PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special Issue *FAATNA20>22: Functional Analysis, Approximation Theory and Numerical Analysis*
Volume 16 · 2023 · Pages 104–117

On approximation properties of some non-positive Bernstein-Durrmeyer type operators modified in the Bezier-King sense

Bianca Ioana VASIAN[a]

Dedicated to Professor Ioan Raşa on the occasion of his 70th anniversary.

Abstract

In this paper we propose some new Bernstein-Durrmeyer type operators modified in Bezier-King sense, which are not positive on the entire interval $[0,1]$. We prove that, even though the operators are not positive on the entire $[0,1]$, they can approximate all continuous functions on $[0,1]$, first, by using the first order modulus of continuity, and then the second order one, with the appropriate K-functionals. Finally, we prove a Voronovkaja type theorem.

Keywords: Nonpositive approximation operators, Bernstein-Durrmeyer operators, Bézier operators, King operators, Voronovskaja type theorem

2010 AMS Classification: 41A25, 41A35, 41A36.

1 Introduction

In order to give a proof of Weierstrass's approximation theorem [22], S. N. Bernstein [9] proposed the following sequence of positive and linear operators

$$B_n(f,x)=\sum_{k=0}^{n}p_{n,k}(x)f\left(\frac{k}{n}\right),\ x\in[0,1],\ f\in C[0,1], \tag{1}$$

where $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$, for $0\le k\le n$, and $p_{n.k}(x)=0$ for $k>n$. It is well-known that these operators can be used to uniformly approximate all continuous functions on $[0,1]$. This operators have been extensively studied and are still a subject of interest, see, for example [11, 18, 19, 21].

After this operators have been introduced, there arose a lot of generalizations, some of which are used in this paper. Let us recall the King operators, introduced by J. P. King in paper [16]: let $V_n^{\tau}(f,x)$ a sequence of positive linear Bernstein-type operators defined for every $f\in C[0,1]$ by:

$$V_n^{\tau}(f,x)=\sum_{k=0}^{n}\binom{n}{k}(\tau(x))^k(1-\tau(x))^{n-k}f\left(\frac{k}{n}\right),\ x\in[0,1], \tag{2}$$

where $\tau(x)$ is a continuous function on $[0,1]$ and $0\le\tau(x)\le 1$.

In his paper, King focused his attention on operators that fix the monomial function $e_2(t)=t^2$. This kind of modification of operators constitutes a matter of interest and were studied in some recent research papers such as [2, 15, 24].

Another topic that presents a lot of interest is represented by Bézier curves which are used for systems that designs free form curves and surfaces. These curves are used in computer graphics to generate smooth curves and surfaces that are adequate for some geometric problems where smoothness is of great importance. The Bézier modification of Bernstein operators is defined as follows:

$$\mathbb{B}_n^{\theta}(f,x)=\sum_{k=0}^{n}Q_{n,k}^{\theta}(x)f\left(\frac{k}{n}\right),\ x\in[0,1],\ f\in C[0,1], \tag{3}$$

where $\theta\ge 1$ is an integer, $Q_{n,k}^{\theta}(x)=\left[J_{n,k}(x)\right]^{\theta}-\left[J_{n,k+1}(x)\right]^{\theta}$, and $J_{n,k}(x)=\sum_{i=k}^{n}p_{n,k}(x)$. More details about studies on Bézier modified operators can be found in [6, 8, 14].

[a]Transilvania University of Braşov, Department of Mathematics and Computer Science

From the research on modifications of Bernstein operators, one that proved very useful was done by Durrmeyer in paper [13], where the author introduced the operators which are now known as Bernstein-Durrmeyer operators and they are defined as:

$$D_n(f,x)=(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt,\ x\in[0,1], \tag{4}$$

for $f\in L_1[0,1]$ and $p_{n,k}$ defined as before. These operators were intensively studied and represented a great interest for many authors, see [7, 10, 3].

In paper [17], D. A. Meleşteu proposed the following modification of Bernstein operators: $S_n^\alpha : C[0,1]\to C\left[0,\frac{n}{n+t}\right]$, $\alpha>0$, $n\in\mathbb{N}$:

$$S_n^\alpha(f,x)=\sum_{k=0}^{n}p_{n,k}^\alpha(x)f\left(\frac{k}{n}\right),\ x\in[0,1], \tag{5}$$

where $p_{n,k}^\alpha(x)=\left(\frac{n+\alpha}{n}\right)^n\binom{n}{k}x^k\left(\frac{n}{n+\alpha}-x\right)^{n-k}$, $0\le k\le n$. For these modified Bernstein type operators, the author proved some approximation results for $x\in[0,1-\varepsilon]$, where operators are positive.

The operators mentioned above are positive and linear. In this paper we will study approximation properties of some linear operators defined using the methods mentioned above. The operators studied are not positive operators on the entire $[0,1]$ interval.

2 Preliminaries

Throughout the paper we will use the following notions:

Let f be a function defined on $[0,1]$. Denote by $C[0,1]$ the class of all continuous functions on $[0,1]$.

Definition 2.1. Let $\tau:[0,1]\to[0,1]$ be a differentiable and increasing function with $\tau(0)=0$ and $\tau(1)=1$.

In order to prove our results we need the following moduli of continuity, K-functionals and the link between them (see [11, 12]). Let f be a bounded function on $[0,1]$. We have the first order modulus of continuity which is defined as:

$$\omega_1(f,t)=\sup_{\substack{|x-y|\le t\\ x,y\in[0,1]}}|f(x)-f(y)|,\ t>0, \tag{6}$$

and the second order modulus of continuity:

$$\omega_2(f,t)=\sup_{0<h\le t}\sup_{h}|f(x+2h)-2f(x+h)+f(x)|,\ t>0. \tag{7}$$

In order to prove our results we will need the following K - functionals which are equivalent to the moduli mentioned above. For the following K - functionals

$$K_i(f,t^i)=\inf_{g\in C^i[0,1]}\{\|f-g\|+t^i\|g^{(i)}\|\},\ t\ge 0,\ i\in 1,2, \tag{8}$$

we have the equivalence for $f\in C[0,1]$, with $C,\tilde{C}>0$ depending only on i, see [12]:

$$\tilde{C}\omega_i(f,t)\le K_i(f,t^i)\le C\omega_i(f,t),\ t>0, \tag{9}$$

Having in mind the operators studied in papers [4, 5, 23]: let $f\in C[0,1]$ and $\alpha\ge 0$:

$$D_n^\alpha(f,x)=(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}p_{n,k}^\alpha(x)\int_0^{\frac{n}{n+\alpha}}p_{n,k}^\alpha(t)f(t)dt,$$

where $p_{n,k}^\alpha(x)=\left(\frac{n+\alpha}{n}\right)^n\binom{n}{k}x^k\left(\frac{n}{n+\alpha}-x\right)^{n-k}$, we propose the following King-Bézier modification of this operators:

$$D_{n,\tau}^{\alpha,\theta}(f,x)=(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}Q_{n,k}^{\alpha,\tau,\theta}(x)\int_0^{\frac{n}{n+\alpha}}p_{n,k}^\alpha(t)\left(f\circ\tau^{-1}\right)(t)dt,\ x\in[0,1], \tag{10}$$

where, $\alpha\ge 0$, $Q_{n,k}^{\alpha,\tau,\theta}(x)=\left[J_{n,k}^{\alpha,\tau}(x)\right]^\theta-\left[J_{n,k+1}^{\alpha,\tau}(x)\right]^\theta$, with $\theta\ge 1$ an integer, $J_{n,k}^{\alpha,\tau}(x)=\sum_{j=k}^{n}p_{n,k}^{\alpha,\tau}(x)$, and $p_{n,k}^{\alpha,\tau}(x)=\left(\frac{n+\alpha}{n}\right)^n\binom{n}{k}\tau^k(x)\left(\frac{n}{n+\alpha}-\tau(x)\right)^{n-k}$.

Remark 1.
1. If the index θ is missing, we assumed that $\theta=1$;
2. If the index τ is missing, then we considered $\tau(x)=x$.

With the established notations and definitions we can immediately state the following remarks.

Remark 2. From the definition, it can be seen that the operators $D_{n,\tau}^{\alpha,\theta}$ are linear operators on $C[0,1]$.

Remark 3. There is $\xi_n\in(0,1)$ having the property $\tau(\xi_n)=\frac{n}{n+\alpha}$, such that $\tau(x)>\frac{n}{n+\alpha}$ for $x\in(\xi_n,1]$ and $\tau(x)\le\frac{n}{n+\alpha}$ for $x\in[0,\xi_n]$, therefore the operators $D_{n,\tau}^{\alpha,\theta}$ are not positive on the entire interval $[0,1]$.

3 Auxiliary results

In order to prove our results concerning these operators, we will need the following results concerning operators $D_{n,\tau}^{\alpha}$, i.e. when $\theta = 1$.

Lemma 3.1. *We have the following:*

$$\int_0^{\frac{n}{n+\alpha}} t^s p_{n,k}^{\alpha}(t)\,dt = \left(\frac{n}{n+\alpha}\right)^{s+1} \binom{n}{k} \beta(k+s+1, n-k+1), \tag{11}$$

where $\beta(a,b)$ is Euler's Beta function.

Proof. The formula can be obtained by changing the variable $u = \frac{n+\alpha}{n} t$. □

Another useful result is the recurrence relation of functions $p_{n,k}^{\alpha,\tau}(x)$, which is:

Lemma 3.2. *For the functions $p_{n,k}^{\alpha,\tau}(x) = \left(\frac{n+\alpha}{n}\right)^n \binom{n}{k} \tau^k(x) \left(\frac{n}{n+\alpha} - \tau(x)\right)^{n-k}$ we have the following:*

$$\tau(x)\left(\frac{n}{n+\alpha} - \tau(x)\right)\left(p_{n,k}^{\alpha,\tau}(x)\right)' = n\tau'(x)\left(\frac{k}{n+\alpha} - \tau(x)\right) p_{n,k}^{\alpha,\tau}(x),\ x \in [0,1]. \tag{12}$$

Proof. Let us compute the derivative of $p_{n,k}^{\alpha,\tau}(x)$:

$$\left(p_{n,k}^{\alpha,\tau}(x)\right)' = \left(\frac{n+\alpha}{n}\right)^n \binom{n}{k} \left[k\tau^{k-1}(x)\tau'(x)\left(\frac{n}{n+\alpha} - \tau(x)\right)^{n-k} - (n-k)\tau^k(x)\tau'(x)\left(\frac{n}{n+\alpha} - \tau(x)\right)^{n-k-1}\right].$$

Now, if we multiply the expression above with $\tau(x)\left(\frac{n}{n+\alpha} - \tau(x)\right)$, we get the desired result. □

Lemma 3.3. *The operators $D_{n,\tau}^{\alpha}$ satisfy the following relations:*

1. $D_{n,\tau}^{\alpha}(e_0, x) = 1$, *where* $e_0(t) = t^0$;
2. $D_{n,\tau}^{\alpha}(\tau, x) = \frac{1}{n+2}\left(n\tau(x) + \frac{n}{n+\alpha}\right)$;
3. $D_{n,\tau}^{\alpha}(\tau^2, x) = \frac{1}{(n+2)(n+3)}\left(n(n-1)\tau^2(x) + \frac{4n^2}{n+\alpha}\tau(x) + \frac{2n^2}{(n+\alpha)^2}\right)$;

where τ is defined as before and $x \in [0,1]$.

Proof. The results can be obtained by direct computation using relation (11). □

Denote by $\mu_{n,m}^{\tau,\alpha}(x)$ the m-th order central moment of the operators $D_{n,\tau}^{\alpha}$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, which is defined as follows

$$\mu_{n,m}^{\tau,\alpha}(x) = D_{n,\tau}^{\alpha}\left((\tau(t) - \tau(x))^m, x\right),\ x \in [0,1].$$

Lemma 3.4. *The following recurrence relation holds:*

$$\begin{gathered}(m+n+2)\tau'(x)\mu_{n,m+1}^{\tau,\alpha}(x) = \\ \tau(x)\left(\frac{n}{n+\alpha} - \tau(x)\right)\left[2m\tau'(x)\mu_{n,m-1}^{\tau,\alpha}(x) + \left(\mu_{n,m}^{\tau,\alpha}\right)'(x)\right] + (m+1)\tau'(x)\left(\frac{n}{n+\alpha} - 2\tau(x)\right)\mu_{n,m}^{\tau,\alpha}(x).\end{gathered} \tag{13}$$

Proof. We have that:

$$\begin{gathered}\mu_{n,m}^{\tau,\alpha}(x) = D_{n,\tau}^{\alpha}\left((\tau(t) - \tau(x))^m, x\right) = \\ (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^{\alpha,\tau}(x) \int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)(t-\tau(x))^m\,dt.\end{gathered}$$

Now, we compute the derivative of $\mu_{n,m}^{\tau,\alpha}(x)$ and we get:

$$\begin{gathered}\left(\mu_{n,m}^{\tau,\alpha}(x)\right)' = \\ (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}\left(p_{n,k}^{\alpha,\tau}(x)\right)' \int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)(t-\tau(x))^m\,dt - m\tau'(x)\mu_{n,m-1}^{\tau,\alpha}(x).\end{gathered}$$

In order to apply relation (12) we will consider:

$$\tau(x)\left(\frac{n+\alpha}{n}-\tau(x)\right)\left[m\tau'(x)\mu_{n,m-1}^{\tau,\alpha}(x)+\left(\mu_{n,m}^{\tau,\alpha}(x)\right)'\right]=$$
$$(n+1)\left(\frac{n+\alpha}{n}\right)\tau'(x)\sum_{k=0}^{n}n\left(\frac{k}{n+\alpha}-\tau(x)\right)p_{n,k}^{\alpha,\tau}(x)\int_0^{\frac{n}{n+\alpha}}p_{n,k}^{\alpha}(t)(t-\tau(x))^m\,dt.$$

Now, if we take into account that the recurrence relation for $p_{n,k}^{\alpha,\tau}(x)$, when we take $\tau(x)=x$, becomes a recurrence relation for $p_{n,k}^{\alpha}(t)$, and by rearranging the terms, we get:

$$\tau(x)\left(\frac{n+\alpha}{n}-\tau(x)\right)\left[m\tau'(x)\mu_{n,m-1}^{\tau,\alpha}(x)+\left(\mu_{n,m}^{\tau,\alpha}(x)\right)'\right]=$$
$$(n+1)\left(\frac{n+\alpha}{n}\right)\tau'(x)\sum_{k=0}^{n}p_{n,k}^{\alpha,\tau}(x)\left[\int_0^{\frac{n}{n+\alpha}}n\left(\frac{k}{n+\alpha}-t\right)p_{n,k}^{\alpha}(t)(t-\tau(x))^m\,dt+n\int_0^{\frac{n}{n+\alpha}}p_{n,k}^{\alpha}(t)(t-\tau(x))^{m+1}\,dt\right]$$
$$=(n+1)\left(\frac{n+\alpha}{n}\right)\tau'(x)\sum_{k=0}^{n}p_{n,k}^{\alpha,\tau}(x)\int_0^{\frac{n}{n+\alpha}}t\left(\frac{n}{n+\alpha}-t\right)\left(p_{n,k}^{\alpha}(t)\right)'(t-\tau(x))^m\,dt+n\tau'(x)\mu_{n,m+1}^{\tau,\alpha}(x).$$

We rewrite the integral part as:

$$\int_0^{\frac{n}{n+\alpha}}t\left(\frac{n}{n+\alpha}-t\right)\left(p_{n,k}^{\alpha}(t)\right)'(t-\tau(x))^m\,dt=$$
$$\int_0^{\frac{n}{n+\alpha}}\left[\tau(x)\left(\frac{n}{n+\alpha}-\tau(x)\right)+\left(\frac{n}{n+\alpha}-2\tau(x)\right)(t-\tau(x))-(t-\tau(x))^2\right]\left(p_{n,k}^{\alpha}(t)\right)'(t-\tau(x))^m\,dt.$$

Now, by integration by parts formula we get:

$$\tau(x)\left(\frac{n+\alpha}{n}-\tau(x)\right)\left[m\tau'(x)\mu_{n,m-1}^{\tau,\alpha}(x)+\left(\mu_{n,m}^{\tau,\alpha}(x)\right)'\right]=$$
$$n\tau'(x)\mu_{n,m+1}^{\tau,\alpha}(x)-m\tau'(x)\tau(x)\left(\frac{n}{n+\alpha}-\tau(x)\right)\mu_{n,m-1}^{\tau,\alpha}(x)-$$
$$(m+1)\tau'(x)\left(\frac{n}{n+\alpha}-2\tau(x)\right)\mu_{n,m}^{\tau,\alpha}(x)+$$
$$(m+2)\tau'(x)\mu_{n,m+1}^{\tau,\alpha},$$

which completes our proof. □

Remark 4. For the simplicity of the results, we will make the following notation: $\phi_\tau(x):=\tau(x)\left(\frac{n}{n+\alpha}-\tau(x)\right)$.

Remark 5. The function $\phi_\tau(x)$ attains its maximum for $\tau(x)=\frac{n}{2(n+\alpha)}$ and its maximum value is $\max\phi_\tau=\frac{1}{4}\left(\frac{n}{n+\alpha}\right)^2$.

Lemma 3.5. *We have the following expressions for some central moments of the operators $D_{n,\tau}^{\alpha}$:*

1. $\mu_{n,0}^{\tau,\alpha}(x)=1;$
2. $\mu_{n,1}^{\tau,\alpha}(x)=-\frac{2}{n+2}\tau(x)+\frac{n}{(n+2)(n+\alpha)};$
3. $\mu_{n,2}^{\tau,\alpha}(x)=\frac{2}{(n+2)(n+3)}\left[(n-3)\phi_\tau(x)+\left(\frac{n}{n+\alpha}\right)^2\right];$
4. $\mu_{n,3}^{\tau,\alpha}(x)=\frac{6}{(n+2)(n+3)(n+4)}\left(\frac{n}{n+\alpha}-2\tau(x)\right)\left[2(n-1)\phi_\tau(x)+\left(\frac{n}{n+\alpha}\right)^2\right];$
5. $\mu_{n,4}^{\tau,\alpha}(x)=\frac{12}{(n+2)(n+3)(n+4)(n+5)}\left\{\phi_\tau^2(x)\left(n^2-12n+10\right)+6(n-3)\phi_\tau(x)\left(\frac{n}{n+\alpha}\right)^2+2\left(\frac{n}{n+\alpha}\right)^4\right\}.$

Proof. By direct computation and using the recurrence formula (13). □

Proposition 3.6. *We have the following:*

$$\left\|D_{n,\tau}^{\alpha}f\right\|\le e^{2\alpha}\|f\|, \tag{14}$$

for all $f\in C[0,1]$.

Proof. In order to prove the result we will consider two cases, i.e. $x\in[0,\xi_n]$ and $x\in[\xi_n,1]$, where ξ_n is as in Remark 3.

Case 1. Consider $x \in [0, \xi_n]$. In this case, the operators have the property that they are positive. We have:

$$\begin{aligned}\left|D^{\alpha}_{n,\tau}(f,x)\right| &\le (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p^{\alpha,\tau}_{n,k}(x)\int_0^{\frac{n}{n+\alpha}} p^{\alpha}_{n,k}(t)\left|\left(f\circ\tau^{-1}\right)(t)\right|dt\\ &\le \left\|f\circ\tau^{-1}\right\|(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p^{\alpha,\tau}_{n,k}(x)\int_0^{\frac{n}{n+\alpha}} p^{\alpha}_{n,k}(t)\,dt\\ &= \left\|f\circ\tau^{-1}\right\| D^{\alpha}_{n,\tau}(1,x) = \|f\|.\end{aligned}$$

Case 2. Consider $x \in [\xi_n, 1]$. In this case the operators are not positive. Hence, we have:

$$\begin{aligned}\left|D^{\alpha}_{n,\tau}(f,x)\right| &\le (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} \left|p^{\alpha,\tau}_{n,k}(x)\right|\int_0^{\frac{n}{n+\alpha}} p^{\alpha}_{n,k}(t)\left|\left(f\circ\tau^{-1}\right)(t)\right|dt\\ &\le \left\|f\circ\tau^{-1}\right\|(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} \left|p^{\alpha,\tau}_{n,k}(x)\right|\int_0^{\frac{n}{n+\alpha}} p^{\alpha}_{n,k}(t)\,dt\\ &= \left\|f\circ\tau^{-1}\right\|(n+1)\left(\frac{n+\alpha}{n}\right)^{n+1}\sum_{k=0}^{n}\left\{\binom{n}{k}\tau^k(x)\left(\tau(x)-\frac{n}{n+\alpha}\right)^{n-k}\right.\\ &\qquad\left.\left(\frac{n+\alpha}{n}\right)^{n}\binom{n}{k}\left(\frac{n}{n+\alpha}\right)^{n+1}\frac{k!\,(n-k)!}{(n+1)!}\right\}\\ &= \left\|f\circ\tau^{-1}\right\|\left(\frac{n+\alpha}{n}\right)^{n}\sum_{k=0}^{n}\binom{n}{k}\tau^k(x)\left(\tau(x)-\frac{n}{n+\alpha}\right)^{n-k}\\ &= \left\|f\circ\tau^{-1}\right\|\left(\frac{n+\alpha}{n}\right)^{n}\left(2\tau(x)-\frac{n}{n+\alpha}\right)^{n}\\ &= \left\|f\circ\tau^{-1}\right\|\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n}.\end{aligned}$$

By the choice of $\tau(x)$ we know that $\tau(x) \le 1$. Therefore, we get:

$$\left|D^{\alpha}_{n,\tau}(f,x)\right| \le \left\|f\circ\tau^{-1}\right\|\left(1+\frac{2\alpha}{n}\right)^{n} \le e^{2\alpha}\left\|f\circ\tau^{-1}\right\| = e^{2\alpha}\|f\|.$$

Now, combining both cases and taking into account that $\alpha \ge 0$, we get

$$\left|D^{\alpha}_{n,\tau}(f,x)\right| \le \max\left(1, e^{2\alpha}\right)\|f\| = e^{2\alpha}\|f\| \text{ for all } x \in [0,1],$$

which leads to

$$\left\|D^{\alpha}_{n,\tau}f\right\| \le e^{2\alpha}\|f\|.$$

□

Before we proceed to our main results, we need the following:

Remark 6. For $a, b \in [-1, 1]$ and $\theta \ge 1$ integer, the inequality

$$\left|a^{\theta}-b^{\theta}\right| \le \theta\,|a-b| \tag{15}$$

holds.

Remark 7. We have the following inequality

$$|Q^{\alpha,\tau,\theta}_{n,k}(x)| = \left|\left[J^{\alpha,\tau}_{n,k}(x)\right]^{\theta}-\left[J^{\alpha,\tau}_{n,k+1}(x)\right]^{\theta}\right| \le \theta\left|J^{\alpha,\tau}_{n,k}(x)-J^{\alpha,\tau}_{n,k+1}(x)\right| = \theta|p^{\alpha,\tau}_{n,k}(x)|, \tag{16}$$

obtained as a consequence of Remark 6, where $\theta \ge 1$ is an integer.

Using the results stated above, we get the following results concerning the operators $D^{\alpha,\theta}_{n,\tau}$.

Proposition 3.7. *We have the following:*

$$\left\|D^{\alpha,\theta}_{n,\tau}f\right\| \le \theta e^{2\alpha}\|f\|, \tag{17}$$

for all $f \in C[0,1]$.

Proof. The proof is similar to the one above but using the inequality in Remark 7, with $\theta \ge 1$ an integer:

$$|Q^{\alpha,\tau,\theta}_{n,k}(x)| \le \theta|p^{\alpha,\tau}_{n,k}(x)|.$$

Now, if we consider $x \in [0,1]$ we will split the proof as in Proposition 3.6, for $x \in [0,\xi_n]$ and for $x \in [\xi_n, 1]$.

Case 1. Consider $x \in [0, \xi_n]$, then we have:

$$\begin{aligned} \left|D_{n,\tau}^{\alpha,\theta}(f,x)\right| &\le (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} Q_{n,k}^{\alpha,\tau,\theta}\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)\left|\left(f\circ\tau^{-1}\right)(t)\right|dt \\ &\le \theta\left\|f\circ\tau^{-1}\right\|(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^{\alpha,\tau}(x)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)\,dt \\ &= \theta\left\|f\circ\tau^{-1}\right\| D_{n,\tau}^{\alpha}(1,x) = \theta\|f\|, \end{aligned}$$

where we could drop the absolute value on $Q_{n,k}^{\alpha,\tau,\theta}$ because in this case the operators are positive.

Case 2. For the second part of the proof we will consider $x \in [\xi_n, 1]$ and take into account that the operators are not positive on this interval and that $\tau(x) > \frac{n}{n+\alpha}$. We get:

$$\begin{aligned} \left|D_{n,\tau}^{\alpha,\theta}(f,x)\right| &\le (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}\left|Q_{n,k}^{\alpha,\tau,\theta}\right|\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)\left|\left(f\circ\tau^{-1}\right)(t)\right|dt \\ &\le \theta\left\|f\circ\tau^{-1}\right\|(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}\left|p_{n,k}^{\alpha,\tau}(x)\right|\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)\,dt. \end{aligned}$$

From this point, the proof is similar to the one in Proposition 3.6, Case 2, and this concludes our result. □

Remark 8. We have $D_{n,\tau}^{\alpha,\theta}(e_0,x) = 1$ for all $x \in [0,1]$. Indeed, computing $D_{n,\tau}^{\alpha,\theta}(e_0,x)$, we get

$$\begin{aligned} D_{n,\tau}^{\alpha,\theta}(e_0,x) &= \sum_{k=0}^{n} Q_{n,k}^{\alpha,\tau,\theta}(x) = \sum_{k=0}^{n}\left\{\left[J_{n,k}^{\alpha,\tau}(x)\right]^{\theta} - \left[J_{n,k+1}^{\alpha,\tau}(x)\right]^{\theta}\right\} \\ &= \left[J_{n,0}^{\alpha,\tau}(x)\right]^{\theta} = \left[\sum_{k=0}^{n} p_{n,k}^{\alpha,\tau}(x)\right]^{\theta} = 1, \end{aligned}$$

for all $x \in [0,1]$.

4 Quantitative approximation

In the following we will establish some quantitative results using different types of moduli of continuity: the classical modulus of continuity ω_1 and a combination of ω_1 and the modulus of smoothness ω_2.

Theorem 4.1. *For $f \in C[0,1]$ we have*

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x) - f(x)\right| \le \left\{1 + \frac{1}{\delta}\frac{n}{n+\alpha}\sqrt{\frac{\theta(n+1)}{2(n+2)(n+3)}}\right\}\omega_1\left(f\circ\tau^{-1},\delta\right), \tag{18}$$

for $x \in [0,\xi_n]$, $\delta > 0$, and

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x) - f(x)\right| \le \theta e^{2\alpha}\left\{1 + \frac{1}{\delta'}\left[\frac{2\alpha}{n} + \frac{n}{(n+2)(n+\alpha)}\right]\right\}\omega_1\left(f\circ\tau^{-1},\delta'\right), \tag{19}$$

for $x \in [\xi_n, 1]$, $\delta' > 0$.

Proof. We will prove the result considering two cases.

Case 1. We take $x \in [0,\xi_n]$. We know that $D_{n,\tau}^{\alpha,\theta}(e_0,x) = 1$ for all $x \in [0,1]$. We have:

$$\begin{gathered} \left|D_{n,\tau}^{\alpha,\theta}(f,x) - f(x)\right| \le \\ (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} Q_{n,k}^{\alpha,\tau,\theta}(x)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)\left|\left(f\circ\tau^{-1}\right)(t) - f(x)\right|dt \le \\ \left[(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} Q_{n,k}^{\alpha,\tau,\theta}(x)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)\left(1 + \frac{|t-\tau(x)|}{\delta}\right)dt\right]\cdot\omega_1\left(f\circ\tau^{-1},\delta\right). \end{gathered}$$

Now, by applying Cauchy-Schwarz inequality to the integral part, we get:

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right| \leq \left\{1+\frac{1}{\delta}\left[D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right]^{\frac{1}{2}}\right\}\omega_1\left(f\circ\tau^{-1},\delta\right)$$
$$\leq \left\{1+\frac{1}{\delta}\left[\theta D_{n,\tau}^{\alpha}\left((\tau(t)-\tau(x))^2,x\right)\right]^{\frac{1}{2}}\right\}\omega_1\left(f\circ\tau^{-1},\delta\right)$$
$$=\left\{1+\frac{1}{\delta}\left[\frac{2\theta}{(n+2)(n+3)}\left[(n-3)\phi_\tau(x)+\left(\frac{n}{n+\alpha}\right)^2\right]\right]^{\frac{1}{2}}\right\}\omega_1\left(f\circ\tau^{-1},\delta\right).$$

Since $\phi_\tau(x)=\tau(x)\left(\frac{n}{n+\alpha}-\tau(x)\right)$ attains its maximum for $\tau(x)=\frac{n}{2(n+\alpha)}$, we get:

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right| \leq \left\{1+\frac{1}{\delta}\frac{n}{n+\alpha}\sqrt{\frac{\theta(n+1)}{2(n+2)(n+3)}}\right\}\omega_1\left(f\circ\tau^{-1},\delta\right).$$

Case 2. For the second part of the result we will consider $x\in[\xi_n,1]$ and take into account that the operators are not positive on this interval and that $\tau(x)>\frac{n}{n+\alpha}$. We get:

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right| \leq$$
$$(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}\left|Q_{n,k}^{\alpha,\tau,\theta}(x)\right|\int_0^{\frac{n}{n+\alpha}}p_{n,k}^{\alpha}(t)\left|\left(f\circ\tau^{-1}\right)(t)-f(x)\right|dt$$
$$\leq\left[(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}\left|Q_{n,k}^{\alpha,\tau,\theta}(x)\right|\int_0^{\frac{n}{n+\alpha}}p_{n,k}^{\alpha}(t)\left(1+\frac{|t-\tau(x)|}{\delta'}\right)dt\right]\omega_1\left(f\circ\tau^{-1},\delta'\right).$$

Now, we use $\left|Q_{n,k}^{\alpha,\tau,\theta}(x)\right|\leq\theta\left|p_{n,k}^{\alpha,\tau}(t)\right|=\theta\left(\frac{n+\alpha}{n}\right)^n\binom{n}{k}\tau^k(x)\left(\tau(x)-\frac{n}{n+\alpha}\right)^{n-k}$, and the fact that for $t\in\left[0,\frac{n}{n+\alpha}\right]$, $|t-\tau(x)|=\tau(x)-t$. Hence:

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right|\leq\theta\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^n\omega\left(f\circ\tau^{-1},\delta\right)+$$
$$\frac{\theta}{\delta'}\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n-1}\times$$
$$\left(2\tau^2(x)\frac{n+\alpha}{n}-2\tau(x)+\frac{n}{(n+\alpha)(n+2)}\right)\omega_1\left(f\circ\tau^{-1},\delta'\right).$$

In order to get the desired result we will use that $\tau(x)\leq 1$, but only in terms $\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^n$ and $\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n-1}$:

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right|\leq$$
$$\leq\theta\left[e^{2\alpha}+\frac{e^{2\alpha}}{\delta'}\left(2\tau^2(x)\frac{n+\alpha}{n}-2\tau(x)+\frac{n}{(n+\alpha)(n+2)}\right)\right]\omega_1\left(f\circ\tau^{-1},\delta'\right).$$

Now, consider the quadratic expression $g_\tau(x)=2\tau^2(x)\frac{n+\alpha}{n}-2\tau(x)+\frac{n}{(n+\alpha)(n+2)}$ which attains its minimum for $\tau(x)=\frac{1}{2}\frac{n}{n+\alpha}$. But, we established that $x\in[\xi_n,1]$ so $\tau(x)>\frac{n}{n+\alpha}$, hence function $g_\tau(x)$ is increasing on that section, and also $\tau(x)$ is an increasing function, therefore $g_\tau(x)$ attains its maximum for $\tau(x)=1$. We obtained that $g_\tau(x)\leq\frac{2\alpha}{n}+\frac{n}{(n+\alpha)(n+2)}$. Therefore:

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right|\leq\theta e^{2\alpha}\left[1+\frac{1}{\delta'}\left(\frac{2\alpha}{n}+\frac{n}{(n+\alpha)(n+2)}\right)\right]\omega_1\left(f\circ\tau^{-1},\delta'\right).$$

□

Corollary 4.2. *Let $f\in C[0,1]$. We have:*

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right|\leq 2\omega_1\left(f\circ\tau^{-1}|_{[0,\xi_n]},\frac{n}{n+\alpha}\sqrt{\frac{\theta(n+1)}{2(n+2)(n+3)}}\right),\tag{20}$$

for $x\in[0,\xi_n]$, and

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right|\leq 2\theta e^{2\alpha}\omega_1\left(f\circ\tau^{-1}|_{[\xi_n,1]},\frac{2\alpha}{n}+\frac{n}{(n+2)(n+\alpha)}\right),\tag{21}$$

for $x\in[\xi_n,1]$.

Proof. Taking into account Theorem 4.1, we get the desired result by taking $\delta=\frac{n}{n+\alpha}\sqrt{\frac{\theta(n+1)}{2(n+2)(n+3)}}$, when $x\in[0,\xi_n]$, and $\delta'=\frac{2\alpha}{n}+\frac{n}{(n+2)(n+\alpha)}$ when $x\in[\xi_n,1]$. □

Lemma 4.3. *For $x \in [0,1]$, we have the following:*

$$D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right) \le \theta \frac{n+1}{2(n+2)(n+3)}\left(\frac{n}{n+\alpha}\right)^2,\ for\ x \in [0,\xi_n], \tag{22}$$

and

$$\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right| \le \theta e^{2\alpha}\frac{2n\left[n^2-n(2-\alpha)-3\alpha\right]}{(n+2)(n+3)(n+\alpha)^2},\ for\ x \in [\xi_n,1]. \tag{23}$$

Proof. **Case 1**. Let us first consider the case $x \in [0,\xi_n]$. In this case the operators $D_{n,\tau}^{\alpha,\theta}$ are positive. Then:

$$D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right) =$$
$$(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} Q_{n,k}^{\alpha,\tau,\theta}(x)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)(t-\tau(x))^2(t)\,dt$$
$$\le \theta\mu_{n,2}^{\tau,\alpha}(x) = \theta\frac{2}{(n+2)(n+3)}\left[(n-3)\phi_\tau(x)+\left(\frac{n}{n+\alpha}\right)^2\right].$$

If we consider the fact that $\phi_\tau(x)$ attains its maximum for $\tau(x)=\frac{n}{2(n+\alpha)}$ and the maximum value is $\frac{1}{4}\left(\frac{n}{n+\alpha}\right)^2$, we get:

$$D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right) \le \theta\frac{n+1}{2(n+2)(n+3)}\left(\frac{n}{n+\alpha}\right)^2.$$

Case 2. Now, let us consider $x \in [\xi_n,1]$:

$$\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right|$$
$$\le (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}\left|Q_{n,k}^{\alpha,\tau,\theta}(x)\right|\left|\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)(t-\tau(x))^2(t)\,dt\right|$$
$$\le \theta(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}\left|p_{n,k}^{\alpha,\tau}(x)\right|\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)(t-\tau(x))^2(t)\,dt.$$

Taking $\left|p_{n,k}^{\alpha,\tau}(x)\right| = \left(\frac{n+\alpha}{n}\right)^n\binom{n}{k}\tau^k(x)\left(\tau(x)-\frac{n}{n+\alpha}\right)^{n-k}$, we get:

$$\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right| \le$$
$$\le \theta(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}\left(\frac{n+\alpha}{n}\right)^n\binom{n}{k}\tau^k(x)\left(\tau(x)-\frac{n}{n+\alpha}\right)^{n-k}\times$$
$$\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)(t-\tau(x))^2(t)\,dt.$$

Now we treat the integral part of the inequality:

$$I = \int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)(t-\tau(x))^2(t)\,dt =$$
$$\left(\frac{n+\alpha}{n}\right)^n\binom{n}{k}\left[\int_0^{\frac{n}{n+\alpha}} t^{k+2}\left(t-\frac{n}{n+\alpha}\right)^{n-k}dt - 2\tau(x)\int_0^{\frac{n}{n+\alpha}} t^{k+1}\left(t-\frac{n}{n+\alpha}\right)^{n-k}dt+\right.$$
$$\left.\tau^2(x)\int_0^{\frac{n}{n+\alpha}} t^{k}\left(t-\frac{n}{n+\alpha}\right)^{n-k}dt\right].$$

Using the formula (11), we get

$$I = \left(\frac{n}{n+\alpha}\right)^3\frac{(k+1)(k+2)}{(n+1)(n+2)(n+3)} - 2\tau(x)\left(\frac{n}{n+\alpha}\right)^2\frac{k+1}{(n+1)(n+2)} + \tau^2(x)\left(\frac{n}{n+\alpha}\right)\frac{1}{n+1}.$$

We obtain:

$$\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right| \le$$
$$\theta(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}\left(\frac{n+\alpha}{n}\right)^n\binom{n}{k}\tau^k(x)\left(\tau(x)-\frac{n}{n+\alpha}\right)^{n-k}\times$$
$$\left[\left(\frac{n}{n+\alpha}\right)^3\frac{(k+1)(k+2)}{(n+1)(n+2)(n+3)} - 2\tau(x)\left(\frac{n}{n+\alpha}\right)^2\frac{k+1}{(n+1)(n+2)}+\right.$$
$$\left.\tau^2(x)\left(\frac{n}{n+\alpha}\right)\frac{1}{n+1}\right]$$

Returning to the expression of $\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right|$ with $(k+1)(k+2)=k(k-1)+4k+2$ and making the computations, we obtain:

$$\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right| \le \theta\left\{\tau^2(x)\frac{n(n-1)}{(n+2)(n+3)}\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n-2}+\right.$$
$$\left(4\tau(x)\frac{n^2}{(n+\alpha)(n+2)(n+3)}-2\tau^2(x)\frac{n}{n+2}\right)\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n-1}+$$
$$\left.\left(\tau^2(x)-2\tau(x)\frac{n}{(n+2)(n+\alpha)}+2\left(\frac{n}{n+\alpha}\right)^2\frac{1}{(n+2)(n+3)}\right)\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n}\right\}.$$

Now, we can write:

$$\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right| \le$$
$$\theta\left\{\tau^2(x)\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n-2}\left[\frac{n(n-1)}{(n+2)(n+3)}-2\frac{n}{n+2}\left(2\tau(x)\frac{n+\alpha}{n}-1\right)+\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^2\right]+\right.$$
$$2\tau(x)\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n-1}\frac{n}{(n+2)(n+\alpha)}\left[2\frac{n}{n+3}-\left(2\tau(x)\frac{n+\alpha}{n}-1\right)\right]+$$
$$\left.2\left(\frac{n}{n+\alpha}\right)^2\frac{1}{(n+2)(n+3)}\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n}\right\}$$
$$=\theta\left\{\tau^2(x)\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n-2}g_1(t)+2\tau(x)\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n-1}\frac{n}{(n+2)(n+\alpha)}g_2(t)+\right.$$
$$\left.2\left(\frac{n}{n+\alpha}\right)^2\frac{1}{(n+2)(n+3)}\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n}\right\},$$

where we denoted $t=2\tau(x)\frac{n+\alpha}{n}-1$ and

$$g_1(t)=\frac{n(n-1)}{(n+2)(n+3)}-2\frac{n}{n+2}t+t^2,$$

and

$$g_2(t)=2\frac{n}{n+3}-t,$$

where $t\in\left[1,1+\frac{2\alpha}{n}\right]$. The function $g_1(t)$ has the roots $t_1=\frac{n}{n+2}+\frac{1}{n+2}\sqrt{\frac{n(n+1)}{n+3}}>1+\frac{2\alpha}{n}$ and $t_2=\frac{n}{n+2}-\frac{1}{n+2}\sqrt{\frac{n(n+1)}{n+3}}<1$, so the function is negative on this interval, hence we can take $g_1(t)\le 0$.

For $g_2(t)$ we see that is a decreasing function, hence it attains its maximum for $t=1$, so the maximum value is $g_2(1)=\frac{n-3}{n+3}$.

Returning to the inequality, we have:

$$\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right| \le$$
$$\theta\left\{2\tau(x)\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n-1}\frac{n}{(n+2)(n+\alpha)}\frac{n-3}{n+3}+2\left(\frac{n}{n+\alpha}\right)^2\frac{1}{(n+2)(n+3)}\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n}\right\}.$$

Now, because $\tau(x)\le 1$, we have $\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n-1}\le\left(2\tau(x)\frac{n+\alpha}{n}-1\right)^{n}\le e^{2\alpha}$, so we get:

$$\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right| \le$$
$$\theta e^{2\alpha}\frac{2}{(n+2)(n+3)}\frac{n}{n+\alpha}\left[\tau(x)(n-3)+\frac{n}{n+\alpha}\right],$$

and, using again that $\tau(x)\le 1$, we obtain:

$$\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right| \le \theta e^{2\alpha}\frac{2n\left[n^2-n(2-\alpha)-3\alpha\right]}{(n+2)(n+3)(n+\alpha)^2}. \tag{24}$$

□

Another result that will be proved is expressed in terms of ω_1 and ω_2. In order to obtain this result we have to impose some restrictions to function $\tau(x)$ as follows:

- $\tau(x)\in C^2[0,1]$;
- $\inf_{x\in[0,1]}\tau'(x)\ge l,\ l\in\mathbb{R}_+$.
- $\sup_{x\in[0,1]}|\tau''(x)|\le\beta,\ \beta\in\mathbb{R}_+$.

Theorem 4.4. *For $f \in C[0,1]$, we have:*

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right| \leq \tag{25}$$

$$\leq \frac{\theta e^{2\alpha}+1}{2}\left[C_1\omega_1\left(f,\frac{2\zeta_1\left(1+\frac{\beta}{2l}\zeta_1\right)}{\theta e^{2\alpha}+1}\right)+C_2\omega_2\left(f,\sqrt{\frac{\zeta_1^2}{\theta e^{2\alpha}+1}}\right)\right],\ x\in[0,\xi_n],$$

where $\zeta_1=\frac{\sqrt{\theta}}{l}\frac{n}{n+\alpha}\sqrt{\frac{n+1}{2(n+2)(n+3)}}$ *and* $C_1,\ C_2$ *are constants not depending on n, and*

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right| \leq \tag{26}$$

$$\leq \frac{\theta e^{2\alpha}+1}{2}\left[C_1^*\omega_1\left(f,\frac{2\zeta_2\left(1+\frac{\beta}{2l}\zeta_2\right)}{\theta e^{2\alpha}+1}\right)+C_2^*\omega_2\left(f,\sqrt{\frac{\zeta_2^2}{\theta e^{2\alpha}+1}}\right)\right],\ x\in[\xi_n,1],$$

where $\zeta_2=\frac{1}{l}\sqrt{\theta e^{2\alpha}\frac{2n\left[n^2-n(2-\alpha)-3\alpha\right]}{(n+2)(n+3)(n+\alpha)^2}}$, *and* $C_1^*,\ C_2^*$ *are constants not depending on* n.

Proof. We will use the following representation

$$g(t)=\left(g\circ\tau^{-1}\right)(\tau(t))=\left(g\circ\tau^{-1}\right)(\tau(x))+\left(g\circ\tau^{-1}\right)'(\tau(x))(\tau(t)-\tau(x))$$
$$+\int_{\tau(x)}^{\tau(t)}\left(g\circ\tau^{-1}\right)''(u)(\tau(t)-u)\,du.$$

If we apply $D_{n,\tau}^{\alpha,\theta}$ to the expression above and take into account that $D_{n,\tau}^{\alpha,\theta}(e_0,x)=1$, we get:

$$D_{n,\tau}^{\alpha,\theta}(g,x)-g(x)=\left(g\circ\tau^{-1}\right)'(\tau(x))D_{n,\tau}^{\alpha,\theta}(\tau(t)-\tau(x),x)+ \tag{27}$$
$$D_{n,\tau}^{\alpha,\theta}\left(\int_{\tau(x)}^{\tau(t)}\left(g\circ\tau^{-1}\right)''(u)(\tau(t)-u)\,du,x\right).$$

We treat separately the absolute value of the integral:

$$\left|\int_{\tau(x)}^{\tau(t)}\left(g\circ\tau^{-1}\right)''(u)(\tau(t)-u)\,du\right|=$$
$$\left|\int_x^t\left(\frac{g''(y)}{(\tau'(y))^2}-g'(y)\frac{\tau''(y)}{(\tau'(y))^3}\right)(\tau(t)-\tau(y))\tau'(y)\,dy\right|$$
$$\leq\left(\frac{\|g''\|}{l^2}+\beta\frac{\|g'\|}{l^3}\right)\frac{(\tau(t)-\tau(x))^2}{2}.$$

Now, taking the absolute value of the expression (27) we get:

$$\left|D_{n,\tau}^{\alpha,\theta}(g,x)-g(x)\right|\leq$$
$$\frac{\|g'\|}{l}D_{n,\tau}^{\alpha,\theta}(|\tau(t)-\tau(x)|,x)+\frac{1}{2}\left(\frac{\|g''\|}{l^2}+\beta\frac{\|g'\|}{l^3}\right)\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right|.$$

Hence, using Cauchy-Schwarz inequality for $D_{n,\tau}^{\alpha,\theta}(|\tau(t)-\tau(x)|,x)$, we obtain:

$$\left|D_{n,\tau}^{\alpha,\theta}(g,x)-g(x)\right|\leq\frac{\|g'\|}{l}\sqrt{\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right|}+$$
$$\frac{1}{2}\left(\frac{\|g''\|}{l^2}+\beta\frac{\|g'\|}{l^3}\right)\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right|.$$

At this point we will split the proof in two cases.

Case 1. Let $x\in[0,\xi_n]$. In this case, our operators are positive. Then, using Lemma 4.3, we have:

$$\left|D_{n,\tau}^{\alpha,\theta}(g,x)-g(x)\right| \leq$$
$$\frac{\sqrt{\theta}}{l}\left\|g'\right\|\frac{n}{n+\alpha}\sqrt{\frac{n+1}{2(n+2)(n+3)}}+\frac{\theta}{2}\left(\frac{\|g''\|}{l^2}+\beta\frac{\|g'\|}{l^3}\right)\left(\frac{n}{n+\alpha}\right)^2\frac{n+1}{2(n+2)(n+3)}=$$
$$\frac{\sqrt{\theta}}{l}\frac{n}{n+\alpha}\left\|g'\right\|\sqrt{\frac{n+1}{2(n+2)(n+3)}}\left(1+\frac{\sqrt{\theta}\beta}{2l^2}\frac{n}{n+\alpha}\sqrt{\frac{n+1}{2(n+2)(n+3)}}\right)+$$
$$\frac{\theta}{2}\frac{\|g''\|}{l^2}\left(\frac{n}{n+\alpha}\right)^2\frac{n+1}{2(n+2)(n+3)}.$$

For $f\in C[0,1]$ we take

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right| \leq \left|D_{n,\tau}^{\alpha,\theta}(f-g,x)\right|+|f(x)-g(x)|+\left|D_{n,\tau}^{\alpha,\theta}(g,x)-g(x)\right|$$
$$\leq\left(\theta e^{2\alpha}+1\right)\|f-g\|+\frac{\sqrt{\theta}}{l}\frac{n}{n+\alpha}\sqrt{\frac{n+1}{2(n+2)(n+3)}}\left\|g'\right\|\left(1+\frac{\sqrt{\theta}\beta}{2l^2}\frac{n}{n+\alpha}\sqrt{\frac{n+1}{2(n+2)(n+3)}}\right)+$$
$$\frac{\theta}{2}\frac{\|g''\|}{l^2}\left(\frac{n}{n+\alpha}\right)^2\frac{n+1}{2(n+2)(n+3)}.$$

Further, we rearrange the terms to match the K-functionals K_1 and K_2. In order to simplify the relation, denote $\zeta_1=\frac{\sqrt{\theta}}{l}\frac{n}{n+\alpha}\sqrt{\frac{n+1}{2(n+2)(n+3)}}$. We have:

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right| \leq\left(\theta e^{2\alpha}+1\right)\|f-g\|+\zeta_1\left(1+\frac{\beta}{2l}\zeta_1\right)\|g'\|+\frac{1}{2}\zeta_1^2\|g''\|$$
$$=\frac{\theta e^{2\alpha}+1}{2}\left[\|f-g\|+\frac{2\zeta_1\left(1+\frac{\beta}{2l}\zeta_1\right)}{\theta e^{2\alpha}+1}\|g'\|+\|f-g\|+\frac{\zeta_1^2}{\theta e^{2\alpha}+1}\|g''\|\right].$$

Now, passing to infimum over $g\in C^2[0,1]$ we get:

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right| \leq \frac{\theta e^{2\alpha}+1}{2}\left[K^1\left(f,\frac{2\zeta\left(1+\frac{\beta}{2l}\zeta\right)}{\theta e^{2\alpha}+1}\right)+K^2\left(f,\sqrt{\frac{\zeta^2}{\theta e^{2\alpha}+1}}\right)\right]. \tag{28}$$

Using the equivalence between K - functionals and moduli of continuity, we get:

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right| \leq \frac{\theta e^{2\alpha}+1}{2}\left[C_1\omega_1\left(f,\frac{2\zeta_1\left(1+\frac{\beta}{2l}\zeta_1\right)}{\theta e^{2\alpha}+1}\right)+C_2\omega_2\left(f,\sqrt{\frac{\zeta_1^2}{\theta e^{2\alpha}+1}}\right)\right],$$

where $C_1,\ C_2$ are constants not depending on n.

Case 2. Let us take $x\in[\xi_n,1]$. Then:

$$\left|D_{n,\tau}^{\alpha,\theta}(g,x)-g(x)\right| \leq$$
$$\frac{\|g'\|}{l}\sqrt{\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right|}+\frac{1}{2}\left(\frac{\|g''\|}{l^2}+\beta\frac{\|g'\|}{l^3}\right)D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\leq$$
$$\frac{\|g'\|}{l}\sqrt{\theta e^{2\alpha}\frac{2n\left[n^2-n(2-\alpha)-3\alpha\right]}{(n+2)(n+3)(n+\alpha)^2}}+\frac{1}{2}\left(\frac{\|g''\|}{l^2}+\beta\frac{\|g'\|}{l^3}\right)\theta e^{2\alpha}\frac{2n\left[n^2-n(2-\alpha)-3\alpha\right]}{(n+2)(n+3)(n+\alpha)^2}.$$

In order to simplify the notations, we denote $\zeta_2=\frac{1}{l}\sqrt{\theta e^{2\alpha}\frac{2n\left[n^2-n(2-\alpha)-3\alpha\right]}{(n+2)(n+3)(n+\alpha)^2}}$, so we can write:

$$\left|D_{n,\tau}^{\alpha,\theta}(g,x)-g(x)\right| \leq \zeta_2\left\|g'\right\|+\frac{1}{2}\zeta_2^2\left(\left\|g''\right\|+\beta\frac{\|g'\|}{l}\right).$$

Now, let us take

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right| \leq \left|D_{n,\tau}^{\alpha,\theta}(f-g,x)\right|+|f(x)-g(x)|+\left|D_{n,\tau}^{\alpha,\theta}(g,x)-g(x)\right|$$
$$\leq\left(\theta e^{2\alpha}+1\right)\|f-g\|+\zeta_2\left\|g'\right\|+\frac{1}{2}\zeta_2^2\left(\left\|g''\right\|+\beta\frac{\|g'\|}{l}\right)$$
$$=\frac{\theta e^{2\alpha}+1}{2}\left[\|f-g\|+\frac{2\zeta_2\left(1+\frac{\beta\zeta_2}{2l}\right)}{\theta e^{2\alpha}+1}\left\|g'\right\|+\right.$$
$$\left.\|f-g\|+\frac{\zeta_2^2}{\theta e^{2\alpha}+1}\left\|g''\right\|\right].$$

Passing to infimum over $g\in C^2[0,1]$

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right|\le$$
$$\frac{\theta e^{2\alpha}+1}{2}\left[K^1\left(f,\frac{2\zeta_2\left(1+\frac{\beta}{2l}\zeta_2\right)}{\theta e^{2\alpha}+1}\right)+K^2\left(f,\sqrt{\frac{\zeta_2^2}{\theta e^{2\alpha}+1}}\right)\right],$$

and using the equivalence between K - functionals and moduli of continuity ω_1 and ω_2 we get:

$$\left|D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right|\le$$
$$\frac{\theta e^{2\alpha}+1}{2}\left[C_1^*\omega_1\left(f,\frac{2\zeta_2\left(1+\frac{\beta}{2l}\zeta_2\right)}{\theta e^{2\alpha}+1}\right)+C_2^*\omega_2\left(f,\sqrt{\frac{\zeta_2^2}{\theta e^{2\alpha}+1}}\right)\right],$$

where $C_1^*,\ C_2^*$ are constants not depending on n. □

5 Voronovskaja type result

In this section we will prove a Voronovskaja type result for the operators $D_{n,\tau}^{\alpha,\theta}$.

Lemma 5.1. *Let $f\in C^2[0,1]$.Then:*

$$\left|n\left[D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right]\right|\le \tag{29}$$
$$\theta\frac{2n}{n+2}\left|\frac{f'(x)}{\tau'(x)}\right|\left(\tau(x)+\frac{n}{n+\alpha}\right)+$$
$$\theta\frac{n}{(n+2)(n+3)}\left|\frac{f''(x)}{(\tau'(x))^2}-f'(x)\frac{\tau''(x)}{(\tau'(x))^3}\right|\left[(n-3)\phi_\tau(x)+\left(\frac{n}{n+\alpha}\right)^2\right]+\Lambda_n(x);\ x\in[0,\xi_n],$$

where $\Lambda_n(x)\to 0$ as $n\to\infty$.

Proof. Let $f\in C^2[0,1]$ and consider the Taylor expansion of f as follows:

$$f(t)=\left(f\circ\tau^{-1}\right)(\tau(t))= \tag{30}$$
$$\left(f\circ\tau^{-1}\right)(\tau(x))+\left(f\circ\tau^{-1}\right)'(\tau(x))(\tau(t)-\tau(x))+$$
$$\frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))(\tau(t)-\tau(x))^2+R(t,x)(\tau(t)-\tau(x))^2,$$

where the remainder $R(t,x)$ satisfies $\lim_{t\to x}R(t,x)=0$. Now, consider the difference:

$$n(f(t)-f(x))= \tag{31}$$
$$n\left(f\circ\tau^{-1}\right)'(\tau(x))(\tau(t)-\tau(x))+\frac{n}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))(\tau(t)-\tau(x))^2+$$
$$nR(t,x)(\tau(t)-\tau(x))^2.$$

By applying the operator $D_{n,\tau}^{\alpha,\theta}$ we get:

$$\left|n\left[D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right]\right|\le n\left|\left(f\circ\tau^{-1}\right)'(\tau(x))\right|\left|D_{n,\tau}^{\alpha,\theta}(\tau(t)-\tau(x),x)\right|+ \tag{32}$$
$$\frac{n}{2}\left|\left(f\circ\tau^{-1}\right)''(\tau(x))\right|\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^2,x\right)\right|+$$
$$n\left|D_{n,\tau}^{\alpha,\theta}\left(R(t,x)(\tau(t)-\tau(x))^2,x\right)\right|\le$$
$$\theta n\left|\left(f\circ\tau^{-1}\right)'(\tau(x))\right|\left|\mu_{n,1}^{\alpha}(x)\right|+\theta\frac{n}{2}\left|\left(f\circ\tau^{-1}\right)''(\tau(x))\right|\left|\mu_{n,2}^{\alpha}(x)\right|+$$
$$n\left|D_{n,\tau}^{\alpha,\theta}\left(R(t,x)(\tau(t)-\tau(x))^2,x\right)\right|.$$

We will treat separately $nD_{n,\tau}^{\alpha,\theta}\left(R(t,x)(\tau(t)-\tau(x))^2,x\right)$ by using Cauchy-Schwarz inequality. Now, because $x\in[0,\xi_n]$, we have:

$$n\left|D_{n,\tau}^{\alpha,\theta}\left(R(t,x)(\tau(t)-\tau(x))^2,x\right)\right|\le$$
$$\sqrt{\left|D_{n,\tau}^{\alpha,\theta}\left(R^2(t,x),x\right)\right|}\sqrt{n^2\left|D_{n,\tau}^{\alpha,\theta}\left((\tau(t)-\tau(x))^4,x\right)\right|}\le$$
$$\sqrt{\left|D_{n,\tau}^{\alpha,\theta}\left(R^2(t,x),x\right)\right|}\sqrt{n^2\theta\left|\mu_{n,4}^{\alpha}(x)\right|}.$$

We have that $n^2\left|\mu_{n,4}^{\alpha}(x)\right| = n^2 O\left(\frac{1}{n^2}\right) = O(1)$. The next step is to show that $\left|D_{n,\tau}^{\alpha,\theta}\left(R^2(t,x),x\right)\right| \to 0$, but from the approximation properties $D_{n,\tau}^{\alpha,\theta}$ possess, we obtain:

$$\lim_{n\to\infty} D_{n,\tau}^{\alpha,\theta}\left(R^2(t,x),x\right) = R^2(x,x) = 0. \tag{33}$$

Denote $n\left|D_{n,\tau}^{\alpha,\theta}\left(R(t,x)(\tau(t)-\tau(x))^2,x\right)\right| := \Lambda_n(x)$, and by using the expression of the central moments $\mu_{n,1}^{\alpha}(x)$ and $\mu_{n,2}^{\alpha}(x)$ from Lemma 3.5, we get:

$$\begin{aligned} \left|n\left[D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right]\right| &\leq \\ \theta\frac{2n}{n+2}\left|\left(f\circ\tau^{-1}\right)'(\tau(x))\right|&\left(\tau(x)+\frac{n}{n+\alpha}\right)+ \\ \theta\frac{n}{(n+2)(n+3)}\left|\left(f\circ\tau^{-1}\right)''(\tau(x))\right|&\left[(n-3)\phi_\tau(x)+\left(\frac{n}{n+\alpha}\right)^2\right]+\Lambda_n(x);\ x\in[0,\xi_n]. \end{aligned} \tag{34}$$

In order to have a more explicit expression for the result, we will use the following:

$$\begin{aligned} \left(f\circ\tau^{-1}\right)'(\tau(x)) &= \frac{f'(x)}{\tau'(x)}, \\ \left(f\circ\tau^{-1}\right)''(\tau(x)) &= \frac{f''(x)}{(\tau'(x))^2} - f'(x)\frac{\tau''(x)}{(\tau'(x))^3}. \end{aligned} \tag{35}$$

□

Theorem 5.2. *For $f\in C^2[0,1]$ and $x\in[0,1)$, we have:*

$$\begin{aligned} \limsup_{n\to\infty}\left(n\left[D_{n,\tau}^{\alpha,\theta}(f,x)-f(x)\right]\right) &= \\ 2\theta\left|\frac{f'(x)}{\tau'(x)}\right|(\tau(x)+1)+\theta\left|\frac{f''(x)}{(\tau'(x))^2}-f'(x)\frac{\tau''(x)}{(\tau'(x))^3}\right|&\tau(x)(1-\tau(x));\ x\in[0,1). \end{aligned} \tag{36}$$

Proof. From the Lemma 5.1 passing to lim sup with $n\to\infty$ we get the desired result but taking into account that as $n\to\infty$, $\xi_n\to 1$. □

References

[1] Acar T., Agrawal P. N., Neer T., *Bezier variant of the Bernstein-Durrmeyer type operators*, Results in Mathematics, 2016.

[2] Acar T., Montano M. C., Garrancho P., Leonessa V., *On Sequences of J. P. King-Type Operators*, Journal of Function Spaces, vol. 2019, 2019.

[3] Acar T., Aral A., Gupta V., On Approximation Properties of a New type Bernstein-Durrmeyer Operators, Mathematica Slovaca, 65 (5), 2015, 1107-1122.

[4] Acar T., Aral A., Raşa I., Modified Bernstein-Durrmeyer operators, General Mathematics, 22 (1), 2014, 27-41.

[5] Acar T., Asymptotic Formulas for Generalized Szász-Mirakyan Operators, Applied Mathematics and Computation, 263, 2015, 223-239.

[6] Neer T., Acu A. M., Agrawal P. N., *Bézier variant of genuine-Durrmeyer type operators based on Pólya distribution*, Carpathian J, 33, 2017.

[7] Adell J.A., J. de la Cal, *Bernstein-Durrmeyer operators*, Computers & Mathematics with Applications, Volume 30, Issues 36, 1995.

[8] Agrawal, P.N., Bhardwaj N., Bawa P. Bézier variant of modified α-Bernstein operators. Rend. Circ. Mat. Palermo, II. Ser 71, 807-827, 2022.

[9] Bernstein S. N., *Démonstration du théorém de Weierstrass fondeé sur la calcul desprobabilitiés*. Commun Soc Math Charkow Sér 2(13), 1912

[10] Derriennic M. M., *Sur l'approixmation des fonctions integrables sur* $[0,1]$ *par des polynomes de Bernstein modifies*, J. Approx. Theory, 31, 1981.

[11] DeVore R. A., Lorentz G. G., Constructive Approximation, Springer-Verlag Berlin Heidelberg, 1993.

[12] Ditzian Z., Totik V., *Moduli of Smoothness*, Springer Series in Computational Mathematics, vol. 9, 1987.

[13] Durrmeyer J. L., *Une formule d'inversion de la transformée de Laplace-applications á la théorie des moments*, Thése de 3e cycle, Faculté des Science de l'Université de Paris, 1967.

[14] Kajla A., Acar T., Bezier-Bernstein-Durrmeyer type operators, RACSAM (Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas), 114, 31, 2020.

[15] Gonska H., Pitul P., *Remarks on a article of J.P. King*, Comment. Math. Univ. Carol. 2005, 46, 645-652.

[16] King J.P., *Positive linear operators which preserve* x^2. Acta Math. Hung. 2003, 99, 203-208.

[17] Meleşteu D. A., *Generalized Bernstein type operators*, Bulletin of the *Transilvania* University of Braşov, Vol 13(62), No. 2, 2020.

[18] Nursel A ., Radu V. A., *Approximation by generalized Bernstein-Stancu operators*, Turkish Journal of Mathematics: Vol. 43, 2019.

[19] Păltănea R., *Approximation theory using positive linear operators*. Birkhaüser, Boston, 2004.

[20] Pop O.T., Indrea A.D., Braica P.I., *Durrmeyer operators of King-type*. Ann. Univ. Craiova Math. Comput. Sci. Ser. 2012, 39, 288-298.

[21] Totik V., *Approximation by Bernstein Polynomials*. American Journal of Mathematics, 116(4), 995-1018, 1994.

[22] Weierstrass K.G., *U die analytische Darstellbarkeit sogenannter licher Funktionen einer reellen Veranderlichen,* Sitzungsber. Akad. Berlin 2, 633-639, 1885.

[23] Vasian B.I., *On approximation properties of some non-positive Bernstein-Durrmeyer*, An. Şt. Univ. Ovidius Constanţa, Vol. 31(1), 2023.

[24] Vasian B.I., Garoiu Ş.L., Păcurar C.M., *On New Classes of Stancu-Kantorovich-Type Operators*. Mathematics 2021.