PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special Issue *FAATNA20>22: Functional Analysis, Approximation Theory and Numerical Analysis*
Volume 16 · 2023 · Pages 118–134

Maximal operators on variable Lebesgue and Hardy spaces and applications in Fourier analysis

Ferenc Weisz[a]

Abstract

We summarize some results about the variable Lebesgue, Hardy and Hardy-Lorentz spaces $L_{p(\cdot)}(\mathbb{X})$, $H_{p(\cdot)}(\mathbb{X})$ and $H_{p(\cdot),q}(\mathbb{X})$, where $\mathbb{X}=\mathbb{T}$ or $\mathbb{X}=\mathbb{R}$ or $\mathbb{X}=[0,1)$. Besides the usual Hardy-Littlewood and Doob maximal functions, we introduce new dyadic maximal functions and show that they are bounded on $L_{p(\cdot)}(\mathbb{X})$ provided that $1<p_-\le\infty$. We present the atomic decompositions of the spaces $H_{p(\cdot)}(\mathbb{X})$ and $H_{p(\cdot),q}(\mathbb{X})$. As application in Fourier analysis, we consider the Fejér, Cesàro and Riesz summability of trigonometric and Walsh-Fourier series and Fourier transforms. Under some conditions, we prove that the maximal operators of the corresponding means are bounded from $H_{p(\cdot)}(\mathbb{X})$ to $L_{p(\cdot)}(\mathbb{X})$ and from $H_{p(\cdot),q}(\mathbb{X})$ to $L_{p(\cdot),q}(\mathbb{X})$. This implies some norm and almost everywhere convergence results of the Fejér, Cesàro and Riesz means.

1 Introduction

In this survey paper we summarize some results about variable Lebesgue, Hardy and Hardy-Lorentz spaces and about the summability of trigonometric and Walsh-Fourier series and Fourier transforms. Let $p(\cdot):\mathbb{R}\to(0,\infty]$ be a measurable function satisfying the globally log-Hölder condition. We will consider the variable Lebesgue space $L_{p(\cdot)}(\mathbb{X})$ with $\mathbb{X}=[0,1)$, $\mathbb{X}=\mathbb{T}$ or $\mathbb{X}=\mathbb{R}$. It generalizes the classical Lebesgue space: when $p(\cdot)\equiv p$ is a constant, then $L_{p(\cdot)}(\mathbb{X})=L_p(\mathbb{X})$. Interest in the variable Lebesgue spaces has increased since the 1990s because of their use in a variety of applications, such as in fluid dynamics, image processing, partial differential equations, variational calculus and harmonic analysis (see e.g. [1, 2, 6, 9, 13, 29, 31, 43, 57, 73, 90]).

We also introduce the variable martingale or (tempered) distribution Hardy and Hardy-Lorentz spaces $H_{p(\cdot)}(\mathbb{X})$ and $H_{p(\cdot),q}(\mathbb{X})$. These spaces are investigated very intensively in the literature nowadays (see e.g. Cruz-Uribe and Fiorenza [10], Diening at al. [12], Kempka and VybÃral [37], Kokilashvili et al. [38, 39], Nakai and Sawano [49, 58], Jiao at al. [32, 36], Yan at al. [89] and Liu at al. [45, 44]).

We introduce the usual Hardy-Littlewood maximal operator and its dyadic version, the Doob's maximal operator. It is known that under the log-Hölder continuity of the variable exponent $p(\cdot)$, these operators are bounded on $L_{p(\cdot)}(\mathbb{X})$ if $1<p_-\le\infty$, where p_- denotes the infimum of $p(\cdot)$ while p_+ the supremum of $p(\cdot)$ and $\mathbb{X}=\mathbb{R}$, or $\mathbb{X}=\mathbb{T}$ or $\mathbb{X}=[0,1)$. We generalize the dyadic maximal operator by

$$M_{\gamma,s}f(x):=\sup_{x\in I}\sum_{m=0}^{n}\sum_{j=0}^{m}2^{(j-n)\gamma}\sum_{i=j}^{m}2^{(j-i)s}\frac{1}{\lambda(I^{j,i})}\left|\int_{I^{j,i}}f\,d\lambda\right|,$$

where I is a dyadic interval with length 2^{-n}, γ,s are positive constants and

$$I^{j,i}:=I\dot{+}[0,2^{-i})\dot{+}2^{-j-1}.$$

Here $\dot{+}$ denotes the dyadic addition. $M_{\gamma,s}$ contains the Doob's maximal operator as special case. We prove that this maximal operator is also bounded on $L_{p(\cdot)}[0,1)$ under the sufficient and necessary condition

$$1<p_-\le p_+\le\infty,\qquad \frac{1}{p_-}-\frac{1}{p_+}<\gamma+s.$$

Weak type inequalities are also given for $p_-=1$. The aim of these new maximal operators is to be able to prove the summability results in the last section.

We study the Hardy and Hardy-Lorentz spaces $H_{p(\cdot)}[0,1)$ and $H_{p(\cdot),q}[0,1)$ containing martingales, the $H_{p(\cdot)}(\mathbb{T})$ and $H_{p(\cdot),q}(\mathbb{T})$ spaces containing distributions as well as the $H_{p(\cdot)}(\mathbb{R})$ and $H_{p(\cdot),q}(\mathbb{R})$ spaces containing tempered distributions. We present the atomic decomposition of these spaces which is used several times in the proofs of the results of the last two sections. Some

[a]Department of Numerical Analysis, Eötvös L. University, H-1117 Budapest, Pázmány P. sétány 1/C., Hungary, e-mail: weisz@inf.elte.hu

equivalent characterizations of these spaces are given. Amongst others, we obtain that the operator $M_{\gamma,s}$ generates equivalent norms on $H_{p(\cdot)}[0,1)$ and $H_{p(\cdot),q}[0,1)$.

In the last two sections, we study some applications in Fourier analysis. We investigate the convergence of partial sums and some summability means of trigonometric and Walsh-Fourier series and Fourier transforms. It is known (see Lebesgue [42] and also Fejér [16]) that the Fejér means

$$\sigma_n f(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) \widehat{f}(k) e^{2\pi \imath kx}$$

of the trigonometric Fourier series converge almost everywhere to f, whenever $f \in L_p(\mathbb{T})$ for some $1 \le p < \infty$. The corresponding result holds for the Fourier transforms, too (see e.g. Stein, Taibleson and Weiss [70]). Fine [17] proved the analogous almost everywhere convergence of the Fejér means for Walsh-Fourier series. Later Schipp [59] obtained the same result by proving the weak type inequality of the maximal operator σ_* of the Fejér means. Next Fujii [18] showed that σ_* is bounded from the dyadic Hardy space $H_1[0,1)$ to $L_1[0,1)$ (see also Schipp and Simon [61]). Moreover, the maximal operator σ_* of the Fejér means is bounded from the Hardy space $H_p(\mathbb{R})$ to $L_p(\mathbb{R})$ and from $H_p(\mathbb{T})$ to $L_p(\mathbb{T})$ if $1/2 < p < \infty$ (see Grafakos [27] and Lu [46] or Weisz [81, 87]). Later the author ([77, 79]) proved that σ_* is bounded from $H_p[0,1)$ to $L_p[0,1)$ for $1/2 < p < \infty$. Similar summability results for Vilenkin and Walsh-Kaczmarz systems were proved in Gát [19, 20, 21, 22, 24, 23], Goginava [25, 26], Persson and Tephnadze [52, 53, 54, 55] and Simon [64, 65, 66, 67].

In this paper, we will generalize these results to variable Hardy and Hardy-Lorentz spaces. We consider two generalizations of the Fejér summability, the Cesàro and the Riesz means,

$$\sigma_n^{\alpha} f(x) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k=-n}^{n} A_{n-1-|k|}^{\alpha} \widehat{f}(k) e^{2\pi \imath kx}$$

and

$$\sigma_n^{\alpha,\gamma} f(x) := \frac{1}{n^{\alpha\gamma}} \sum_{k=-n}^{n} (n^{\gamma} - |k|^{\gamma})^{\alpha} \widehat{f}(k) e^{2\pi \imath kx},$$

where

$$A_n^{\alpha} := \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} \qquad (n \in \mathbb{N}).$$

We state that the maximal operator σ_*^{α} $(0 < \alpha \le 1)$ of the Cesàro or Riesz means of the trigonometric Fourier series or Fourier transforms is bounded from $H_{p(\cdot)}(\mathbb{X})$ to $L_{p(\cdot)}(\mathbb{X})$ and from $H_{p(\cdot),q}(\mathbb{X})$ to $L_{p(\cdot),q}(\mathbb{X})$, where $\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = \mathbb{R}$ and

$$\frac{1}{\alpha+1} < p_- \le p_+ < \infty.$$

It is a very surprising result, that this result do not hold for Walsh-Fourier series, though the results are usually very similar for the trigonometric and Walsh-Fourier series. However, if we suppose in addition that

$$\frac{1}{p_-} - \frac{1}{p_+} < 1,$$

then the corresponding theorem already holds for Walsh-Fourier series, i.e., σ_*^{α} is bounded from $H_{p(\cdot)}[0,1)$ to $L_{p(\cdot)}[0,1)$ and from $H_{p(\cdot),q}[0,1)$ to $L_{p(\cdot),q}[0,1)$. Both conditions are sufficient and necessary. These results imply some norm and almost everywhere convergence results of the Cesàro and Riesz means. This paper contains the results of my talk given at the conference Functional Analysis, Approximation Theory and Numerical Analysis, Matera, Italy, 2022.

2 Variable Lebesgue spaces

In this section, we recall some basic notations on variable Lebesgue spaces and give some elementary and necessary facts about these spaces. Our main references are the books Cruz-Uribe and Fiorenza [10] and Diening et al. [12].

In this paper we consider the measure space $(\mathbb{X}, \lambda)$, where $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = [0,1)$ and λ is the Lebesgue measure. For a constant p, the $L_p(\mathbb{X})$ space is equipped with the quasi-norm

$$\|f\|_{L_p(\mathbb{X})} := \left(\int_{\mathbb{X}} |f(x)|^p \, dx\right)^{1/p} \qquad (0 < p < \infty),$$

with the usual modification for $p = \infty$.

We generalize these spaces as follows. A measurable function $p(\cdot) : \mathbb{X} \to (0,\infty]$ is called a variable exponent. For a measurable set $A \subset \mathbb{X}$, we denote

$$p_-(A) := \operatorname{ess\,inf}_{x \in A} p(x), \quad p_+(A) := \operatorname{ess\,sup}_{x \in A} p(x)$$

and for convenience

$$p_- := p_-(\mathbb{X}), \quad p_+ := p_+(\mathbb{X}).$$

Denote by $\mathcal{P}(\mathbb{X})$ the collection of all variable exponents $p(\cdot)$ such that

$$0 < p_- \leq p_+ \leq \infty.$$

To introduce the variable Lebesgue spaces let

$$\rho_{\mathbb{X}}(f) := \int_{\mathbb{X}\setminus\Omega_\infty^{\mathbb{X}}} |f(x)|^{p(x)} dx + \|f\|_{L_\infty(\Omega_\infty^{\mathbb{X}})},$$

where $\Omega_\infty^{\mathbb{X}} = \{x \in \mathbb{X} : p(x) = \infty\}$. The variable Lebesgue space $L_{p(\cdot)}(\mathbb{X})$ is the collection of all measurable functions f for which there exists $\nu > 0$ such that

$$\rho_{\mathbb{X}}(f/\nu) < \infty.$$

The space $L_{p(\cdot)}(\mathbb{X})$ is equipped with the quasi-norm

$$\|f\|_{L_{p(\cdot)}(\mathbb{X})} := \inf\{\nu > 0 : \rho_{\mathbb{X}}(f/\nu) \leq 1\}.$$

It is easy to see that if $p(\cdot) = p$ is a constant, then we get back the definition of the usual $L_p(\mathbb{X})$ spaces. For any $f \in L_{p(\cdot)}(\mathbb{X})$, we have $\rho_{\mathbb{X}}(f) \leq 1$ if and only if $\|f\|_{L_{p(\cdot)}(\mathbb{X})} \leq 1$. It is known that $\|\nu f\|_{L_{p(\cdot)}(\mathbb{X})} = |\nu| \|f\|_{L_{p(\cdot)}(\mathbb{X})}$ and

$$\||f|^s\|_{L_{p(\cdot)}(\mathbb{X})} = \|f\|^s_{L_{sp(\cdot)}(\mathbb{X})},$$

where $p(\cdot) \in \mathcal{P}(\mathbb{X})$, $s \in (0, \infty)$, $\nu \in \mathbb{C}$ and $f \in L_{p(\cdot)}(\mathbb{X})$. Details can be found in the monographs Cruz-Uribe and Fiorenza [10] and Diening et al. [12]. The variable exponent $p'(\cdot)$ is defined pointwise by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{X}.$$

The next result is well known, see Cruz-Uribe and Fiorenza [10] or Diening et al. [12].

Theorem 2.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{X})$ with $1 \leq p_- \leq p_+ \leq \infty$. For all $f \in L_{p(\cdot)}(\mathbb{X})$ and $g \in L_{p'(\cdot)}(\mathbb{X})$,*

$$\int_{\mathbb{X}} |fg| \, d\lambda \lesssim \|f\|_{L_{p(\cdot)}(\mathbb{X})} \|g\|_{L_{p'(\cdot)}(\mathbb{X})}.$$

Moreover,

$$\|f\|_{L_{p(\cdot)}(\mathbb{X})} \sim \sup_{\|g\|_{L_{p'(\cdot)}(\mathbb{X})} \leq 1} \left| \int_{\mathbb{X}} f g \, d\lambda \right|,$$

where $\sim$ denotes the equivalence of the numbers.

In this paper the constants C are absolute constants and the constants $C_{p(\cdot)}$ are depending only on $p(\cdot)$ and may denote different constants in different contexts. For two positive numbers A and B, we use also the notation $A \lesssim B$, which means that there exists a constant C or $C_{p(\cdot)}$ such that $A \leq C_{p(\cdot)} B$.

Usually, the proofs of the generalizations for the variable Lebesgue spaces are much more complicated than those of the original theorems. However, not all theorems can be generalized. For example, the translation operator $T_x f(\cdot) := f(\cdot - x)$ is trivially bounded on the $L_p(\mathbb{X})$ spaces, but not on $L_{p(\cdot)}(\mathbb{X})$. More exactly, T_x is bounded on $L_{p(\cdot)}(\mathbb{X})$ if and only if $p(\cdot) = p$ almost everywhere. Moreover, the well known theorem about the boundedness of the Hardy-Littlewood maximal operator on $L_p(\mathbb{X})$ $(1 < p \leq \infty)$ (see Theorem 3.1) cannot be generalized to variable Lebesgue spaces without additional conditions (see e.g. Cruz-Uribe and Fiorenza [10]). To be able to generalize this theorem, we introduce the following condition.

We denote by $C^{\log}(\mathbb{X})$ the set of all variable exponents $p(\cdot)$ satisfying the so-called globally log-Hölder continuous condition, namely, there exist two positive constants $C_{\log}(p)$ and C_∞, and $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{X}$,

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/|x - y|)} \tag{1}$$

and

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}. \tag{2}$$

The globally log-Hölder continuous condition is a commonly used condition in the literature. Of course, for $\mathbb{X} = \mathbb{T}$ and $\mathbb{X} = [0, 1)$, we can omit (2). There are a lot of functions satisfying the globally log-Hölder continuous condition. For example, it is easy to see that a Lipschitz function of order α $(0 < \alpha \leq 1)$ satisfies (1).

Under the condition $0 < p_- \leq p_+ < \infty$, we know that $p(\cdot) \in C^{\log}(\mathbb{X})$ if and only if $1/p(\cdot) \in C^{\log}(\mathbb{X})$. Usually, it is not easy to compute the variable Lebesgue norm of a function, even for characteristic functions. The following theorem was proved in Cruze-Uribe and Fiorenza [10] and Diening et al. [12, Corollary 4.5.9] (see also Hao and Jiao [28]).

Theorem 2.2. *Let $1/p(\cdot) \in C^{\log}(\mathbb{X})$, $I \subset \mathbb{X}$ be an arbitrary interval and $x \in I$. Then there exists a constant $0 < \beta < 1$ such that*

$$\lambda(I)^{1/p_+(I) - 1/p_-(I)} \leq \frac{1}{\beta}. \tag{3}$$

Moreover,

$$\|\chi_I\|_{L_{p(\cdot)}(\mathbb{X})} \sim \begin{cases} \lambda(I)^{1/p(x)} & \text{if } \lambda(I) \leq 2; \\ \lambda(I)^{1/p_\infty} & \text{if } \lambda(I) \geq 1, \end{cases}$$

where χ_I denotes the characteristic function of I.

For probability spaces, there is no topology usually. So for general martingale Hardy spaces, we assume that every σ-algebra is generated by finitely many atoms. Moreover, instead of the log-Hölder continuity condition, we supposed in [28, 32, 35, 72, 86] the slightly more general condition (3) for all atoms of the σ-algebras.

The variable Lorentz spaces were introduced and investigated by Kempka and VybÃral [37]. $L_{p(\cdot),q}(\mathbb{X})$ is defined to be the space of all measurable functions f such that

$$\|f\|_{L_{p(\cdot),q}(\mathbb{X})} := \begin{cases} \left(\int_0^\infty \rho^q \left\| \chi_{\{x \in [0,1):\ |f(x)| > \rho\}} \right\|_{L_{p(\cdot)}(\mathbb{X})}^q \frac{d\rho}{\rho} \right)^{1/q}, & \text{if } 0 < q < \infty; \\ \sup_{\rho \in (0,\infty)} \rho \left\| \chi_{\{x \in [0,1):\ |f(x)| > \rho\}} \right\|_{L_{p(\cdot)}(\mathbb{X})}, & \text{if } q = \infty \end{cases}$$

is finite. If $p(\cdot)$ is a constant, we get back the classical Lorentz spaces (see Bergh and Löfström [4]).

3 Maximal operators

In this section, we will investigate the usual Hardy-Littlewood maximal functions for $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = \mathbb{T}$ and a generalization of the dyadic maximal function for $\mathbb{X} = [0,1)$ and prove their boundedness on variable Lebesgue spaces.

3.1 Hardy-Littlewood maximal functions

Let $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = \mathbb{T}$. One of the most important operators of harmonic analysis is the Hardy-Littlewood maximal operator M. Given a locally integrable function f, M is defined by

$$M(f)(x) := \sup_{x \in I} \frac{1}{\lambda(I)} \left| \int_I f(y)\, dy \right| \qquad (x \in \mathbb{X}),$$

where the supremum is taken over all intervals I of $\mathbb{X}$ containing x. It is well known that M is bounded on $L_p(\mathbb{X})$ if $1 < p < \infty$ and is of weak type $(1,1)$ (see e.g. Stein [69]).

Theorem 3.1. *Let $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = \mathbb{T}$. If $f \in L_p(\mathbb{X})$ with $1 < p \leq \infty$, then*

$$\|M(f)\|_{L_p(\mathbb{X})} \lesssim \|f\|_{L_p(\mathbb{X})}. \tag{4}$$

If $1 \leq p < \infty$, then

$$\sup_{\rho \in (0,\infty)} \rho\, \lambda(\{x \in \mathbb{X} : M(f)(x) > \rho\})^{1/p} \lesssim \|f\|_{L_p(\mathbb{X})}. \tag{5}$$

The expression of the left hand side of (5) is called the weak $L_p(\mathbb{X})$-norm. Obviously, the weak $L_p(\mathbb{X})$-norm is smaller than the $L_p(\mathbb{X})$-norm, so (5) follows from (4) if $1 < p < \infty$.

For the next theorem, the globally log-Hölder continuity condition is sufficient, not necessary, but in some sense it is the best possible condition (see e.g. Cruz-Uribe and Fiorenza [10] and Diening et al. [12]).

Theorem 3.2. *Suppose that $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = \mathbb{T}$, $1/p(\cdot) \in C^{\log}(\mathbb{X})$ and $f \in L_{p(\cdot)}(\mathbb{X})$. If $1 < p_- \leq \infty$, then*

$$\|M(f)\|_{L_{p(\cdot)}(\mathbb{X})} \lesssim \|f\|_{L_{p(\cdot)}(\mathbb{X})}. \tag{6}$$

If $p_- = 1$, then

$$\sup_{\rho \in (0,\infty)} \rho \left\| \chi_{\{x \in \mathbb{X}: M(f)(x) > \rho\}} \right\|_{L_{p(\cdot)}(\mathbb{X})} \lesssim \|f\|_{L_{p(\cdot)}(\mathbb{X})}. \tag{7}$$

If $p(\cdot)$ is a constant, (6) and (7) give back (4) and (5), respectively. Another result which cannot be generalized to variable spaces even if we suppose the globally log-Hölder continuity condition, is the following modular inequality which is similar to (6) (see e.g. Cruz-Uribe and Fiorenza [10]):

$$\int_{\mathbb{X}} M(f)(x)^{p(x)}\, dx \lesssim \int_{\mathbb{X}} |f(x)|^{p(x)}\, dx$$

holds if and only if $p(\cdot) = p$ almost everywhere and $1 < p \leq \infty$.

3.2 Dyadic maximal functions

Let $\mathbb{X} = [0,1)$. By a dyadic interval we mean one of the form $[k2^{-n},(k+1)2^{-n})$ for some $k,n \in \mathbb{N}$, $0 \le k < 2^n$. For a fixed $x \in [0,1)$ and $n \in \mathbb{N}$, let us denote the unique dyadic interval $[k2^{-n},(k+1)2^{-n})$ which contains x by $I_n(x)$. We will define the dyadic maximal operator not only for integrable functions but also for martingales.

Let $\mathcal{F}_n$ be the σ-algebra

$$\mathcal{F}_n = \sigma\{[k2^{-n},(k+1)2^{-n}) : 0 \le k < 2^n\},$$

where $\sigma(\mathcal{H})$ denotes the σ-algebra generated by an arbitrary set system $\mathcal{H}$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E_n. It is easy to see that for an integrable function f,

$$E_n(f)(x) = \frac{1}{\lambda(I_n(x))} \left| \int_{I_n(x)} f \, d\lambda \right| \qquad (x \in [0,1)).$$

A sequence of integrable functions $f = (f_n)_{n\in\mathbb{N}}$ is called a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if

(i) f_n is $\mathcal{F}_n$ measurable

(ii) $E_n(f_{n+1}) = f_n$ for every $n \in \mathbb{N}$.

Martingales with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ are called dyadic martingales.

For a dyadic martingale $f = (f_n)_{n\in\mathbb{N}}$, the Doob maximal function is defined by

$$M(f) := \sup_{n\in\mathbb{N}} |f_n|.$$

The maximal operator can also be written in the form

$$M(f)(x) := \sup_{n\in\mathbb{N}} \frac{1}{\lambda(I_n(x))} \left| \int_{I_n(x)} f_n \, d\lambda \right| \qquad (x \in [0,1)).$$

If $f \in L_1[0,1)$, then we can replace f_n by f in the integral. Then Theorems 3.1 and 3.2 hold for the dyadic maximal operator M, too. For general martingales, Theorem 3.2 was proved in Jiao et al. [34, 32] for $1 \le p_- < \infty$ and in [86] for $1 \le p_- \le \infty$.

In [85], we generalized the dyadic maximal operator as follows. Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \qquad (0 \le x_k < 2,\ x_k \in \mathbb{N}).$$

If there are two different forms, choose the one for which $\lim_{k\to\infty} x_k = 0$. The so called dyadic addition is defined by

$$x \dot{+} y = \sum_{k=0}^{\infty} \frac{z_k}{2^{k+1}}, \qquad \text{where } z_k := x_k + y_k \bmod 2,\ (k \in \mathbb{N}).$$

For a dyadic interval I with length 2^{-n}, $i,j,n \in \mathbb{N}$, let us use the notation

$$I^{j,i} := I \dot{+} [0,2^{-i}) \dot{+} 2^{-j-1}.$$

Parallel, we denote $I_n(x)^{j,i} := (I_n(x))^{j,i}$. Let γ and s be two positive constants. For a martingale $f = (f_n)$, let

$$M_{\gamma,s}(f)(x) := \sup_{n\in\mathbb{N}} \sum_{m=0}^{n} \sum_{j=0}^{m} 2^{(j-n)\gamma} \sum_{i=j}^{m} 2^{(j-i)s} \frac{1}{\lambda(I_n(x)^{j,i})} \left| \int_{I_n(x)^{j,i}} f_n d\lambda \right|.$$

Of course, if $f \in L_1[0,1)$, then we can write in the definition f instead of f_n. Let us define $I_{k,n} := [k2^{-n},(k+1)2^{-n})$, where $0 \le k < 2^n$, $n \in \mathbb{N}$. The definition of $M_{\gamma,s}(f)$ can be rewritten to

$$M_{\gamma,s}(f) = \sup_{n\in\mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}} \sum_{m=0}^{n} \sum_{j=0}^{m} 2^{(j-n)\gamma} \sum_{i=j}^{m} 2^{(j-i)s} \frac{1}{\lambda(I_{k,n}^{j,i})} \left| \int_{I_{k,n}^{j,i}} f_n d\lambda \right|,$$

where $I_{k,n}^{j,i} := (I_{k,n})^{j,i}$. Now we point out four special cases of this operator.

In the first special case, if $j = i = n = m$, then $I_n(x)^{n,n} = I_n(x)$ $(n \in \mathbb{N})$ and so we get back the dyadic maximal operator:

$$\begin{aligned} M_{\gamma,s}^{(1)}(f)(x) &:= \sup_{n\in\mathbb{N}} \frac{1}{\lambda(I_n(x)^{n,n})} \left| \int_{I_n(x)^{n,n}} f_n d\lambda \right| \\ &= \sup_{n\in\mathbb{N}} \frac{1}{\lambda(I_n(x))} \left| \int_{I_n(x)} f_n d\lambda \right| \\ &= M(f)(x). \end{aligned}$$

If $j = i = m$, we have

$$M^{(2)}_{\gamma,s}(f)(x) := \sup_{n\in\mathbb{N}} \sum_{m=0}^{n} 2^{(m-n)\gamma} \frac{1}{\lambda(I_n(x)^{m,m})} \left| \int_{I_n(x)^{m,m}} f_n \, d\lambda \right|$$
$$= \sup_{n\in\mathbb{N}} \sum_{m=0}^{n} 2^{(m-n)\gamma} \frac{1}{\lambda(I_m(x))} \left| \int_{I_m(x)} f_n \, d\lambda \right|.$$

Here $I_n(x)^{m,m} = I_n(x) \dot{+} [0, 2^{-m}) \dot{+} 2^{-m-1} = x \dot{+} [0, 2^{-m}) = I_m(x)$. It is easy to see that

$$M(f) = M^{(1)}_{\gamma,s}(f) \le M^{(2)}_{\gamma,s}(f) \le CM(f)$$

and so Theorem 3.2 holds also for these two operators.

If $m = n$ and $i = n$, we get that

$$M^{(3)}_{\gamma,s}(f)(x) := \sup_{n\in\mathbb{N}} \sum_{j=0}^{n} 2^{(j-n)(\gamma+s)} \frac{1}{\lambda(I_n(x)^{j,n})} \left| \int_{I_n(x)^{j,n}} f_n \, d\lambda \right|.$$

Note that $I_n(x)^{j,n} = I_n(x) \dot{+} [0, 2^{-n}) \dot{+} 2^{-j-1} = I_n(x) \dot{+} 2^{-j-1}$.

If $m = n$, we obtain the last special case,

$$M^{(4)}_{\gamma,s}(f)(x) := \sup_{n\in\mathbb{N}} \sum_{j=0}^{n} 2^{(j-n)\gamma} \sum_{i=j}^{n} 2^{(j-i)s} \frac{1}{\lambda(I_n(x)^{j,i})} \left| \int_{I_n(x)^{j,i}} f_n \, d\lambda \right|.$$

The maximal operators $M^{(3)}_{\gamma,s}(f)$ and $M^{(4)}_{\gamma,s}(f)$ as well as $M_{\gamma,s}(f)$ cannot be estimated by $M(f)$ from above pointwise. In [32], we investigated the operators $M^{(3)}_{\gamma,s}$ and $M^{(4)}_{\gamma,s}$. Their boundedness on $L_{p(\cdot)}[0,1)$ was the key point in the proof of boundedness and convergence results for the Fejér means of the Walsh-Fourier series.

It is easy to see that

$$M(f) \le M^{(j)}_{\gamma,s}(f) \le M_{\gamma,s}(f) \qquad (j = 1, \dots, 4).$$

In [85], we generalized (6) and proved the next theorem.

Theorem 3.3. *Suppose that* $1/p(\cdot) \in C^{\log}[0,1)$, $0 < \gamma, s < \infty$, $f \in L_{p(\cdot)}[0,1)$ *and*

$$\frac{1}{p_-} - \frac{1}{p_+} < \gamma + s. \tag{8}$$

If $1 < p_- \le p_+ \le \infty$, *then*

$$\left\| M_{\gamma,s}(f) \right\|_{L_{p(\cdot)}[0,1)} \lesssim \|f\|_{L_{p(\cdot)}[0,1)}.$$

If $p_- = 1$, *then*

$$\sup_{\rho\in(0,\infty)} \rho \left\| \chi_{\{x\in[0,1):\, M_{\gamma,s}(f)(x)>\rho\}} \right\|_{L_{p(\cdot)}[0,1)} \lesssim \|f\|_{L_{p(\cdot)}[0,1)}.$$

Inequality (8) and Theorem 3.3 hold if $p_- > \max(1/(\gamma+s), 1)$. If $p_- < 1/(\gamma+s)$, then (8) is equivalent to

$$p_+ < \frac{p_-}{1 - (\gamma+s)p_-}.$$

In special cases, we proved in [32, 85] that condition (8) is important, the results are not true without this condition.

4 Variable Hardy and Hardy-Lorentz spaces

In this section, we introduce three types of Hardy spaces with variable exponents, the martingale Hardy space $H_{p(\cdot)}[0,1)$, the space $H_{p(\cdot)}(\mathbb{T})$ containing distributions and the space $H_{p(\cdot)}(\mathbb{R})$ containing tempered distributions.

4.1 Hardy and Hardy-Lorentz spaces $H_{p(\cdot)}[0,1)$ and $H_{p(\cdot),q}[0,1)$

These spaces contain the following dyadic martingales. For $p(\cdot) \in \mathcal{P}[0,1)$ and $0 < q \le \infty$, we define the variable martingale Hardy and Hardy-Lorentz spaces by

$$H_{p(\cdot)}[0,1) \quad := \quad \left\{ f = (f_n)_{n\in\mathbb{N}} : \|f\|_{H_{p(\cdot)}[0,1)} := \|M(f)\|_{L_{p(\cdot)}[0,1)} < \infty \right\}$$

and

$$H_{p(\cdot),q}[0,1) \quad := \quad \left\{ f = (f_n)_{n\in\mathbb{N}} : \|f\|_{H_{p(\cdot),q}[0,1)} := \|M(f)\|_{L_{p(\cdot),q}[0,1)} < \infty \right\},$$

respectively. Let us denote by $\mathcal{H}_{p(\cdot),\infty}[0,1)$ the closure of the dyadic step functions in $H_{p(\cdot),\infty}[0,1)$. These spaces can also be defined via equivalent norms. For a martingale $f=(f_n)_{n\geq 0}$, let

$$d_n f = f_n - f_{n-1} \qquad (n \geq 0)$$

denote the martingale differences, where $f_{-1}:=0$. The square function and the conditional square function of f are defined by

$$S(f)=\left(\sum_{n=0}^{\infty}|d_n f|^2\right)^{1/2}, \qquad s(f)=\left(|d_0 f|^2+\sum_{n=0}^{\infty}E_n|d_{n+1}f|^2\right)^{1/2}.$$

We have shown the following theorem in [32].

Theorem 4.1. *Let $p(\cdot)\in C^{\log}[0,1)$, $0<p_-\leq p_+<\infty$ and $0<q\leq\infty$. Then $f\in H_{p(\cdot)}[0,1)$ if and only if $S(f)\in L_{p(\cdot)}[0,1)$ or $s(f)\in L_{p(\cdot)}[0,1)$. Moreover, $f\in H_{p(\cdot),q}[0,1)$ if and only if $S(f)\in L_{p(\cdot),q}[0,1)$ or $s(f)\in L_{p(\cdot),q}[0,1)$. We have the following equivalences of norms:*

$$\|f\|_{H_{p(\cdot)}[0,1)}\sim\|S(f)\|_{L_{p(\cdot)}[0,1)}\sim\|s(f)\|_{L_{p(\cdot)}[0,1)}$$

and

$$\|f\|_{H_{p(\cdot),q}[0,1)}\sim\|S(f)\|_{L_{p(\cdot),q}[0,1)}\sim\|s(f)\|_{L_{p(\cdot),q}[0,1)}.$$

If in addition $1<p_-\leq p_+<\infty$, then the Hardy spaces are equivalent to the Lebesgue spaces and the Hardy-Lorentz spaces to the Lorentz spaces (see Theorem 4.4). The next result shows that the maximal functions $M_{\gamma,s}(f)$ generate equivalent characterizations for the variable Hardy and Hardy-Lorentz spaces.

Theorem 4.2. *Let $p(\cdot)\in C^{\log}[0,1)$, $0<\gamma,s<\infty$, $0<p_-\leq p_+<\infty$ and $0<q\leq\infty$. If (8) holds, then*

$$\|f\|_{H_{p(\cdot)}[0,1)}\leq\|M_{\gamma,s}(f)\|_{L_{p(\cdot)}[0,1)}\lesssim\|f\|_{H_{p(\cdot)}[0,1)} \qquad (f\in H_{p(\cdot)}[0,1))$$

and

$$\|f\|_{H_{p(\cdot),q}[0,1)}\leq\|M_{\gamma,s}(f)\|_{L_{p(\cdot),q}[0,1)}\lesssim\|f\|_{H_{p(\cdot),q}[0,1)} \qquad (f\in H_{p(\cdot),q}[0,1)).$$

4.2 Hardy and Hardy-Lorentz spaces $H_{p(\cdot)}(\mathbb{T})$ and $H_{p(\cdot),q}(\mathbb{T})$

Denote by $S(\mathbb{R}^d)$ the set of all Schwartz functions, by $S'(\mathbb{R})$ the set of all tempered distributions and by $D(\mathbb{T})$ the set of all distributions. For a distribution $f\in D(\mathbb{T})$, the nth Fourier coefficient is defined by

$$\widehat{f}(n):=f(e_n), \qquad \text{where} \qquad e_n(x):=e^{-2\pi\imath nx} \tag{9}$$

and $n\in\mathbb{Z}, x\in\mathbb{T}, \imath=\sqrt{-1}$. In special case, if $f\in L_1(\mathbb{T})$, then

$$\widehat{f}(n)=\int_{\mathbb{T}}f(x)e^{-2\pi\imath nx}\,dx \qquad (n\in\mathbb{Z}).$$

For $f\in D(\mathbb{T})$,

$$f=\sum_{n\in\mathbb{N}}\widehat{f}(n)e_n \qquad \text{in } D(\mathbb{T})$$

and $\widehat{f}(n)=O(|n|^k)$, where $k\in\mathbb{N}$ is the order of f (see Edwards [14, p. 68]). Conversely, if $c_n=O(|n|^k)$, then

$$f=\sum_{n\in\mathbb{N}}c_n e_n \qquad \text{in } D(\mathbb{T}).$$

We define the convolution of $f\in D(\mathbb{T})$ and $\psi\in L_1(\mathbb{R})$ by

$$f*\psi:=\sum_{n\in\mathbb{N}}\widehat{f}(n)\widehat{\psi}(n)e_n \qquad \text{in } D(\mathbb{T}),$$

where $\widehat{\psi}$ denotes the Fourier transform of $\psi\in L_1(\mathbb{R})$,

$$\widehat{\psi}(x):=\int_{\mathbb{R}}\psi(t)e^{-2\pi\imath xt}\,dt \qquad (x\in\mathbb{R}).$$

The Fourier transform can be extended to all tempered distributions in the usual way. For $t\in(0,\infty)$ and $\xi\in\mathbb{T}$, let

$$\psi_t(\xi):=\frac{1}{t}\psi\left(\frac{\xi}{t}\right).$$

For $f\in D(\mathbb{T})$ and $\psi\in L_1(\mathbb{R})$, we have

$$f*\psi_t=\sum_{n\in\mathbb{N}}\widehat{f}(n)\widehat{\psi}(tn)e_n \qquad \text{in } D(\mathbb{T}). \tag{10}$$

The convergence in (10) does exist because $\widehat{\psi} \in L_\infty(\mathbb{R})$. Moreover, if $\psi \in S(\mathbb{R})$, then (10) converges absolutely in each point as well. It is easy to see that

$$f * \psi_t(x) = \int_{\mathbb{R}} f(x-u)\psi_t(u)\,du \tag{11}$$

for $f \in L_1(\mathbb{T})$ and $\psi \in L_1(\mathbb{R})$.

Fix $\psi \in S(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(x)dx \neq 0$. We define the radial maximal function and the non-tangential maximal function of the distribution $f \in D(\mathbb{R}^n)$ associated to ψ by

$$\psi_+^*(f)(x) := \sup_{t\in(0,\infty)} |f * \psi_t(x)| \tag{12}$$

and

$$\psi_\triangledown^*(f)(x) := \sup_{t\in(0,\infty),|y-x|<t} |f * \psi_t(y)|, \tag{13}$$

respectively. For $N \in \mathbb{N}$, let

$$\mathcal{F}_N(\mathbb{R}) := \left\{ \psi \in S(\mathbb{R}) : \sup_{x\in\mathbb{R},\|\beta\|_1\le N} (1+|x|)^{N+d} |\partial^\beta \psi(x)| \le 1 \right\},$$

where $\|\beta\|_1 = \beta_1 + \cdots + \beta_d$. For any $N \in \mathbb{N}$, the radial grand maximal function and the non-tangential grand maximal function of $f \in D(\mathbb{R})$ are defined by

$$f_+^*(x) := \sup_{\psi\in\mathcal{F}_N(\mathbb{R})} \sup_{t\in(0,\infty)} |f * \psi_t(y)| \tag{14}$$

and

$$f_\triangledown^*(x) := \sup_{\psi\in\mathcal{F}_N(\mathbb{R})} \sup_{t\in(0,\infty),|y-x|<t} |f * \psi_t(y)|, \tag{15}$$

respectively. Let $p(\cdot) \in \mathcal{P}(\mathbb{X})$ and $0 < q \le \infty$. We introduce the number $d_{p(\cdot)} := \lfloor d(1/p_- - 1) \rfloor$ and fix a positive integer $N > d_{p(\cdot)}$, where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$. The variable Hardy and Hardy-Lorentz spaces $H_{p(\cdot)}(\mathbb{T})$ and $H_{p(\cdot),q}(\mathbb{T})$ are defined to be the sets of all $f \in D(\mathbb{R})$ such that $f_\triangledown^* \in L_{p(\cdot)}(\mathbb{T})$ and $f_\triangledown^* \in L_{p(\cdot),q}(\mathbb{T})$ equipped with the quasi-norms

$$\|f\|_{H_{p(\cdot)}(\mathbb{T})} := \left\|f_\triangledown^*\right\|_{L_{p(\cdot)}(\mathbb{T})} \qquad \text{and} \qquad \|f\|_{H_{p(\cdot),q}(\mathbb{T})} := \|f_\triangledown^*\|_{L_{p(\cdot),q}(\mathbb{T})},$$

respectively. Let us denote by $\mathcal{H}_{p(\cdot),\infty}(\mathbb{T})$ the closure of the step functions in $H_{p(\cdot),\infty}(\mathbb{T})$. We will see in Theorem 4.3 that the Hardy spaces are independent of N, more exactly, different integers N give the same space with equivalent norms.

4.3 Hardy and Hardy-Lorentz spaces $H_{p(\cdot)}(\mathbb{R})$ and $H_{p(\cdot),q}(\mathbb{R})$

Note that the convolution of $f \in L_p(\mathbb{R})$ and $\psi \in L_1(\mathbb{R})$ $(1 \le p \le \infty)$ can be defined similarly to (11). This definition can be extended to all tempered distributions $f \in S'(\mathbb{R})$ and $\psi \in S(\mathbb{R})$ by

$$(f * \psi)(h) := f\left(\check{\psi} * h\right) \qquad (h \in S(\mathbb{R})),$$

where $\check{\psi}(x) := \psi(-x)$ $(x \in \mathbb{R})$. The convolution is well defined because $\check{\psi} * h \in S(\mathbb{R})$. Indeed, we know that $\widehat{\check{\psi} * h} = \widehat{\check{\psi}}\widehat{f} \in S(\mathbb{R})$. This means that the formulas in (12), (13), (14) and (15) are well defined for tempered distributions, too. A tempered distribution f is in the spaces $H_{p(\cdot)}(\mathbb{R})$ and $H_{p(\cdot),q}(\mathbb{R})$ if

$$\|f\|_{H_{p(\cdot)}(\mathbb{R})} := \left\|f_\triangledown^*\right\|_{L_{p(\cdot)}(\mathbb{R})} \qquad \text{and} \qquad \|f\|_{H_{p(\cdot),q}(\mathbb{R})} := \|f_\triangledown^*\|_{L_{p(\cdot),q}(\mathbb{R})},$$

respectively. We denote again by $\mathcal{H}_{p(\cdot),\infty}(\mathbb{R})$ the closure of the step functions in $H_{p(\cdot),\infty}(\mathbb{R})$. As we mentioned above, for different N's, we get the same space with equivalent norms.

Theorem 4.3. *Let $\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = \mathbb{R}$ and $p(\cdot) \in C^{\log}(\mathbb{X})$, $0 < p_- \le p_+ < \infty$ and $0 < q \le \infty$. Fix $\psi \in S(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(x)dx \neq 0$ and fix a positive integer $N > d_{p(\cdot)}$. Then $f \in H_{p(\cdot)}(\mathbb{X})$ if and only if $f_\triangledown^* \in L_{p(\cdot)}(\mathbb{X})$ or $\psi_\triangledown^*(f) \in L_{p(\cdot)}(\mathbb{X})$ or $\psi_+^*(f) \in L_{p(\cdot)}(\mathbb{X})$. Moreover, $f \in H_{p(\cdot),q}(\mathbb{X})$ if and only if $f_\triangledown^* \in L_{p(\cdot),q}(\mathbb{X})$ or $\psi_\triangledown^*(f) \in L_{p(\cdot),q}(\mathbb{X})$ or $\psi_+^*(f) \in L_{p(\cdot),q}(\mathbb{X})$. We have the following equivalences of norms:*

$$\|f\|_{H_{p(\cdot)}(\mathbb{X})} \sim \|f_+^*\|_{L_{p(\cdot)}(\mathbb{X})} \sim \|\psi_+^*(f)\|_{L_{p(\cdot)}(\mathbb{X})} \sim \|\psi_\triangledown^*(f)\|_{L_{p(\cdot)}(\mathbb{X})}$$

and

$$\|f\|_{H_{p(\cdot),q}(\mathbb{X})} \sim \|f_+^*\|_{L_{p(\cdot),q}(\mathbb{X})} \sim \|\psi_+^*(f)\|_{L_{p(\cdot),q}(\mathbb{X})} \sim \|\psi_\triangledown^*(f)\|_{L_{p(\cdot),q}(\mathbb{X})}.$$

If $1 < p_- < \infty$, then the Hardy and Hardy-Lorentz spaces are equivalent to the Lebesgue and Lorentz spaces, respectively.

Theorem 4.4. *Let $\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = [0,1)$. If $1/p(\cdot) \in C^{\log}(\mathbb{X})$, $1 < p_- \le \infty$ and $0 < q \le \infty$, then*

$$H_{p(\cdot)}(\mathbb{X}) \sim L_{p(\cdot)}(\mathbb{X}), \qquad H_{p(\cdot),q}(\mathbb{X}) \sim L_{p(\cdot),q}(\mathbb{X}).$$

If $p(\cdot)$ is a constant, then we get back the classical Hardy spaces $H_p[0,1)$ and $H_p(\mathbb{R})$ investigated in Fefferman, Stein and Weiss [15, 71, 69], Lu [46], Uchiyama [75] and Weisz [76, 81]. For variable martingale and distribution Hardy spaces see the references Nakai and Sawano [49, 58], Yan et al. [89], Liu et al. [45, 44] and Jiao et al. [36, 32].

5 Atomic decomposition of the Hardy spaces $H_p(\mathbb{X})$

From what follows let $\mathbb{X}=\mathbb{T}$ or $\mathbb{X}=\mathbb{R}$ or $\mathbb{X}=[0,1)$. The atomic decomposition is a useful characterization of the Hardy spaces by the help of which some boundedness results, duality theorems, inequalities and interpolation results can be proved. The atomic decomposition of Hardy spaces with constant p was proved e.g. in Latter [41], Lu [46], Stein [69] and Weisz [81]. In this section we suppose that p is a constant.

Definition 5.1. A measurable function a is called a $H_p(\mathbb{X})$-atom if there exists an interval $I\subset\mathbb{X}$ such that

(i) $\operatorname{supp} a \subset I$,
(ii) $\|a\|_{L_\infty(\mathbb{X})} \le \lambda(I)^{-1/p}$,
(iii) $\int_{\mathbb{X}} a(x)x^\alpha dx = 0$ for all integer α with $0\le\alpha\le 1/p-1$.

If $\mathbb{X}=[0,1)$, then by an interval we mean a dyadic interval and we may suppose in (iii) that $\alpha=0$.

Every function from the Hardy space $H_p(\mathbb{X})$ $(0<p\le 1)$ can be decomposed into the sum of $H_p(\mathbb{X})$-atoms.

Theorem 5.1. *Let $0<p\le 1$ and $\mathbb{X}=\mathbb{T}$ or $\mathbb{X}=\mathbb{R}$ or $\mathbb{X}=[0,1)$. Then f is in $H_p(\mathbb{X})$ if and only if there exist a sequence $\{a_k\}_{k\in\mathbb{N}}$ of $H_p(\mathbb{X})$-atoms and a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ of positive numbers such that*

$$f=\sum_{k\in\mathbb{N}}\lambda_k a_k. \tag{16}$$

Moreover,

$$\|f\|_{H_p(\mathbb{X})} \sim \inf\left(\sum_{k\in\mathbb{N}}\lambda_k^p\right)^{1/p}, \tag{17}$$

where the infimum is taken over all decompositions of f as above.

In (16), we mean the convergence in the sense of distributions, or tempered distributions or martingales if $\mathbb{X}=\mathbb{T}$ or $\mathbb{X}=\mathbb{R}$ or $\mathbb{X}=[0,1)$, respectively. Note that the Hardy space $H_p(\mathbb{T})$ contains distributions, $H_p(\mathbb{R})$ contains tempered distributions and $H_p[0,1)$ contains martingales.

The theorem is not true for $1<p<\infty$ and it cannot be extended to variable Hardy spaces in the present form. However, using the following ideas, we will be able to extend the atomic decomposition to all $0<p<\infty$ and to variable Hardy spaces as well as to Hardy-Lorentz spaces in the next section. First of all, observe that (ii) of Definition 5.1 is the same as

$$\|a\|_{L_\infty(\mathbb{X})} \le \lambda(I)^{-1/p} = \|\chi_I\|_{L_p(\mathbb{X})}^{-1}.$$

Secondly, for $0<p\le 1$, (17) can be written as

$$\|f\|_{H_p(\mathbb{X})} \sim \inf\left(\sum_{k\in\mathbb{N}}\lambda_k^p\right)^{1/p} = \inf\left\|\left(\sum_{k\in\mathbb{N}}\left(\frac{\lambda_k\chi_{I_k}}{\|\chi_{I_k}\|_{L_p(\mathbb{X})}}\right)^p\right)^{1/p}\right\|_{L_p(\mathbb{X})}, \tag{18}$$

where I_k is the support of the p-atom a_k. Writing $L_{p(\cdot)}(\mathbb{X})$ instead of $L_p(\mathbb{X})$, we can generalize this form of the atomic decomposition to variable Hardy spaces.

6 Atomic decomposition of the variable Hardy and Hardy-Lorentz spaces

The atomic decomposition of variable Hardy and Hardy-Lorentz spaces were studied in Nakai and Sawano [49, 58], Yan et al. [89], Liu et al. [45, 44], Jiao et al. [36, 32] and Weisz [84].

Definition 6.1. A measurable function a is called a $H_{p(\cdot)}(\mathbb{X})$-atom if there exists an interval $I\subset\mathbb{X}$ such that

(i) $\operatorname{supp} a \subset I$,
(ii) $\|a\|_{L_\infty(\mathbb{X})} \le \|\chi_I\|_{L_{p(\cdot)}(\mathbb{X})}^{-1}$,
(iii) $\int_{\mathbb{X}} a(x)x^\alpha dx = 0$ for all integer α with $0\le\alpha\le 1/p_- -1$.

If $\mathbb{X}=[0,1)$, then I is a dyadic interval and we suppose in (iii) that $\alpha=0$.

Theorem 6.1. *Let $\mathbb{X}=\mathbb{T}$ or $\mathbb{X}=\mathbb{R}$ or $\mathbb{X}=[0,1)$ and $p(\cdot)\in C^{\log}(\mathbb{X})$, $0<p_-\le p_+<\infty$. Then f is in $H_{p(\cdot)}(\mathbb{X})$ if and only if there exist a sequence $\{a_k\}_{k\in\mathbb{N}}$ of $H_{p(\cdot)}(\mathbb{X})$-atoms with support $\{I_k\}_{k\in\mathbb{N}}$ and a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ of positive numbers such that*

$$f=\sum_{k\in\mathbb{N}}\lambda_k a_k. \tag{19}$$

Moreover,

$$\|f\|_{H_{p(\cdot)}(\mathbb{X})} \sim \inf\left\|\left(\sum_{k\in\mathbb{N}}\left(\frac{\lambda_k\chi_{I_k}}{\|\chi_{I_k}\|_{L_{p(\cdot)}(\mathbb{X})}}\right)^{\underline{p}}\right)^{1/\underline{p}}\right\|_{L_{p(\cdot)}(\mathbb{X})},$$

where the infimum is taken over all decompositions of f as above.

Here we use the notation $\underline{p} = \min\{p_-, 1\}$. If $p(\cdot) = p$ is a constant and $0 < p \le 1$, then $\underline{p} = p$ and we get back (18).

The atomic decomposition of variable Hardy-Lorentz spaces have a slightly different form.

Theorem 6.2. *Let $\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = [0,1)$ and $p(\cdot) \in C^{\log}(\mathbb{X})$, $0 < p_- \le p_+ < \infty$ and $0 < q \le \infty$. Then f is in $H_{p(\cdot),q}(\mathbb{X})$ if and only if there exist a sequence $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ of $H_{p(\cdot)}(\mathbb{X})$-atoms with support $\{I_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ such that*

$$f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j}, \tag{20}$$

where $\sum_{j\in\mathbb{N}} \chi_{I_{i,j}}(x) \le A$ for all $x \in \mathbb{X}$ and $i \in \mathbb{Z}$ and $\lambda_{i,j} := C2^i \|\chi_{I_{i,j}}\|_{L_{p(\cdot)}(\mathbb{X})}$ $(i \in \mathbb{Z}, j \in \mathbb{N})$ with A and C being positive constants. Moreover,

$$\|f\|_{H_{p(\cdot),q}(\mathbb{X})} \sim \inf \left(\sum_{i\in\mathbb{Z}} \left\| \left(\sum_{j\in\mathbb{N}} \left(\frac{\lambda_{i,j} \chi_{I_{i,j}}}{\|\chi_{I_{i,j}}\|_{L_{p(\cdot)}(\mathbb{X})}} \right)^{\underline{p}} \right)^{1/\underline{p}} \right\|_{L_{p(\cdot)}(\mathbb{X})}^q \right)^{1/q},$$

where the infimum is taken over all decompositions of f as above and with the usual modification for $q = \infty$.

Note that for $q = \infty$, we have the ℓ_∞-norm on the right hand side, i.e.,

$$\|f\|_{H_{p(\cdot),\infty}(\mathbb{X})} \sim \inf \left(\sup_{i\in\mathbb{Z}} \left\| \left(\sum_{j\in\mathbb{N}} \left(\frac{\lambda_{i,j} \chi_{I_{i,j}}}{\|\chi_{I_{i,j}}\|_{L_{p(\cdot)}(\mathbb{X})}} \right)^{\underline{p}} \right)^{1/\underline{p}} \right\|_{L_{p(\cdot)}(\mathbb{X})} \right).$$

In (19) and (20), the convergences are understood again in the sense of distributions, or tempered distributions or martingales if $\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = [0,1)$, respectively.

In the dyadic case, i.e., if $\mathbb{X} = [0,1)$, we [32, 33] have proved Theorem 6.2 under the condition that the intervals $I_{i,j}$ are disjoint for every fixed $i \in \mathbb{Z}$. Taking a closer look at the proofs in those papers, we can prove the present form of Theorem 6.2 in the same way for $\mathbb{X} = [0,1)$. These two theorems will be applied in the next sections.

7 Fourier series and Fourier transforms

Now we turn to some applications in Fourier analysis. For $\mathbb{X} = \mathbb{T}$, we will consider the trigonometric Fourier series, for $\mathbb{X} = [0,1)$, the Walsh Fourier series and for $\mathbb{X} = \mathbb{R}$, the inverse Fourier transform of the Fourier transform.

7.1 Trigonometric and Walsh-Fourier series

For $\mathbb{X} = \mathbb{T}$, we introduce the trigonometric system in the usual way by

$$w_n(x) := e_n(x) = e^{-2\pi \imath n x} \qquad (x \in \mathbb{T}, n \in \mathbb{Z}).$$

For $\mathbb{X} = [0,1)$, we introduce the Walsh system as follows. The Rademacher functions are defined by

$$r_n(x) := r(2^n x) \qquad (x \in [0,1), n \in \mathbb{N}),$$

where

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2}, 1). \end{cases}$$

It is clear that, for any $n \in \mathbb{N}$, r_n is $\mathcal{F}_{n+1}$ measurable. The product system generated by the Rademacher functions is the Walsh system:

$$w_n := \prod_{k=0}^{\infty} r_k^{n_k} \qquad (n \in \mathbb{N}),$$

where

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad (n_k = 0, 1).$$

Let

$$w_n = 0 \qquad \text{if} \qquad n \in \mathbb{Z} \setminus \mathbb{N}.$$

Thus for $\mathbb{X} = \mathbb{T}$, $(w_n)_{n\in\mathbb{N}}$ denotes the trigonometric system and for $\mathbb{X} = [0,1)$, the Walsh system.

If $f \in L_1(\mathbb{X})$ and $n \in \mathbb{N}$, the number

$$\widehat{f}(n) := \int_{\mathbb{X}} f w_n \, d\lambda$$

is called the nth trigonometric-Fourier (resp. Walsh-Fourier) coefficient of f. If $\mathbb{X}=\mathbb{T}$ and f is a distribution, $\widehat{f}(n)$ was defined in (9). If $\mathbb{X}=[0,1)$, we can extend this definition to martingales as follows. If $f:=(f_k)_{k\in\mathbb{N}}$ is a martingale, then let

$$\widehat{f}(n):=\lim_{k\to\infty}\int_0^1 f_k w_n\,d\lambda \qquad (n\in\mathbb{N}).$$

Denote by $s_n f$ the nth partial sum of the trigonometric- or Walsh–Fourier series of a martingale f, namely,

$$s_n f:=\sum_{k=-n}^{n}\widehat{f}(k)w_k \qquad (n\in\mathbb{N}),$$

where w_n denotes the trigonometric or Walsh system.

It is easy to see that, for any martingale $f=(f_n)$,

$$s_{2^n}f=E_n(f)=f_n \qquad (n\in\mathbb{N}).$$

Hence, by the martingale convergence theorem, we know that, for $1\le p<\infty$ and $f\in L_p[0,1)$,

$$\lim_{n\to\infty} s_{2^n}f=f \qquad \text{in the } L_p[0,1)\text{-norm}.$$

This result was generalized in Zygmund [91], Paley [50] and Schipp et al. [62, Theorem 4.1].

Theorem 7.1. *Let $\mathbb{X}=\mathbb{T}$ or $\mathbb{X}=[0,1)$. If $f\in L_p(\mathbb{X})$ for some $1<p<\infty$, then*

$$\sup_{n\in\mathbb{N}}\|s_n f\|_{L_p(\mathbb{X})}\lesssim\|f\|_{L_p(\mathbb{X})}$$

and

$$\lim_{n\to\infty} s_n f=f \qquad \textit{in the } L_p(\mathbb{X})\textit{-norm}.$$

Using the conjugate functions, the theorem was verified e.g. in Zygmund [91] or Grafakos [27] for trigonometric Fourier series. For Walsh-Fourier series, we can prove that $s_n f$ is a martingale transform of $f w_n$ and the theorem follows from the boundedness of the martingale transform on $L_p[0,1)$ and from Theorem 4.1 (see Schipp et al. [62], Persson et al. [55] or Weisz [88]).

One of the deepest results in harmonic analysis is Carleson's result (see Carleson [8] and Hunt [30] for Fourier series and Billard [5] and Sjölin [68] for Walsh-Fourier series). More detailed proof for Fourier series can also be found in Arias de Reyna [3], Grafakos [27], Muscalu and Schlag [48], Lacey [40] or Demeter [11]. Using the theory of tree martingales, Schipp [60] gave a nice proof for Walsh-Fourier series (see also [63, 76] and Persson et al. [51, 55]).

Theorem 7.2. *Let $\mathbb{X}=\mathbb{T}$ or $\mathbb{X}=[0,1)$. If $f\in L_p(\mathbb{X})$ for some $1<p<\infty$, then*

$$\Big\|\sup_{n\in\mathbb{N}}|s_n f|\Big\|_{L_p(\mathbb{X})}\lesssim\|f\|_{L_p(\mathbb{X})}$$

and

$$\lim_{n\to\infty} s_n f(x)=f(x) \qquad \textit{for a.e. } x\in\mathbb{X}.$$

7.2 Fourier transforms

Recall that the Fourier transform was defined in Subsection 4.2. If $f\in L_p(\mathbb{R})$ for some $1\le p\le 2$, then

$$f(x)=\int_{\mathbb{R}}\widehat{f}(t)e^{2\pi\imath x\cdot t}\,dt \qquad (x\in\mathbb{R})$$

holds if $\widehat{f}\in L_1(\mathbb{R})$. This motivates the definition of the Dirichlet integrals,

$$s_T f(x):=\int_{-T}^{T}\widehat{f}(t)e^{2\pi\imath xt}\,dt \qquad (x\in\mathbb{R}).$$

The analogues of Theorems 7.1 and 7.2 reads as follows (see Grafakos [27] or Weisz [81]).

Theorem 7.3. *If $f\in L_p(\mathbb{R})$ for some $1<p<\infty$, then*

$$\sup_{T>0}\|s_T f\|_{L_p(\mathbb{R})}\lesssim\|f\|_{L_p(\mathbb{R})}$$

and

$$\lim_{T\to\infty} s_T f=f \qquad \textit{in the } L_p(\mathbb{R})\textit{-norm}.$$

Theorem 7.4. *If $f\in L_p(\mathbb{R})$ for some $1<p<\infty$, then*

$$\Big\|\sup_{T>0}s_T f\Big\|_{L_p(\mathbb{R})}\lesssim\|f\|_{L_p(\mathbb{R})}$$

and

$$\lim_{T\to\infty} s_T f=f \qquad \textit{for a.e. } x\in\mathbb{R}.$$

8 Summability of Fourier series and Fourier transforms

Though Theorems 7.1–7.4 are not true for $p=1$ and $p=\infty$, with the help of some summability methods they can be generalized for these endpoint cases, too. Obviously, summability means have better convergence properties than the original Fourier series. Summability is intensively studied in the literature, see e.g. the books Stein and Weiss [71], Butzer and Nessel [7], Trigub and Belinsky [74], Grafakos [27] and Weisz [79, 80, 81, 87, 55] and the references therein.

8.1 Summability of trigonometric and Walsh-Fourier series

Fejér [16] investigated the arithmetic means of the partial sums of the trigonometric Fourier series, the so-called Fejér means, and proved that if the left and right limits $f(x-0)$ and $f(x+0)$ exist at a point x, then the Fejér means converge to $(f(x-0)+f(x+0))/2$. One year later Lebesgue [42] extended this theorem and obtained that every integrable function is Fejér summable almost everywhere.

Recall that the Fejér means are defined by

$$\sigma_n f(x) := \sum_{k=-n}^{n}\left(1-\frac{|k|}{n}\right)\widehat{f}(k)w_k(x)$$

for any $n\in\mathbb{N}$ and $x\in\mathbb{X}$, where $\mathbb{X}=\mathbb{T}$ or $\mathbb{X}=[0,1)$ and (w_n) denotes either the trigonometric system or the Walsh system. It is easy to see that the Fejér means can also be written in the form

$$\sigma_n f(x) = \frac{1}{n}\sum_{j=0}^{n-1} s_j f(x).$$

Now we generalize these means and introduce the Cesàro and Riesz means. For $\alpha\neq -1,-2,\ldots$, let $A_{-1}^{\alpha}:=0$ and

$$A_n^{\alpha} := \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} \qquad (n\in\mathbb{N}).$$

Obviously, $A_0^{\alpha}=1$ and if $\alpha=0$, then $A_n^0=1$, if $\alpha=1$, then $A_n^1=n+1$ $(n\in\mathbb{N})$.

For $n\in\mathbb{N}$ and $\alpha\geq 0$, the nth Cesàro mean $\sigma_n^{\alpha}f$ of the trigonometric or Walsh-Fourier series of f is introduced by

$$\sigma_n^{\alpha} f(x) := \frac{1}{A_{n-1}^{\alpha}}\sum_{k=-n}^{n} A_{n-1-|k|}^{\alpha}\widehat{f}(k)w_k(x).$$

We can easily check that $\alpha=0$ gives back the partial sums and $\alpha=1$ the Fejér means. Using the equalities

$$A_n^{\alpha+\beta+1} = \sum_{k=0}^{n} A_k^{\alpha} A_{n-k}^{\beta}$$

and

$$A_n^{\alpha} = \sum_{k=0}^{n} A_k^{\alpha-1}, \qquad A_n^{\alpha} - A_{n-1}^{\alpha} = A_n^{\alpha-1}$$

$(\alpha,\beta\neq -1,-2,\ldots,n\in\mathbb{N})$, we can show that

$$\sigma_n^{\alpha} f(x) = \frac{1}{A_{n-1}^{\alpha}}\sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} s_j f(x).$$

For an intensive study of the Cesàro summability see Weisz [87].

For $0<\alpha<\infty, 1\leq\gamma<\infty$, we define the Riesz means of f by

$$\sigma_n^{\alpha,\gamma} f(x) := \frac{1}{n^{\alpha\gamma}}\sum_{k=-n}^{n}(n^{\gamma}-|k|^{\gamma})^{\alpha}\widehat{f}(k)w_k(x) \qquad (1\leq n\in\mathbb{N}).$$

If $\alpha=\gamma=1$, these means give back also the Fejér means. We will always suppose that $0<\alpha\leq 1$, since $1<\alpha<\infty$ can be traced back to the case $0<\alpha\leq 1$ for both the Cesàro and Riesz means.

The following theorem follows from Theorem 7.1.

Theorem 8.1. *Let $\mathbb{X}=\mathbb{T}$ or $\mathbb{X}=[0,1)$ and $0<\alpha\leq 1\leq\gamma$. If $f\in L_p(\mathbb{X})$ for some $1\leq p<\infty$, then*

$$\sup_{n\in\mathbb{N}}\left\|\sigma_n^{\alpha} f\right\|_{L_p(\mathbb{X})} \lesssim \|f\|_{L_p(\mathbb{X})}$$

and

$$\lim_{n\to\infty}\sigma_n^{\alpha} f = f \qquad \text{in the } L_p(\mathbb{X})\text{-norm.}$$

The same results hold for the Riesz means $\sigma_n^{\alpha,\gamma} f$.

To obtain almost everywhere convergence for the summability means, we introduce the maximal operators

$$\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha f|, \qquad \sigma_*^{\alpha,\gamma} f := \sup_{n\in\mathbb{N}} |\sigma_n^{\alpha,\gamma} f|.$$

The following result is due to the author of [77, 79, 80, 87].

Theorem 8.2. *Let $\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = [0,1)$ and $0 < \alpha \le 1 \le \gamma$. If*

$$\frac{1}{\alpha+1} < p \le \infty \tag{21}$$

and $f \in H_p(\mathbb{X})$, then

$$\left\|\sigma_*^\alpha f\right\|_{L_p(\mathbb{X})} \lesssim \|f\|_{H_p(\mathbb{X})}. \tag{22}$$

If $f \in L_p(\mathbb{X})$ with $1 \le p \le \infty$, then

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x) \qquad \text{for a.e. } x \in \mathbb{X}. \tag{23}$$

The same results hold for the Riesz means $\sigma_n^{\alpha,\gamma} f$.

The almost everywhere convergence (23) follows from inequality (22) and from a density argument due to Marcinkiewicz and Zygmund [47]. For trigonometric Fourier series, (23) was first proved by Fejér [16] and Lebesgue [42] with $\alpha = 1$, by Riesz [56] for other α's and, for Walsh-Fourier series, by Fine [17], Schipp [59] and Weisz [78].

For variable Hardy and Hardy-Lorentz spaces, we replace the condition $1/(\alpha+1) < p < \infty$ by $1/(\alpha+1) < p_- < \infty$ and we need an additional condition, too. The proof of the next theorem is very complicated and it is based on the atomic decomposition Theorems 6.1 and 6.2 and on the boundedness of the Hardy-Littlewood and Doob's maximal operator, namely, on Theorems 3.2, 3.3 and 4.2 (see [32, 72, 83, 84].

Theorem 8.3. *Suppose that $0 < \alpha \le 1 \le \gamma$, $p(\cdot) \in C^{\log}(\mathbb{X})$, $0 < q < \infty$ and*

$$\frac{1}{\alpha+1} < p_- \le p_+ < \infty. \tag{24}$$

Let $\mathbb{X} = \mathbb{T}$. Then

$$\left\|\sigma_*^\alpha f\right\|_{L_{p(\cdot)}(\mathbb{X})} \lesssim \|f\|_{H_{p(\cdot)}(\mathbb{X})} \qquad (f \in H_{p(\cdot)}(\mathbb{X}))$$

and

$$\left\|\sigma_*^\alpha f\right\|_{L_{p(\cdot),q}(\mathbb{X})} \lesssim \|f\|_{H_{p(\cdot),q}(\mathbb{X})} \qquad (f \in H_{p(\cdot),q}(\mathbb{X})).$$

If $f \in L_{p(\cdot)}(\mathbb{X})$ (resp. $f \in L_{p(\cdot),q}(\mathbb{X})$) with $p_- > 1$, then

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x) \qquad \text{for a.e. } x \in \mathbb{X}$$

as well as in the $L_{p(\cdot)}(\mathbb{X})$-norm (resp. in the $L_{p(\cdot),q}(\mathbb{X})$-norm). The almost everywhere convergence holds also if $p_- = 1$. The theorem holds for $q = \infty$ as well if we change $H_{p(\cdot),\infty}(\mathbb{X}^d)$ by $\mathcal{H}_{p(\cdot),\infty}(\mathbb{X}^d)$. The same results hold for the Riesz means $\sigma_n^{\alpha,\gamma} f$.

The same results hold also for Walsh-Fourier series, i.e., for $\mathbb{X} = [0,1)$, if we suppose in addition that

$$\frac{1}{p_-} - \frac{1}{p_+} < 1. \tag{25}$$

This theorem obviously generalizes Theorem 8.2 as well as the famous Lebesgue theorem about the almost everywhere convergence of the Fejér means to the original integrable function which was mentioned in the beginning of Section 8.

Now we give some special cases for (25) to hold. If $1 \le p_- < \infty$, then (25) holds obviously for all p_+. Moreover, if $p_- < 1$, then

$$\frac{1}{\alpha+1} < p_- \iff \frac{1}{\alpha} < \frac{p_-}{1-p_-} < \infty$$

and

$$\frac{1}{p_-} - \frac{1}{p_+} < 1 \iff p_+ < \frac{p_-}{1-p_-}.$$

Thus if

$$\frac{1}{\alpha+1} < p_- \le p_+ < \frac{p_-}{1-p_-},$$

then (25) holds.

The conditions (24) and (25) are both necessary. If one of the conditions does not hold, then the corresponding theorem is not true (see [32]). Condition (25) is a very surprising condition because we need it for the Walsh-Fourier series but we do not need it for the trigonometric Fourier series or for the Fourier transforms (see also the next subsection). Usually the theorems are very similar for Walsh-Fourier series and for trigonometric Fourier series, there are only a few cases when there are differences in the corresponding theorems. Though the theorems are similar for the two function systems, the proofs are entirely different usually.

8.2 Summability of Fourier transforms

For $T>0$, we define the Fejér means of Fourier transforms by

$$\sigma_T f(x) := \int_{-T}^{T} \left(1-\frac{|t|}{T}\right)\widehat{f}(t)e^{2\pi\imath xt}\,dt = \frac{1}{T}\int_0^T s_t f(x)\,dt.$$

We can generalize these means again by the Riesz means

$$\sigma_T f(x) := \int_{-T}^{T} \left(1-\left(\frac{|t|}{T}\right)^{\gamma}\right)^{\alpha}\widehat{f}(t)e^{2\pi\imath xt}\,dt,$$

where $0<\alpha<\infty, 1\le\gamma<\infty$, while the Cesàro means cannot be defined. The maximal Riesz operator is given by

$$\sigma_*^{\alpha,\gamma} f := \sup_{T>0}\left|\sigma_T^{\alpha,\gamma} f\right|.$$

The corresponding theorems of the preceding subsection read as follows.

Theorem 8.4. *Let $0<\alpha\le 1\le\gamma$. If $f\in L_p(\mathbb{R})$ for some $1\le p<\infty$, then*

$$\sup_{T>0}\left\|\sigma_T^{\alpha,\gamma} f\right\|_{L_p(\mathbb{R})} \lesssim \|f\|_{L_p(\mathbb{R})}$$

and

$$\lim_{T\to\infty}\sigma_T^{\alpha,\gamma} f = f \qquad \text{in the } L_p(\mathbb{R})\text{-norm}.$$

The following result was proved in Weisz [79, 81].

Theorem 8.5. *Let $0<\alpha\le 1\le\gamma$. If (21) holds and $f\in H_p(\mathbb{R})$, then*

$$\left\|\sigma_*^{\alpha,\gamma} f\right\|_{L_p(\mathbb{R})} \lesssim \|f\|_{H_p(\mathbb{R})}.$$

If $f\in L_p(\mathbb{R})$ with $1\le p<\infty$, then

$$\lim_{T\to\infty}\sigma_T^{\alpha,\gamma} f(x) = f(x) \qquad \text{for a.e. } x\in\mathbb{R}.$$

The following result generalizes Theorems 8.4 and 8.5 to variable Hardy and Hardy-Lorentz spaces (see [44, 45, 82]). We point out that we do not need condition (25) in the next theorem. In the proof, we use again the atomic of decomposition Theorem 6.2 and the boundedness of Theorem 3.2.

Theorem 8.6. *If $0<\alpha\le 1\le\gamma$, $p(\cdot)\in C^{\log}(\mathbb{R})$, $0<q<\infty$ and (24) holds, then*

$$\left\|\sigma_*^{\alpha,\gamma} f\right\|_{L_{p(\cdot)}(\mathbb{R})} \lesssim \|f\|_{H_{p(\cdot)}(\mathbb{R})} \qquad (f\in H_{p(\cdot)}(\mathbb{R}))$$

and

$$\left\|\sigma_*^{\alpha,\gamma} f\right\|_{L_{p(\cdot),q}(\mathbb{R})} \lesssim \|f\|_{H_{p(\cdot),q}(\mathbb{R})} \qquad (f\in H_{p(\cdot),q}(\mathbb{R})).$$

If $f\in L_{p(\cdot)}(\mathbb{R})$ (resp. $f\in L_{p(\cdot),q}(\mathbb{R})$) with $p_->1$, then

$$\lim_{T\to\infty}\sigma_T^{\alpha,\gamma} f(x) = f(x) \qquad \text{for a.e. } x\in\mathbb{R}$$

as well as in the $L_{p(\cdot)}(\mathbb{R})$-norm (resp. in the $L_{p(\cdot),q}(\mathbb{R})$-norm). The almost everywhere convergence holds also if $p_-=1$. The theorem holds for $q=\infty$ as well if we change $H_{p(\cdot),\infty}(\mathbb{R}^d)$ by $\mathcal{H}_{p(\cdot),\infty}(\mathbb{R}^d)$.

References

[1] E. Acerbi and G. Mingione. Gradient estimates for the $p(x)$-Laplacian system. *J. Reine Angew. Math.*, 584:117–148, 2005.

[2] A. Almeida, P. Harjulehto, P. Hästö, and T. Lukkari. Riesz and Wolff potentials and elliptic equations in variable exponent weak Lebesgue spaces. *Ann. Mat. Pura Appl. (4)*, 194(2):405–424, 2015.

[3] J. Arias de Reyna. *Pointwise Convergence of Fourier Series.* Berlin: Springer, 2002.

[4] J. Bergh and J. Löfström. *Interpolation Spaces, an Introduction*. Springer, Berlin, 1976.

[5] P. Billard. Sur la convergence presque partout des séries de Fourier-Walsh des fonctions de l'espace $L^2[0,1]$. *Studia Math.*, 28:363–388, 1967.

[6] D. Breit, L. Diening, and S. Schwarzacher. Finite element approximation of the $p(\cdot)$-Laplacian. *SIAM J. Numer. Anal.*, 53(1):551–572, 2015.

[7] P. L. Butzer and R. J. Nessel. *Fourier Analysis and Approximation*. Birkhäuser Verlag, Basel, 1971.

[8] L. Carleson. On convergence and growth of partial sums of Fourier series. *Acta Math.*, 116:135–157, 1966.

[9] Y. Chen, S. Levine, and M. Rao. Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.*, 66(4):1383–1406, 2006.

[10] D. V. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue spaces. Foundations and Harmonic Analysis*. New York, NY: Birkhäuser/Springer, 2013.

[11] C. Demeter. A guide to Carleson's theorem. *Rocky Mt. J. Math.*, 45(1):169–212, 2015.

[12] L. Diening, P. Harjulehto, P. Hästö, and M. Ružička. *Lebesgue and Sobolev spaces with Variable Exponents.* Springer, Berlin, 2011.

[13] L. Diening and S. Schwarzacher. Global gradient estimates for the $p(\cdot)$-Laplacian. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 106:70–85, 2014.

[14] R. E. Edwards. *Fourier Series, A Modern Introduction*, volume 2. Springer, Berlin, 1982.

[15] C. Fefferman and E. M. Stein. H^p spaces of several variables. *Acta Math.*, 129:137–194, 1972.

[16] L. Fejér. Untersuchungen über Fouriersche Reihen. *Math. Ann.*, 58:51–69, 1904.

[17] N. J. Fine. Cesàro summability of Walsh-Fourier series. *Proc. Nat. Acad. Sci. USA*, 41:558–591, 1955.

[18] N. Fujii. A maximal inequality for H^1-functions on a generalized Walsh-Paley group. *Proc. Amer. Math. Soc.*, 77:111–116, 1979.

[19] G. Gát. On $(C,1)$ summability for Vilenkin-like systems. *Studia Math.*, 144:101–120, 2001.

[20] G. Gát. On the pointwise convergence of Cesàro means of two-variable functions with respect to unbounded Vilenkin systems. *J. Approx. Theory*, 128(1):69–99, 2004.

[21] G. Gát. Pointwise convergence of cone-like restricted two-dimensional $(C,1)$ means of trigonometric Fourier series. *J. Approx. Theory.*, 149:74–102, 2007.

[22] G. Gát and U. Goginava. Norm convergence of logarithmic means on unbounded Vilenkin groups. *Banach J. Math. Anal.*, 12(2):422–438, 2018.

[23] G. Gát and U. Goginava. Maximal operators of Cesaro means with varying parameters of Walsh-Fourier series. *Acta Math. Hung.*, 159(2):653–668, 2019.

[24] G. Gát, U. Goginava, and K. Nagy. On the Marcinkiewicz-Fejér means of double Fourier series with respect to Walsh-Kaczmarz system. *Studia Sci. Math. Hungar.*, 46:399–421, 2009.

[25] U. Goginava. Marcinkiewicz-Fejér means of d-dimensional Walsh-Fourier series. *J. Math. Anal. Appl.*, 307:206–218, 2005.

[26] U. Goginava. Weak type inequality for the maximal operator of the (C,α) means of two-dimensional Walsh-Fourier series. *Anal. Math.*, 36:1–31, 2010.

[27] L. Grafakos. *Classical and Modern Fourier Analysis*. Pearson Education, New Jersey, 2004.

[28] Z. Hao and Y. Jiao. Fractional integral on martingale Hardy spaces with variable exponents. *Fract. Calc. Appl. Anal.*, 18(5):1128–1145, 2015.

[29] P. Harjulehto and P. Hästö. *Orlicz spaces and generalized Orlicz spaces.*, volume 2236. Cham: Springer, 2019.

[30] R. A. Hunt. On the convergence of Fourier series. In *Orthogonal Expansions and their Continuous Analogues, Proc. Conf. Edwardsville, Ill., 1967*, pages 235–255. Illinois Univ. Press Carbondale, 1968.

[31] M. Izuki, E. Nakai, and Y. Sawano. Function spaces with variable exponents – an introduction. *Sci. Math. Jpn.*, 77(2):187–315, 2014.

[32] Y. Jiao, F. Weisz, L. Wu, and D. Zhou. Variable martingale Hardy spaces and their applications in Fourier analysis. *Dissertationes Math.*, 550:1–67, 2020.

[33] Y. Jiao, F. Weisz, L. Wu, and D. Zhou. Dual spaces for variable martingale Lorentz-Hardy spaces. *Banach J. Math. Anal.*, 15:53, 2021.

[34] Y. Jiao, D. Zhou, Z. Hao, and W. Chen. Martingale Hardy spaces with variable exponents. *Banach J. Math*, 10:750–770, 2016.

[35] Y. Jiao, D. Zhou, F. Weisz, and Z. Hao. Corrigendum: Fractional integral on martingale Hardy spaces with variable exponents. *Fract. Calc. Appl. Anal.*, 20:1051–1052, 2017.

[36] Y. Jiao, Y. Zuo, D. Zhou, and L. Wu. Variable Hardy-Lorentz spaces $H^{p(\cdot),q}(\mathbb{R}^n)$. *Math. Nachr.*, 292:309–349, 2019.

[37] H. Kempka and J. Vybíral. Lorentz spaces with variable exponents. *Math. Nachr.*, 287:938–954, 2014.

[38] V. Kokilashvili, A. Meskhi, H. Rafeiro, and S. Samko. *Integral Operators in Non-standard Function Spaces. Volume 1: Variable Exponent Lebesgue and Amalgam Spaces.*, volume 248. Basel: Birkhäuser/Springer, 2016.

[39] V. Kokilashvili, A. Meskhi, H. Rafeiro, and S. Samko. *Integral Operators in Non-standard Function Spaces. Volume 2: Variable Exponent Hölder, Morrey-Campanato and Grand Spaces.*, volume 249. Basel: Birkhäuser/Springer, 2016.

[40] M. T. Lacey. Carleson's theorem: proof, complements, variations. *Publ. Mat., Barc.*, 48(2):251–307, 2004.

[41] R. H. Latter. A characterization of $H^p(\mathbf{R}^n)$ in terms of atoms. *Studia Math.*, 62:92–101, 1978.

[42] H. Lebesgue. Recherches sur la convergence des séries de Fourier. *Math. Ann.*, 61:251–280, 1905.

[43] T. Lin. $L^{p(\cdot),\lambda(\cdot)}$ regularity for fully nonlinear elliptic equations. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 149:117–129, 2017.

[44] J. Liu, F. Weisz, D. Yang, and W. Yuan. Variable anisotropic Hardy spaces and their applications. *Taiwanese J. Math.*, 22:1173–1216, 2018.

[45] J. Liu, F. Weisz, D. Yang, and W. Yuan. Littlewood-Paley and finite atomic characterizations of anisotropic variable Hardy-Lorentz spaces and their applications. *J. Fourier Anal. Appl.*, 25:874–922, 2019.

[46] S. Lu. *Four Lectures on Real H^p Spaces*. World Scientific, Singapore, 1995.

[47] J. Marcinkiewicz and A. Zygmund. On the summability of double Fourier series. *Fund. Math.*, 32:122–132, 1939.

[48] C. Muscalu and W. Schlag. *Classical and Multilinear Harmonic Analysis. Volume II.* Cambridge: Cambridge University Press, 2013.

[49] E. Nakai and Y. Sawano. Hardy spaces with variable exponents and generalized Campanato spaces. *J. Funct. Anal.*, 262(9):3665–3748, 2012.

[50] R. E. A. C. Paley. A remarkable system of orthogonal functions. *Proc. Lond. Math. Soc.*, 34:241–279, 1932.

[51] L. Persson, F. Schipp, G. Tephnadze, and F. Weisz. An analogy of the Carleson-Hunt theorem with respect to Vilenkin systems. *J. Fourier Anal. Appl.*, 28:48, 2022.

[52] L. E. Persson and G. Tephnadze. A sharp boundedness result concerning some maximal operators of Vilenkin–Fejér means. *Mediterr. J. Math.*, 13(4):1841–1853, 2016.

[53] L. E. Persson, G. Tephnadze, and P. Wall. Maximal operators of Vilenkin-Nörlund means. *J. Fourier Anal. Appl.*, 21(1):76–94, 2015.

[54] L. E. Persson, G. Tephnadze, and P. Wall. On the Nörlund logarithmic means with respect to Vilenkin system in the martingale Hardy space H_1. *Acta Math. Hung.*, 154(2):289–301, 2018.

[55] L. E. Persson, G. Tephnadze, and F. Weisz. *Martingale Hardy Spaces and Summability of One-Dimensional Vilenkin-Fourier Series*. Springer, Birkhäuser, Basel, 2022.

[56] M. Riesz. Sur la sommation des séries de Fourier. *Acta Sci. Math. (Szeged)*, 1:104–113, 1923.

[57] M. Ruzicka. *Electrorheological fluids: modeling and mathematical theory.*, volume 1748 of *Lect. Notes Math.* Berlin: Springer, 2000.

[58] Y. Sawano. Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operators. *Integral Equ. Oper. Theory*, 77:123–148, 2013.

[59] F. Schipp. Über gewissen Maximaloperatoren. *Ann. Univ. Sci. Budapest Sect. Math.*, 18:189–195, 1975.

[60] F. Schipp. Universal contractive projections and a.e. convergence. In J. Galambos and I. Kátai, editors, *Probability Theory and Applications, Essays to the Memory of József Mogyoródi*, pages 47–75. Kluwer Academic Publishers, Dordrecht, 1992.

[61] F. Schipp and P. Simon. On some (H, L_1)-type maximal inequalities with respect to the Walsh-Paley system. In *Functions, Series, Operators, Proc. Conf. in Budapest, 1980*, volume 35 of *Coll. Math. Soc. J. Bolyai*, pages 1039–1045. North Holland, Amsterdam, 1981.

[62] F. Schipp, W. R. Wade, P. Simon, and J. Pál. *Walsh Series: An Introduction to Dyadic Harmonic Analysis*. Adam Hilger, Bristol, New York, 1990.

[63] F. Schipp and F. Weisz. Tree martingales and a.e. convergence of Vilenkin-Fourier series. *Math. Pannonica*, 8:17–36, 1997.

[64] P. Simon. Cesàro summability with respect to two-parameter Walsh systems. *Monatsh. Math.*, 131:321–334, 2000.

[65] P. Simon. Two-parameter Vilenkin multipliers and a square function. *Anal. Math.*, 28:231–249, 2002.

[66] P. Simon. (C, α) summability of Walsh-Kaczmarz-Fourier series. *J. Approx. Theory*, 127:39–60, 2004.

[67] P. Simon and F. Weisz. Weak inequalities for Cesàro and Riesz summability of Walsh-Fourier series. *J. Approx. Theory*, 151:1–19, 2008.

[68] P. Sjölin. An inequality of Paley and convergence a.e. of Walsh-Fourier series. *Ark. Mat.*, 7(6):551–570, 1969.

[69] E. M. Stein. *Harmonic Analysis: Real-variable Methods, Orthogonality and Oscillatory Integrals*. Princeton Univ. Press, Princeton, N.J., 1993.

[70] E. M. Stein, M. H. Taibleson, and G. Weiss. Weak type estimates for maximal operators on certain H^p classes. *Rend. Circ. Mat. Palermo, Suppl.*, 1:81–97, 1981.

[71] E. M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton Univ. Press, Princeton, N.J., 1971.

[72] K. Szarvas and F. Weisz. The boundedness of the Cesàro- and Riesz means in variable dyadic Hardy spaces. *Banach J. Math. Anal.*, 13:675–696, 2019.

[73] J. Tiirola. Image decompositions using spaces of variable smoothness and integrability. *SIAM J. Imaging Sci.*, 7(3):1558–1587, 2014.

[74] R. M. Trigub and E. S. Belinsky. *Fourier Analysis and Approximation of Functions*. Kluwer Academic Publishers, Dordrecht, Boston, London, 2004.

[75] A. Uchiyama. *Hardy Spaces on the Euclidean Space*. Springer Monographs in Mathematics. Springer, Berlin, 2001.

[76] F. Weisz. *Martingale Hardy Spaces and their Applications in Fourier Analysis*, volume 1568 of *Lecture Notes in Math.* Springer, Berlin, 1994.

[77] F. Weisz. Cesàro summability of one- and two-dimensional Walsh-Fourier series. *Anal. Math.*, 22:229–242, 1996.

[78] F. Weisz. (C, α) summability of Walsh-Fourier series. *Anal. Math.*, 27:141–155, 2001.

[79] F. Weisz. *Summability of Multi-dimensional Fourier Series and Hardy Spaces*. Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[80] F. Weisz. Summability of multi-dimensional trigonometric Fourier series. *Surv. Approx. Theory*, 7:1–179, 2012.

[81] F. Weisz. *Convergence and Summability of Fourier Transforms and Hardy Spaces*. Applied and Numerical Harmonic Analysis. Springer, Birkhäuser, Basel, 2017.

[82] F. Weisz. Summability of Fourier transforms in variable Hardy and Hardy-Lorentz spaces. *Jaen J. Approx.*, 10:101–131, 2018.

[83] F. Weisz. Boundedness of dyadic maximal operators on variable Lebesgue spaces. *Adv. Oper. Theory*, 5:1588–1598, 2020.

[84] F. Weisz. Summability of Fourier series in periodic Hardy spaces with variable exponent. *Acta Math. Hungar.*, 162:557–583, 2020.

[85] F. Weisz. Characterizations of variable martingale Hardy spaces via maximal functions. *Fract. Calc. Appl. Anal.*, 24:393–420, 2021.

[86] F. Weisz. Doob's and Burkholder-Davis-Gundy inequalities with variable exponent. *Proc. Amer. Math. Soc.*, 149:875–888, 2021.

[87] F. Weisz. *Lebesgue Points and Summability of Higher Dimensional Fourier Series*. Applied and Numerical Harmonic Analysis. Springer, Birkhäuser, Basel, 2021.

[88] F. Weisz. Convergence of Vilenkin-Fourier series in variable Hardy spaces. *Math. Nachr.*, 295:1812–1839, 2022.

[89] X. Yan, D. Yang, W. Yuan, and C. Zhuo. Variable weak Hardy spaces and their applications. *J. Funct. Anal.*, 271:2822–2887, 2016.

[90] D. Yang, Y. Liang, and L. D. Ky. *Real-variable theory of Musielak-Orlicz Hardy spaces.* Number 2182 in Lecture Notes in Mathematics. Springer, 2017.

[91] A. Zygmund. *Trigonometric Series*. Cambridge Press, London, 3rd edition, 2002.