frac{1}{\zeta_{n,i}}),$$

where $A_n$ is bounded by (17) for $n \geq n_1(\varepsilon)$. Observe that

$$\prod_{i=1}^{n} |w - \frac{1}{\zeta_{n,i}}| \leq \prod_{i=1}^{n} (|w| + |\frac{1}{\zeta_{n,i}}|) \leq (2 \max_{w \in \widetilde{E}} |w|)^n \leq (2c_1)^n.$$

Define

$$c_6 := \max_{w \in \widetilde{E}} |1/w|,$$

then

$$\max_{w \in \widetilde{E}} |\widetilde{D}_n(w)| \leq (2c_1 c_3 c_4 c_5 c_6)^n \varepsilon^{n\psi(a)/2} \tag{18}$$

for all $n \geq n_1(\varepsilon)$ or

$$\max_{w \in \widetilde{E}} |\widetilde{D}_n(w)| \leq c(\varepsilon)^n \tag{19}$$

with

$$c(\varepsilon) = 2c_1 c_3 c_4 c_5 c_6 \ \varepsilon^{\psi(a)/2}. \tag{20}$$

Next we use a modification of the Bernstein-Walsh lemma for the growth of the functions $\widetilde{D}_n(w)$ in $\widetilde{\Omega} = \overline{\mathbb{C}} \backslash \widetilde{E}$: Since $0 \in \widetilde{\Omega}$ and cap $\widetilde{E} > 0$, there exists the Green function $g_{\widetilde{\Omega}}(w, 0)$ with pole at 0. $g_{\widetilde{\Omega}}(w, 0)$ satisfies the following properties:

(a) $g_{\widetilde{\Omega}}(w, 0)$ is nonnegative and harmonic in $\widetilde{\Omega} \backslash \{0\}$ and bounded as $w$ stays away from 0,

(b) $\lim_{w \to 0} (g_{\widetilde{\Omega}}(w, 0) + \log|w|)$ exists and $g_{\widetilde{\Omega}}(w, 0) + \log|w|$ is harmonic at the point $w = 0$,

(c) $\lim_{w \to z, w \in \widetilde{\Omega}} g_{\widetilde{\Omega}}(w, 0) = 0$ for quasi-every $z \in \partial \widetilde{\Omega}$

(cf. Saff, Totik [9], chapter II, 4, p.108 ff or Tsuji [12] chapter I, 6, p.14).

Then we use the following lemma

**Lemma (modified Bernstein-Walsh).** *Let $\widetilde{E}$ be a compact set in $\mathbb{C}$ with cap $\widetilde{E} > 0$, $\widetilde{\Omega} := \overline{\mathbb{C}} \backslash \widetilde{E}$ and $0 \in \widetilde{\Omega}$. If*

$$\widetilde{D}_n(w) = \frac{\widetilde{p}_n(w)}{w^n}, \ \widetilde{p}_n \in \mathcal{P}_n,$$

*then for all $w \in \widetilde{\Omega} \backslash \{0\}$:*

$$|\widetilde{D}_n(w)| = ||\widetilde{D}_n||_{\widetilde{E}} \ e^{n g_{\widetilde{\Omega}}(w,0)}.$$

**Proof of the Lemma**: Define

$$g(w) := \frac{1}{n} \log|\widetilde{D}_n(w)| - g_{\widetilde{\Omega}}(w, 0),$$

then $g$ is harmonic in $\widetilde{\Omega}$ and

$$\lim_{w\to z, z\in\partial\widetilde{\Omega}} g(w) \le \frac{1}{n}\log||\widetilde{D}_n||_{\widetilde{E}}$$

for quasi-every point $z \in \partial\widetilde{\Omega}$. Then the maximum principle for harmonic functions implies that

$$g(w) \le \frac{1}{n}\log||\widetilde{D}_n||_{\widetilde{E}}$$

for every $w \in \widetilde{\Omega}\backslash\{0\}$, or

$$|\widetilde{D}_n(w)| \le ||\widetilde{D}_n||_{\widetilde{E}}\ e^{ng_{\widetilde{\Omega}}(w,0)}$$

for all $w \in \widetilde{\Omega}\backslash\{0\}$. □

Now we can continue with the main proof.

Let $K$ be compact in $\mathbb{C}$, then $\widetilde{K} := h(K)$ is compact in $\overline{\mathbb{C}}\backslash\{0\}$. Define

$$m(\widetilde{K}) := \max_{w\in\widetilde{K}}\ g_{\widetilde{\Omega}}(w,0),$$

then $m(\widetilde{K}) > 0$ and the modified Bernstein-Walsh lemma gives

$$\max_{w\in\widetilde{K}}|\widetilde{D}_n(w)| \le ||\widetilde{D}_n||_{\widetilde{E}}e^{m(\widetilde{K})n}$$

and (18) yields

$$\max_{w\in\widetilde{K}}|\widetilde{D}_n(w)| \le c(\varepsilon)^n e^{m(\widetilde{K})n}$$

with $c(\varepsilon)$ as in (20). Then we can choose $\varepsilon$ such that

$$c(\varepsilon)e^{m(\widetilde{K})} < \delta < 1$$

for all $n \ge n_1(\varepsilon)$, where $p_n \neq p_{n-1}$ was satisfied. But the remaining members

$$p_n - p_{n-1}$$

in the telescoping series are 0. Hence, the telescoping series (2.1) converges uniformly on $K$ for any compact set $K$ of $\mathbb{C}$. Consequently, $f$ is an entire function and the first part of the theorem is proved. Concerning the second part for $a \neq \infty$, we notice that $f$ must have a zero at the point $a$ of infinite order. This is only possible if $f$ is identically 0. ■

## 4 Application to $L_p$−approximation

Let $w$ be a weight on $I = [-1,1]$ such that $\int_I w\,dx = 1$ and $w > 0$ almost everywhere on $I$. We consider the space $L_p(I)$, $1 < p < \infty$, with the norm

$$||f||_{L_p(I)} = \left(\int_I |f|^p w\,dx\right)^{1/p}$$

and denote by $B_{n,p}(f)$ the polynomial of best $L_p$− approximation of $f \in L_p(I)$ in $\mathcal{P}_n$. It is well known that $B_n(f)$ is characterized by the orthogonality relation

$$\int_I p_n|f - B_{n,p}(f)|^{p-1}\mathrm{sgn}(f - B_{n,p}(f))dx = 0,\ p_n \in \mathcal{P}_n. \tag{21}$$

Now, let us consider for fixed $f \in L_p(I)$ the telescoping series

$$f = B_{0,p}(f) + \sum_{n=1}^{\infty}(B_{n,p}(f) - B_{n-1,p}(f)). \tag{22}$$

Our intention is to apply Theorem 1, using

$$p_n := B_{n,p}(f),\ n = 0, 1, 2, \dots,$$

on $E = I$.

In the following we assume that $f$ is holomorphic on $I$. Then Walsh ([13],§5.2) proved that the sequence $\{B_{n,p}(f)\}_{n=0}^{\infty}$ converges maximally to $f$. Hence, if $f$ is not identically 0 there exists a point $\xi \in \mathbb{C}\backslash I$ such that $\xi \notin \mathcal{N}$, i.e., the condition $(ii)$ is true for the sequence $B_{n,p}(f)$, $n \in \mathbb{N}_0$.

Next, we will use some notations to formulate a Nikolski-type inequality and we follow the arguments in the paper of Kroó-Swetits ([3], section 2.1): Let $\mu$ be a the equilibrium measure of $I$ and define for $0 \le \varepsilon \le 1$

$$\phi(w, \varepsilon) := \inf\{\int_A w\ dx : A \in I,\ \mu(A) \ge \varepsilon\}.$$

Then $\phi(w, \varepsilon)$ is a continuous, increasing function on $[0, 1]$, positive on $(0, 1)$ and satisfying $\phi(w, 0) = 0$ and $\phi(w, 1) = 1$. Hence, for every $n \in \mathbb{N}$ the equation

$$\phi(w, \varepsilon) = e^{-n\varepsilon} \tag{23}$$

has a unique solution $\varepsilon_n(w)$. Then $0 < \varepsilon_n(w) < 1$ and $\varepsilon_n(w)$ tends monotonically to 0 as $n \to \infty$. The number $\varepsilon_n(w)$ appears in the Nikolski-type inequality

$$||p_n||_I \le e^{Cn\varepsilon_n(w)}||p_n||_{L_p(I)},\ p_n \in \mathcal{P}_n,$$

where $C$ is a fixed positive constant, independent of $n \in \mathbb{N}$ (Kroó, Swetits [3], inequality 6 on page 89). Hence,

$$\limsup_{n\to\infty} ||B_{n,p}(f)||_I^{1/n} = 1$$

and the property $(iv)$ is satisfied for the sequence $\{B_{n,p}(f)\}_{n\in\mathbb{N}}$.

Now, we turn our attention to

$$D_{n,p}(f) := B_{n,p}(f) - B_{n-1,p}(f) \in \mathcal{P}_n. \tag{24}$$

Because of (21), the function

$$\Phi_{n,p}(f) :=$$

$$|f - B_{n-1,p}(f)|^{p-1}\mathrm{sgn}(f - B_{n-1,p}(f)) - |f - B_{n,p}(f)|^{p-1}\mathrm{sgn}(f - B_{n,p}(f)) \tag{25}$$

satisfies the orthogonality condition

$$\int_I p_{n-1}\Phi_{n,p}(f) w\ dx = 0,\ p_{n-1} \in \mathcal{P}_{n-1}. \tag{26}$$

Then Kroó and Swetits ([3], Proposition 2, p. 94) have pointed out a link between $\Phi_{n,p}(f)$ and $D_{n,p}(f)$, namely

$$\mathrm{sgn}\ \Phi_{n,p}(f) = \mathrm{sgn}(B_{n,p}(f) - B_{n-1,p}(f)). \tag{27}$$

Finally we obtain from (26) and (27) that $D_{n,p}(f)$ has exactly $n$ zeros inside $I$ if $B_{n,p}(f) \ne B_{n-1,p}(f)$. Hence, condition $(i)$ is satisfied for $\{B_{n,p}(f)\}_{n\in\mathbb{N}}$.

Therefore, we immediately obtain from the theorem in section 2 for polynomial $L_p$-approximation the following converse result.

**Corollary.** *Let $1 < p < \infty$ and let $f$ be holomorphic on $I = [-1, 1]$, but not a polynomial. If $B_{n,p}(f)$ is the best $L_p$-approximation of $f$ in $\mathcal{P}_n$, then we denote by $\psi_n(\infty, \varepsilon)$ the number of zeros of $B_{n,p}(f)$ in*

$$K(\infty, \varepsilon) = \{z \in \mathbb{C} : |z| \ge 1/\varepsilon\}\ (\varepsilon > 0)$$

*and we set*

$$\psi(\infty) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{\psi_n(\infty, \varepsilon)}{deg(B_{n,p}(f))}.$$

*If $\psi(\infty) > 0$, then the function $f$ can be continued to an entire function.*

Finally, we want to give a more general example for $L_2$-approximation: Let $E$ be compact, convex set in $\mathbb{C}$ and $\mu$ a positive, regular Borel measure on $E$. We consider the space $L_2(\mu)$, i.e., the set of complex-valued functions $f$ endowed with the norm

$$||f||_{L_2(\mu)} = \left( \int |f|^2 d\mu \right)^{1/2}.$$

We consider the polynomials $B_{n,2}(f)$ of best $L_2(\mu)$-approximation of $f \in L_2(\mu)$ in $\mathcal{P}_n$. Then it is well-known that

$$B_{n,2}(f) - B_{n-1,2}(f) = c_n p_{n,\mu},$$

where $c_n \in \mathbb{C}$ and $p_{n,\mu}$ is the n-th orthonormal polynomial corresponding to $\mu$. Due to a theorem of Fejér, $p_{n,\mu}$ has $n$ zeros in the convex hull of $E$ which is $E$ itself, since $E$ is convex. Because of the regularity of $\mu$, we have

$$\limsup_{n\to\infty} ||B_{n,2}(f)||_E^{1/n} = 1,$$

if $f$ is not a polynomial (cf. Stahl, Totik [10]).

Hence, the application of the theorem is straightforward for functions $f$ holomorphic on $E$.

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