



Yet another DE-Sinc indefinite integration formula

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Abstract

Based on the Sinc approximation combined with the tanh transformation, Haber derived an approximation formula for numerical indefinite integration over the finite interval $(-1, 1)$. The formula uses a special function for the basis functions. In contrast, Stenger derived another formula, which does not use any special function but does include a double sum. Subsequently, Muhammad and Mori proposed a formula, which replaces the tanh transformation with the double-exponential transformation in Haber's formula. Almost simultaneously, Tanaka et al. proposed another formula, which was based on the same replacement in Stenger's formula. As they reported, the replacement drastically improves the convergence rate of Haber's and Stenger's formula. In addition to the formulas above, Stenger derived yet another indefinite integration formula based on the Sinc approximation combined with the tanh transformation, which has an elegant matrix-vector form. In this paper, we propose the replacement of the tanh transformation with the double-exponential transformation in Stenger's second formula. We provide a theoretical analysis as well as a numerical comparison.

1 Introduction

Haber [2] derived a numerical indefinite integration formula of the form

$$\int_{-1}^x f(s) ds \approx \sum_{j=-n}^n f(x_j) w_j(x), \quad x \in (-1, 1), \quad (1)$$

where the weight function w_j depends on x , but the sampling node x_j does not. Furthermore, the formula can attain a root-exponential convergence: $O(\exp(-c\sqrt{n}))$, even if the integrand f has an endpoint singularity such as $f(s) = 1/\sqrt{1-s^2}$. Owing to these features, the formulas have been used in many applications, such as initial value problems [13], Volterra integral equations [12], and Volterra integro-differential equations [17].

Haber's formula is derived as follows. The Sinc approximation over the real axis is expressed as

$$F(u) \approx \sum_{j=-n}^n F(jh) \operatorname{sinc}\left(\frac{u-jh}{h}\right), \quad u \in \mathbb{R},$$

where $\operatorname{sinc}(x)$ is the normalized sinc function, and h is the mesh size appropriately selected depending on n . By integrating both sides of the Sinc approximation, we have

$$\int_{-\infty}^t F(u) du \approx \sum_{j=-n}^n F(jh) J(j, h)(t), \quad t \in \mathbb{R}, \quad (2)$$

where

$$J(j, h)(t) = \int_{-\infty}^t \operatorname{sinc}\left(\frac{u-jh}{h}\right) du = \frac{h}{\pi} \left\{ \frac{\pi}{2} + \operatorname{Si}\left(\frac{\pi(t-jh)}{h}\right) \right\}.$$

Here, $\operatorname{Si}(x)$ is the sine integral defined by $\operatorname{Si}(x) = \int_0^x \{(\sin t)/t\} dt$. The approximation (2) is called the Sinc indefinite integration. In the case of the indefinite integral over the finite interval $(-1, 1)$, Haber employed the tanh transformation

$$s = \psi(u) = \tanh\left(\frac{u}{2}\right),$$

and applied (2) as

$$\int_{-1}^x f(s) ds = \int_{-\infty}^{\psi^{-1}(x)} f(\psi(u)) \psi'(u) du \approx \sum_{j=-n}^n f(\psi(jh)) \psi'(jh) J(j, h)(\psi^{-1}(x)), \quad x \in (-1, 1). \quad (3)$$

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In this paper, we refer to this approximation as formula (SE1).

Although the formula is simple and efficient as described above, it involves a drawback related to the basis function $J(j, h)$, which includes a special function $\text{Si}(x)$. The routine of $\text{Si}(x)$ is not always available, and has a higher computational cost than that of elementary functions. In order to remedy this drawback, introducing an auxiliary function $\eta(x) = (1+x)/2$, Stenger [13] derived another formula

$$\int_{-1}^x f(s) ds \approx \mathcal{I}_{\text{SE2}}(f, x) := h \sum_{i=-n}^n \left\{ \sum_{j=-n}^n \left(f(\psi(jh)) - \frac{1}{2} I_{\psi}^* \right) \psi'(jh) \delta_{ij}^{(-1)} \right\} \text{sinc} \left(\frac{\psi^{-1}(x) - ih}{h} \right) + I_{\psi}^* \eta(x), \quad (4)$$

where

$$\sigma_k = \int_0^k \text{sinc}(x) dx = \frac{1}{\pi} \text{Si}(\pi k),$$

$$I_{\psi}^* = h \sum_{k=-n}^n f(\psi(kh)) \psi'(kh),$$

and $\delta_{ij}^{(-1)} = 1/2 + \sigma_{i-j}$. We refer to the approximation (4) as formula (SE2). In view of (4), we need not compute any special function (note that the value table of σ_k is available [13, Table 1.10.1]). In contrast, however, a double sum exists, which increases the computational cost compared to single sum. Therefore, which formula performs better remains unclear. We note that Haber [2] also derived a similar formula which uses only elementary functions for the basis functions, but includes a double sum. Because those two formulas are essentially the same, we concentrate on the formula (SE2) in this work.

After a decade, as an improvement of (SE1), Muhammad and Mori [5] proposed the replacement of the tanh transformation used in (3) with the double-exponential (DE) transformation

$$s = \phi(u) = \tanh \left(\frac{\pi}{2} \sinh u \right),$$

and derived the following formula

$$\int_{-1}^x f(s) ds \approx \sum_{j=-n}^n f(\phi(jh)) \phi'(jh) J(j, h) (\phi^{-1}(x)). \quad (5)$$

We refer to the approximation (5) as formula (DE1). This replacement has been shown to improve the convergence rate from $O(\exp(-c\sqrt{n}))$ to $O(\exp(-cn/\log n))$ in many fields [3, 4]. In fact, it was shown [11] that the formula (DE1) can attain a convergence rate of $O(\exp(-cn/\log n))$. Such a rapid convergence motivated several authors to apply the formula to various applications [6, 7, 9].

Almost simultaneously, as an improvement of (SE2), Tanaka et al. [16] proposed the following formula

$$\int_{-1}^x f(s) ds \approx \mathcal{I}_{\text{DE2}}(f, x) := h \sum_{i=-n}^n \left\{ \sum_{j=-n}^n \left(f(\phi(jh)) - \frac{1}{2} I_{\phi}^* \right) \phi'(jh) \delta_{ij}^{(-1)} \right\} \text{sinc} \left(\frac{\phi^{-1}(x) - ih}{h} \right) + I_{\phi}^* \eta(x), \quad (6)$$

where

$$I_{\phi}^* = h \sum_{k=-n}^n f(\phi(kh)) \phi'(kh).$$

We refer to the approximation (6) as formula (DE2). They also analyzed the convergence rate as $O(\exp(-cn/\log n))$, which is the same as that of (DE1).

In addition to the formulas above, Stenger [13] derived yet another indefinite integration formula, expressed as

$$\int_{-1}^x f(s) ds \approx \sum_{i=-M}^N \left(h \sum_{j=-M}^N \delta_{ij}^{(-1)} f(\psi(jh)) \psi'(jh) \right) \omega_i^{\text{SE}}(x), \quad (7)$$

where

$$\begin{cases} \omega_{-M}^{\text{SE}}(x) = \frac{1}{1 - \eta(\psi(-Mh))} \left\{ (1 - \eta(x)) - \sum_{k=-M+1}^N (1 - \eta(\psi(kh))) \text{sinc} \left(\frac{\psi^{-1}(x) - kh}{h} \right) \right\}, \\ \omega_i^{\text{SE}}(x) = \text{sinc} \left(\frac{\psi^{-1}(x) - ih}{h} \right) \quad (i = -M+1, \dots, N-1), \\ \omega_N^{\text{SE}}(x) = \frac{1}{\eta(\psi(Nh))} \left\{ \eta(x) - \sum_{k=-M}^{N-1} \eta(\psi(kh)) \text{sinc} \left(\frac{\psi^{-1}(x) - kh}{h} \right) \right\}. \end{cases}$$

We refer to the approximation (7) as formula (SE3). This formula not only extends the sum from $\sum_{j=-n}^n$ to $\sum_{j=-M}^N$, but also has an elegant matrix-vector form. Let $m = M + N + 1$, and let $I_m^{(-1)}$ be a square matrix of order m having $\delta_{ij}^{(-1)}$ as its (i, j) th element. Moreover, let

$$\begin{aligned} D_m^{\text{SE}} &= \text{diag}[\psi'(-Mh), \psi'(-(M-1)h), \dots, \psi'((N-1)h), \psi'(Nh)], \\ f_m^{\text{SE}} &= [f(\psi(-Mh)), f(\psi(-(M-1)h)), \dots, f(\psi((N-1)h)), f(\psi(Nh))]^T, \\ \omega_m^{\text{SE}}(x) &= [\omega_{-M}^{\text{SE}}(x), \omega_{-M+1}^{\text{SE}}(x), \dots, \omega_{N-1}^{\text{SE}}(x), \omega_N^{\text{SE}}(x)], \end{aligned}$$

and let $A_m^{\text{SE}} = hI_m^{(-1)}D_m^{\text{SE}}$. Then, the approximation (7) can be written as

$$\int_{-1}^x f(s) ds \approx \omega_m^{\text{SE}}(x) A_m^{\text{SE}} f_m^{\text{SE}}.$$

Its convergence rate was analyzed as $O(\exp(-c\sqrt{n}))$, where $n = \max\{M, N\}$. Of note, this notation may be used for an integral of any order; for example,

$$\begin{aligned} f(x) &\approx \omega_m^{\text{SE}}(x) f_m^{\text{SE}}, \\ \int_{-1}^x f(s) ds &\approx \omega_m^{\text{SE}}(x) A_m^{\text{SE}} f_m^{\text{SE}}, \\ \int_{-1}^x \left(\int_{-1}^s f(t) dt \right) ds &\approx \omega_m^{\text{SE}}(x) (A_m^{\text{SE}})^2 f_m^{\text{SE}}, \\ \int_{-1}^x \left\{ \int_{-1}^s \left(\int_{-1}^t f(u) du \right) dt \right\} ds &\approx \omega_m^{\text{SE}}(x) (A_m^{\text{SE}})^3 f_m^{\text{SE}}, \end{aligned}$$

and so forth. As an application, this beautiful feature has been used to derive an approximation formula for indefinite convolution [14].

The main contribution of this paper is an improvement of the formula (SE3). Based on the same idea as (DE1) and (DE2), we propose the replacement of the tanh transformation used in (SE3) with the DE transformation. Namely, using the notation

$$\begin{cases} \omega_{-M}^{\text{DE}}(x) = \frac{1}{1 - \eta(\phi(-Mh))} \left\{ (1 - \eta(x)) - \sum_{k=-M+1}^N (1 - \eta(\phi(kh))) \operatorname{sinc} \left(\frac{\phi^{-1}(x) - kh}{h} \right) \right\}, \\ \omega_j^{\text{DE}}(x) = \operatorname{sinc} \left(\frac{\phi^{-1}(x) - jh}{h} \right) \quad (j = -M + 1, \dots, N - 1), \\ \omega_N^{\text{DE}}(x) = \frac{1}{\eta(\phi(Nh))} \left\{ \eta(x) - \sum_{k=-M}^{N-1} \eta(\phi(kh)) \operatorname{sinc} \left(\frac{\phi^{-1}(x) - kh}{h} \right) \right\}, \end{cases}$$

we propose the following formula:

$$\int_{-1}^x f(s) ds \approx \omega_m^{\text{DE}}(x) A_m^{\text{DE}} f_m^{\text{DE}}, \quad (8)$$

where

$$\begin{aligned} D_m^{\text{DE}} &= \operatorname{diag}[\phi'(-Mh), \phi'(-(M-1)h), \dots, \phi'((N-1)h), \phi'(Nh)], \\ f_m^{\text{DE}} &= [f(\phi(-Mh)), f(\phi(-(M-1)h)), \dots, f(\phi((N-1)h)), f(\phi(Nh))]^T, \\ \omega_m^{\text{DE}}(x) &= [\omega_{-M}^{\text{DE}}(x), \omega_{-M+1}^{\text{DE}}(x), \dots, \omega_{N-1}^{\text{DE}}(x), \omega_N^{\text{DE}}(x)], \end{aligned}$$

and $A_m^{\text{DE}} = hI_m^{(-1)}D_m^{\text{DE}}$. We refer to the approximation (8) as formula (DE3). Furthermore, we show theoretically that the formula (DE3) can attain the same convergence rate as that of (DE1) and (DE2), i.e., $O(\exp(-cn/\log n))$.

As a subsidiary contribution, we compared the six formulas (SE1), (SE2), (SE3), (DE1), (DE2) and (DE3). Akinola [1] performed a numerical comparison between (SE1), (SE2)¹ and (DE2), and reported that the formula (SE1) exhibited the lowest CPU time among those formulas. However, he compared CPU times with respect to n , which is not suitable for a comparison in terms of efficiency. In this work, we compared CPU times with respect to accuracy.

The remainder of this paper is organized as follows. In Section 2, we present existing and new theoretical results. In Section 3, we show numerical results. In Section 4, we provide the proof of a new theorem stated in Section 2. Section 5 concludes the work. Appendix A gives a rigorous error bound used in Section 4.

2 Existing and new theorems

In this section, we present convergence theorems for the formulas (SE1), (SE2), (SE3), (DE1), (DE2), and (DE3). In those theorems, the integrand translated by the tanh transformation or the DE transformation ($f(\psi(\cdot))$ or $f(\phi(\cdot))$) is assumed to be analytic on a strip complex domain $\mathcal{D}_d = \{\zeta \in \mathbb{C} : |\operatorname{Im} \zeta| < d\}$ for a positive constant d . That is, f should be analytic on the domain $\psi(\mathcal{D}_d) = \{z = \psi(\zeta) : \zeta \in \mathcal{D}_d\}$ or on $\phi(\mathcal{D}_d) = \{z = \phi(\zeta) : \zeta \in \mathcal{D}_d\}$.

2.1 Convergence theorems for (SE1), (SE2), and (SE3)

A convergence theorem of the formula (SE1) was given as follows. In the theorem, the sum is extended to $\sum_{j=-M}^N$ instead of $\sum_{j=-n}^n$ in the original formula (3).

¹In fact, he also considered Haber's second formula, which gave results similar to those of the formula (SE2).

Theorem 2.1 (Okayama et al. [11, Theorem 2.9]). Assume that f is analytic in $\psi(\mathcal{D}_d)$ for d with $0 < d < \pi$, and that there exist positive constants K , α and β such that

$$|f(z)| \leq K|z+1|^{\alpha-1}|z-1|^{\beta-1} \quad (9)$$

holds for all $z \in \psi(\mathcal{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let n be a positive integer, and let h be selected by

$$h = \sqrt{\frac{\pi d}{\mu n}}. \quad (10)$$

Furthermore, let M and N be positive integers defined by

$$\begin{cases} M = n, & N = \left\lceil \frac{\alpha}{\beta} n \right\rceil & (\text{if } \mu = \alpha) \\ N = n, & M = \left\lceil \frac{\beta}{\alpha} n \right\rceil & (\text{if } \mu = \beta) \end{cases} \quad (11)$$

respectively. Then, there exists a constant C independent of n such that

$$\sup_{x \in (-1,1)} \left| \int_{-1}^x f(s) ds - \sum_{j=-M}^N f(\psi(jh)) \psi'(jh) J(j, h) (\psi^{-1}(t)) \right| \leq C \exp(-\sqrt{\pi d \mu n}).$$

A convergence theorem of the formula (SE2) was given as follows.

Theorem 2.2 (Stenger [13, Corollary 4.5.3]). Assume that f is analytic in $\psi(\mathcal{D}_d)$ for d with $0 < d < \pi$, and that there exist positive constants K and μ with $\mu \leq 1$ such that

$$|f(z)| \leq K|z+1|^{\mu-1}|z-1|^{\mu-1} \quad (12)$$

holds for all $z \in \psi(\mathcal{D}_d)$. Let n be a positive integer, and let h be selected by (10). Then, there exists a constant C independent of n such that

$$\sup_{x \in (-1,1)} \left| \int_{-1}^x f(s) ds - \mathcal{I}_{\text{SE2}}(f, x) \right| \leq C \sqrt{n} \exp(-\sqrt{\pi d \mu n}).$$

A convergence theorem of the formula (SE3) was given as follows.

Theorem 2.3 (Stenger [14, Theorem 4.9]). Assume that f is analytic in $\psi(\mathcal{D}_d)$ for d with $0 < d < \pi$, and that there exist positive constants K , α with $\alpha \leq 1$ and β with $\beta \leq 1$ such that (9) holds for all $z \in \psi(\mathcal{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let n be a positive integer, and let h be selected by (10). Furthermore, let M and N be positive integers defined by (11). Then, there exists a constant C independent of n such that

$$\sup_{x \in (-1,1)} \left| \int_{-1}^x f(s) ds - \omega_m^{\text{SE}}(x) A_m^{\text{SE}} f_m^{\text{SE}} \right| \leq C \sqrt{n} \exp(-\sqrt{\pi d \mu n}).$$

2.2 Convergence theorems for (DE1), (DE2), and (DE3)

A convergence theorem of the formula (DE1) was given as follows. In the theorem, the sum is extended to $\sum_{j=-M}^N$ instead of $\sum_{j=-n}^n$ in the original formula (5).

Theorem 2.4 (Okayama et al. [11, Theorem 2.16]). Assume that f is analytic in $\phi(\mathcal{D}_d)$ for d with $0 < d < \pi/2$, and that there exist positive constants K , α and β such that (9) holds for all $z \in \phi(\mathcal{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let n be a positive integer, and let h be selected by

$$h = \frac{\log(2dn/\mu)}{n}. \quad (13)$$

Furthermore, let M and N be positive integers defined by

$$\begin{cases} M = n, & N = n - \left\lfloor \frac{1}{h} \log\left(\frac{\beta}{\alpha}\right) \right\rfloor & (\text{if } \mu = \alpha) \\ N = n, & M = n - \left\lfloor \frac{1}{h} \log\left(\frac{\alpha}{\beta}\right) \right\rfloor & (\text{if } \mu = \beta) \end{cases} \quad (14)$$

respectively. Then, there exists a constant C independent of n such that

$$\sup_{x \in (-1,1)} \left| \int_{-1}^x f(s) ds - \sum_{j=-M}^N f(\phi(jh)) \phi'(jh) J(j, h) (\phi^{-1}(t)) \right| \leq C \frac{\log(2dn/\mu)}{n} \exp\left(-\frac{\pi dn}{\log(2dn/\mu)}\right).$$

For the formula (DE2), we can deduce the following convergence theorem based on the existing result [16, Theorem 3.1 with $B = \pi/2$ and $C = 1$].

Theorem 2.5. Assume that f is analytic in $\phi(\mathcal{D}_d)$ for d with $0 < d < \pi/2$, and that there exist positive constants K and μ with $\mu \leq 1$ such that (12) holds for all $z \in \phi(\mathcal{D}_d)$. Let n be a positive integer, and let h be selected by (13). Then, there exists a constant C independent of n such that

$$\sup_{x \in (-1,1)} \left| \int_{-1}^x f(s) ds - \mathcal{I}_{\text{DE2}}(f, x) \right| \leq C \exp\left(-\frac{\pi dn}{\log(2dn/\mu)}\right).$$

This paper shows the following convergence theorem for the formula (DE3); we provide a proof in Section 4.

Theorem 2.6. Assume that f is analytic in $\phi(\mathcal{D}_d)$ for d with $0 < d < \pi/2$, and that there exist positive constants K , α with $\alpha \leq 1$ and β with $\beta \leq 1$ such that (9) holds for all $z \in \phi(\mathcal{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let n be a positive integer, and let h be selected by (13). Furthermore, let M and N be positive integers defined by (14). Then, there exists a constant C independent of n such that

$$\sup_{x \in (-1,1)} \left| \int_{-1}^x f(s) ds - \omega_m^{\text{DE}}(x) A_m^{\text{DE}} f_m^{\text{DE}} \right| \leq C \exp\left(-\frac{\pi dn}{\log(2dn/\mu)}\right).$$

2.3 Discussion on the convergence rate

Roughly speaking, the convergence rates of the formulas (DE1)–(DE3) (described in Theorems 2.4–2.6) are much higher than those of the formulas (SE1)–(SE3) (described in Theorems 2.1–2.3). The considerable difference originates from the variable transformations employed; the tanh transformation is employed in the formulas (SE1)–(SE3), whereas the DE transformation is employed in the formulas (DE1)–(DE3). In view of those theorems more precisely, the convergence rate of the formula (SE1) is slightly better than that of the formulas (SE2) and (SE3), and the convergence rate of the formula (DE1) is slightly better than that of the formulas (DE2) and (DE3). However, the difference is relatively slight, which may be observed from the results of the numerical examples provided in the next section.

3 Numerical examples

In this section, we present a numerical comparison between the formulas (SE1), (SE2), (SE3), (DE1), (DE2), and (DE3). The computation was performed on MacBook Pro with 2.4 GHz quad-core CPU (Intel Core i5) with 16GB memory, running Mac OS X 10.15.7. The computation programs were implemented in the C programming language with double-precision floating-point arithmetic, and compiled with Apple Clang version 12.0.0 with the ‘-O2’ option. We used the GNU Scientific Library to compute $\text{Si}(x)$ for (SE1) and (DE1). We used CBLAS for matrix-vector multiplication for (SE2), (SE3), (DE2), and (DE3). Source code for all programs is available at <https://github.com/okayamat/sinc-undef>.

We consider the following four examples:

$$\int_{-1}^x \frac{1}{\pi \sqrt{1-s^2}} ds = \frac{1}{\pi} \left(\arcsin x + \frac{\pi}{2} \right), \quad (15)$$

$$\int_{-1}^x \frac{1}{4 \log 2} \log \left(\frac{1+s}{1-s} \right) ds = \frac{1}{4 \log 2} \{ (1+x) \log(1+x) + (1-x) \log(1-x) - 2 \log 2 \}, \quad (16)$$

$$\int_{-1}^x \frac{2}{\pi(1+s^2)} ds = \frac{1}{2} + \frac{2}{\pi} \arctan x, \quad (17)$$

$$\int_{-1}^x \frac{-2[s\{\cos(4 \operatorname{artanh} s) + \cosh(\pi)\} + \sin(4 \operatorname{artanh} s)]}{\sqrt{\cos(4 \operatorname{artanh} s) + \cosh(\pi)}} ds = (1-x^2) \sqrt{\cos(4 \operatorname{artanh} x) + \cosh(\pi)}. \quad (18)$$

The integrand in (15) satisfies the assumptions of Theorems 2.1, 2.2, and 2.3 with $\alpha = \beta = 1/2$ and $d = \pi_-$, where π_- denotes a slightly smaller number than π (note that $d < \pi$). We chose $\pi_- = 3.14$ in the actual computation. It also satisfies the assumptions of Theorems 2.4, 2.5, and 2.6 with $\alpha = \beta = 1/2$ and $d = \pi_-/2$ (note that $d < \pi/2$). The integrand in (16) satisfies the assumptions of Theorems 2.1, 2.2, and 2.3 with $\alpha = \beta = 1_-$ and $d = \pi_-$, where 1_- denotes a slightly smaller number than 1. We chose $1_- = 0.99$ in actual computation. It also satisfies the assumptions of Theorems 2.4, 2.5, and 2.6 with $\alpha = \beta = 1_-$ and $d = \pi_-/2$. The integrand in (17) satisfies the assumptions of Theorems 2.1, 2.2, and 2.3 with $\alpha = \beta = 1$ and $d = \pi_-/2$. It also satisfies the assumptions of Theorems 2.4, 2.5, and 2.6 with $\alpha = \beta = 1$ and $d = \pi_-/6$.

In the case of (18), the situation is different from the examples above. The integrand in (18) satisfies the assumptions of Theorems 2.1, 2.2, and 2.3 with $\alpha = \beta = 1$ and $d = \pi_-/2$, but it does not satisfy the assumptions of Theorems 2.4, 2.5, and 2.6. According to Tanaka et al. [15], even in this case, the formulas with the DE transformation can attain the similar convergence rate to the formulas with the tanh transformation, with the same α and β and $d = \pi_-/6$. Therefore, we used these values in the computations of the formulas (DE1), (DE2), and (DE3) in this case.

We investigated the maximum errors among 1000 equally-spaced points over the interval $(-1, 1)$. The results are shown in Figures 1–8. From Figures 1, 3, 5, and 7, it may be observed that the formulas (SE1)–(SE3) exhibited almost the same convergence profile with respect to n . The same holds true for the formulas (DE1)–(DE3), and they were more accurate than the formulas (SE1)–(SE3). In Figure 7, the rate of the formulas (DE1)–(DE3) worsened. This occurred because the assumptions of Theorems 2.4, 2.5, and 2.6 are not satisfied in the case of (18). However, they still exhibited similar convergence profiles to the formulas (SE1)–(SE3) even in this case.

Figures 2, 4, 6, and 8 show the maximum errors with respect to the computation time. From Figures 2, 4, and 6, it may be observed that the formula (DE2) was the most efficient, and the formula (DE3) was the second-most efficient. Although the same

cannot be said for Figure 8, the two formulas still exhibited a similar convergence profile to the formulas (SE2) and (SE3). The formulas (SE1) and (DE1) were not so efficient for all examples. This may be attributed to the special function $\text{Si}(x)$ included in the basis functions of the two formulas.

4 Proofs

In this section, we provide a proof of Theorem 2.6.

4.1 Sketch of the proof

First, we explain the organization of the proof. Let $F(u) = f(\phi(u))\phi'(u)$ and

$$g(x) = \int_{-1}^x f(s) ds = \int_{-\infty}^{\phi^{-1}(x)} f(\phi(u))\phi'(u) du = \int_{-\infty}^{\phi^{-1}(x)} F(u) du.$$

The main strategy in the proof of Theorem 2.6 is to split the error into three terms as follows:

$$\begin{aligned} \left| \int_{-1}^x f(s) ds - \omega_m^{\text{DE}}(x) A_m^{\text{DE}} f_m \right| &= \left| g(x) - \sum_{i=-M}^N \left(\sum_{j=-M}^N F(jh) J(j, h)(ih) \right) \omega_i^{\text{DE}}(x) \right| \\ &\leq \left| g(x) - \sum_{i=-M}^N g(\phi(ih)) \omega_i^{\text{DE}}(x) \right| \\ &\quad + \left| \sum_{i=-M}^N \left(\int_{-\infty}^{ih} F(u) du - \sum_{j=-\infty}^{\infty} F(jh) J(j, h)(ih) \right) \omega_i^{\text{DE}}(x) \right| \\ &\quad + \left| \sum_{i=-M}^N \left(\sum_{j=-\infty}^{\infty} F(jh) J(j, h)(ih) - \sum_{j=-M}^N F(jh) J(j, h)(ih) \right) \omega_i^{\text{DE}}(x) \right|. \end{aligned}$$

In Section 4.2, we bound the first term as

$$\left| g(x) - \sum_{i=-M}^N g(\phi(ih)) \omega_i^{\text{DE}}(x) \right| \leq C_1 \exp\left(-\frac{\pi dn}{\log(2dn/\mu)}\right). \quad (19)$$

In Section 4.3, we bound the second term as

$$\left| \sum_{i=-M}^N \left(\int_{-\infty}^{ih} F(u) du - \sum_{j=-\infty}^{\infty} F(jh) J(j, h)(ih) \right) \omega_i^{\text{DE}}(x) \right| \leq C_2 \log(n+1) \frac{\log(2dn/\mu)}{n} \exp\left(-\frac{\pi dn}{\log(2dn/\mu)}\right). \quad (20)$$

In Section 4.4, we bound the third term as

$$\left| \sum_{i=-M}^N \left(\sum_{j=-\infty}^{\infty} F(jh) J(j, h)(ih) - \sum_{j=-M}^N F(jh) J(j, h)(ih) \right) \omega_i^{\text{DE}}(x) \right| \leq C_3 \log(n+1) \exp(-\pi dn). \quad (21)$$

Here, C_1 , C_2 , and C_3 are constants, independent of n . This completes the proof of Theorem 2.6.

4.2 Bound of the first term

The following function space is crucial in the analysis.

Definition 4.1. Let \mathcal{D} be a bounded and simply-connected domain (or Riemann surface) that contains the interval (a, b) . Let α and β be positive constants with $\alpha \leq 1$ and $\beta \leq 1$. Then, $\mathbf{M}_{\alpha, \beta}(\mathcal{D})$ denotes the family of all functions f that are analytic and bounded on \mathcal{D} and satisfy the following inequalities with a constant C

$$\begin{aligned} |f(z) - f(a)| &\leq C|z - a|^\alpha, \\ |f(b) - f(z)| &\leq C|b - z|^\beta \end{aligned}$$

for all $z \in \mathcal{D}$.

Hereafter, we set $(a, b) = (-1, 1)$. The next error analysis was given for functions belonging to this function space.

Theorem 4.1 (Okayama [8, Theorem 7]). Assume that $f \in \mathbf{M}_{\alpha, \beta}(\phi(\mathcal{D}_d))$ for d with $0 < d < \pi/2$. Let $\mu = \min\{\alpha, \beta\}$, let n be a positive integer, and let h be selected by (13). Furthermore, let M and N be positive integers defined by (14). Then, there exists a constant C independent of n such that

$$\sup_{x \in (-1, 1)} \left| f(x) - \sum_{i=-M}^N f(\phi(ih)) \omega_i^{\text{DE}}(x) \right| \leq C \exp\left(-\frac{\pi dn}{\log(2dn/\mu)}\right).$$

From this theorem, we obtain (19) using the following lemma.

Lemma 4.2. Assume that f is analytic in $\phi(\mathcal{D}_d)$ for d with $0 < d < \pi/2$, and that there exist positive constants K , α with $\alpha \leq 1$ and β with $\beta \leq 1$ such that (9) holds for all $z \in \phi(\mathcal{D}_d)$. Then, setting $g(x) = \int_{-1}^x f(s) ds$, we have $g \in \mathbf{M}_{\alpha, \beta}(\phi(\mathcal{D}_d))$.

We omit the proof of this lemma because it is a straightforward extension of an existing result [10, Lemma 5] (from $\mathbf{M}_{\alpha, \alpha}(\phi(\mathcal{D}_d))$ to $\mathbf{M}_{\alpha, \beta}(\phi(\mathcal{D}_d))$). This completes the proof of (19).

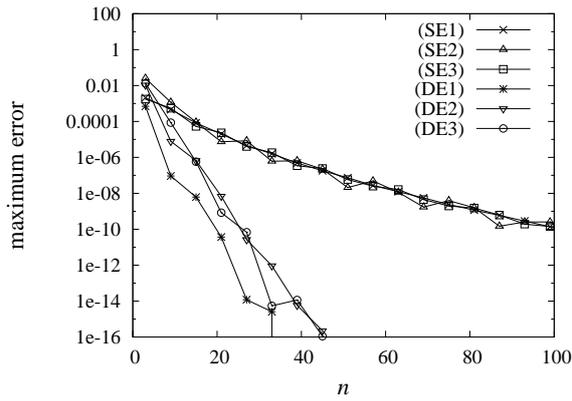


Figure 1: Convergence with respect to n for (15).

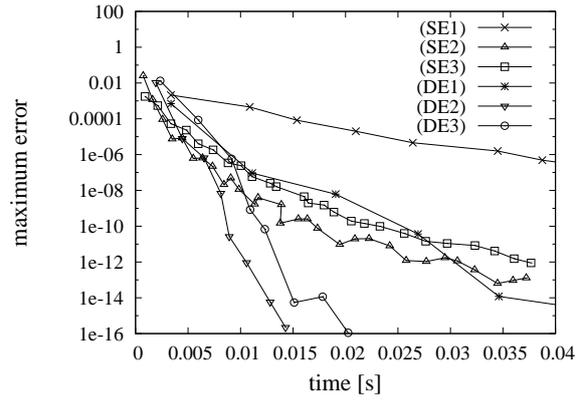


Figure 2: Convergence with respect to computation time for (15).

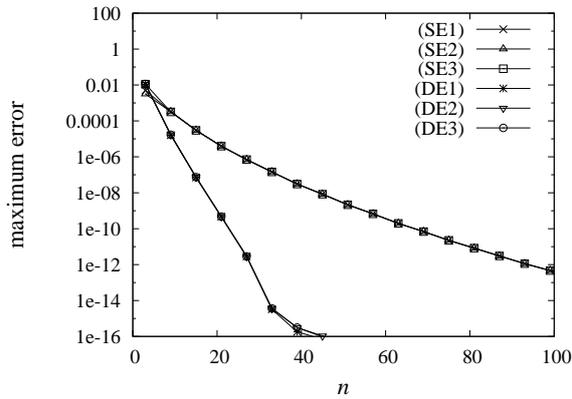


Figure 3: Convergence with respect to n for (16).

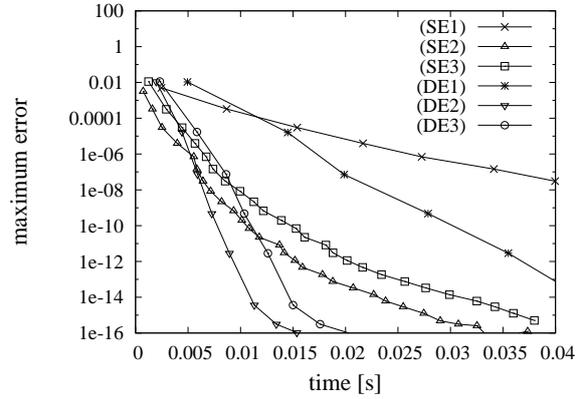


Figure 4: Convergence with respect to computation time for (16).

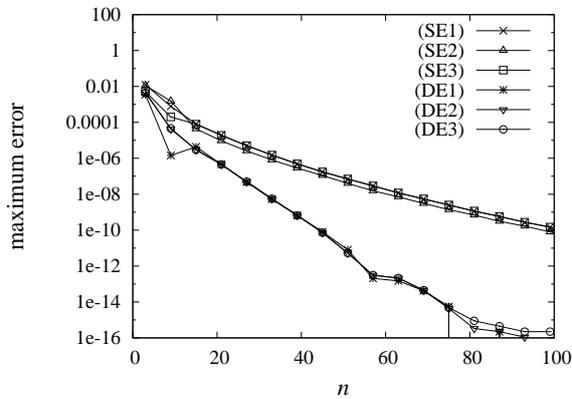


Figure 5: Convergence with respect to n for (17).

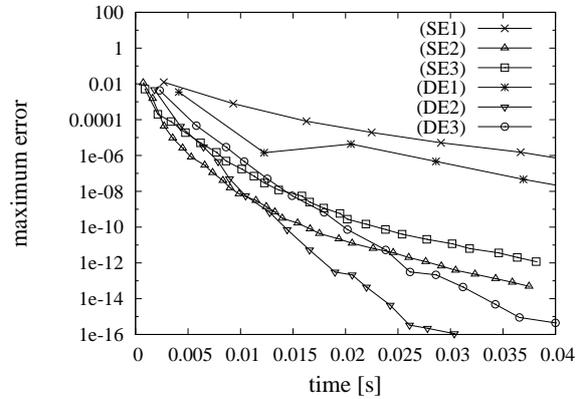


Figure 6: Convergence with respect to computation time for (17).

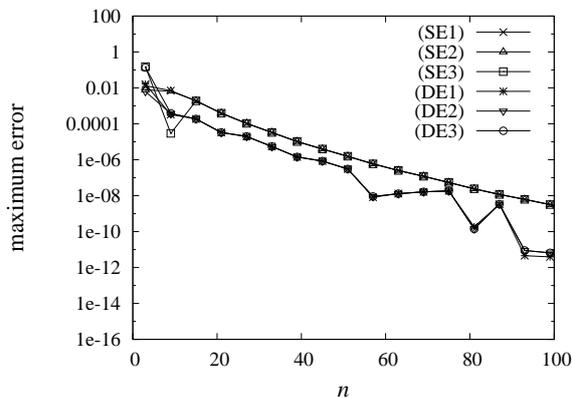


Figure 7: Convergence with respect to n for (18).

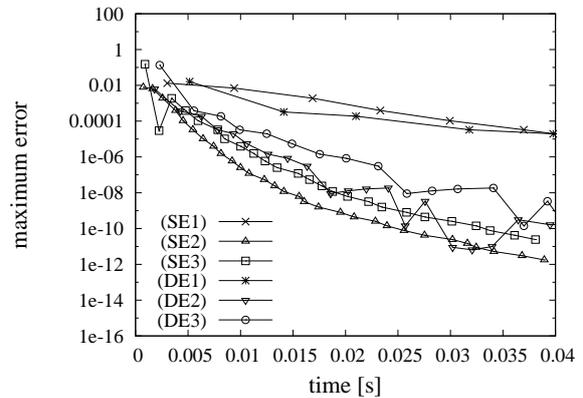


Figure 8: Convergence with respect to computation time for (18).

4.3 Bound of the second term

The following function space is crucial in the analysis.

Definition 4.2. Let d be a positive real number. Then, $B(\mathcal{D}_d)$ denotes the family of all functions f that are analytic on \mathcal{D}_d and satisfy

$$\Lambda(f, d) := \lim_{c \rightarrow d-0} \int_{-\infty}^{\infty} (|f(x+ic)| + |f(x-ic)|) dx < \infty \quad (22)$$

and

$$\lim_{x \rightarrow \pm\infty} \int_{-d}^d |f(x+iy)| dy = 0. \quad (23)$$

The theorem below holds for functions belonging to this function space; a proof is provided in Section A.

Theorem 4.3. Assume that $f \in B(\mathcal{D}_d)$. Then, for any $h > 0$, we have

$$\sup_{x \in \mathbb{R}} \left| \int_{-\infty}^x f(t) dt - \sum_{j=-\infty}^{\infty} f(jh)J(j, h)(x) \right| \leq \frac{4h e^{-\pi d/h}}{\pi d(1 - e^{-2\pi d/h})} \Lambda(f, d). \quad (24)$$

We see that $F \in B(\mathcal{D}_d)$ because the next lemma holds.

Lemma 4.4. Assume that f is analytic in $\phi(\mathcal{D}_d)$ for d with $0 < d < \pi/2$, and that there exist positive constants K , α and β such that (9) holds for all $z \in \phi(\mathcal{D}_d)$. Then, putting $F(u) = f(\phi(u))\phi'(u)$, we have $F \in B(\mathcal{D}_d)$ and

$$\Lambda(F, d) \leq \frac{2^{\alpha+\beta+1}K}{\mu \cos^{\alpha+\beta}((\pi/2) \sin d) \cos d},$$

where $\mu = \min\{\alpha, \beta\}$.

We omit the proof of this lemma as being identical with an existing result [11, Lemma 4.6]. Using this lemma and Theorem 4.3, we have

$$\left| \sum_{i=-M}^N \left(\int_{-\infty}^{ih} F(u) du - \sum_{j=-\infty}^{\infty} F(jh)J(j, h)(ih) \right) \omega_i^{\text{DE}}(x) \right| \leq \left(\sum_{i=-M}^N |\omega_i^{\text{DE}}(x)| \right) \cdot \frac{4h e^{-\pi d/h}}{\pi d(1 - e^{-2\pi d/h})} \Lambda(F, d). \quad (25)$$

Furthermore, the next lemma holds.

Lemma 4.5. Let $h > 0$, and let $n = \max\{M, N\}$. Then, there exists a constant C independent of n such that

$$\sup_{x \in (-1, 1)} \sum_{i=-M}^N |\omega_i^{\text{DE}}(x)| \leq C \log(n+1).$$

This lemma is shown by using the next result.

Lemma 4.6 (Stenger [13, p. 142]). Let $h > 0$. Then, it holds that

$$\sup_{t \in \mathbb{R}} \sum_{i=-n}^n \left| \text{sinc} \left(\frac{t - ih}{h} \right) \right| \leq \frac{2}{\pi} (3 + \log n).$$

Therefore, the right-hand side of (25) is further bounded as

$$\left(\sum_{i=-M}^N |\omega_i^{\text{DE}}(x)| \right) \cdot \frac{4h e^{-\pi d/h}}{\pi d(1 - e^{-2\pi d/h})} \Lambda(F, d) \leq C \log(n+1) \cdot \frac{4h e^{-\pi d/h}}{\pi d(1 - e^{-2\pi d/h})} \Lambda(F, d).$$

Substituting (13) to h , we obtain (20).

4.4 Bound of the third term

Note that the left-hand side of (21) is rewritten as

$$\left| \sum_{i=-M}^N \left(\sum_{j=-\infty}^{\infty} F(jh)J(j, h)(ih) - \sum_{j=-M}^N F(jh)J(j, h)(ih) \right) \omega_i^{\text{DE}}(x) \right| = \left| \sum_{i=-M}^N \left(\sum_{j=-\infty}^{-M-1} F(jh)J(j, h)(ih) + \sum_{j=N+1}^{\infty} F(jh)J(j, h)(ih) \right) \omega_i^{\text{DE}}(x) \right|.$$

The next lemma is crucial in the analysis.

Lemma 4.7 (Okayama et al. [11, Lemma 4.20]). Assume that f is analytic in $\phi(\mathcal{D}_d)$ for d with $0 < d < \pi/2$, and that there exist positive constants K , α and β such that (9) holds for all $z \in \phi(\mathcal{D}_d)$. Let $\mu = \min\{\alpha, \beta\}$, let n be a positive integer, let M and N be positive integers defined by (14) Then, putting $F(u) = f(\phi(u))\phi'(u)$, we have

$$\sup_{x \in \mathbb{R}} \left| \sum_{j=-\infty}^{-M-1} F(jh)J(j, h)(x) + \sum_{j=N+1}^{\infty} F(jh)J(j, h)(x) \right| \leq \tilde{C} \exp\left(-\frac{\pi}{2} \mu \exp(nh)\right),$$

where \tilde{C} is a positive constant independent of n .

Using this lemma, we obtain

$$\left| \sum_{i=-M}^N \left(\sum_{j=-\infty}^{-M-1} F(jh)J(j,h)(ih) + \sum_{j=N+1}^{\infty} F(jh)J(j,h)(ih) \right) \omega_i^{\text{DE}}(x) \right| \leq \left(\sum_{i=-M}^N |\omega_i^{\text{DE}}(x)| \right) \cdot \tilde{C} \exp\left(-\frac{\pi}{2} \mu \exp(nh)\right).$$

Furthermore, using Lemma 4.5 and substituting (13) to h , we obtain (21).

5 Concluding remarks

Based on the Sinc indefinite integration, several numerical approximation formulas of the form (1) have been presented in the literature. The formula (SE1) was derived by combining the Sinc indefinite integration with the tanh transformation. To remove a special function $\text{Si}(x)$ from the basis functions, another formula (SE2) was derived. In addition, yet another formula (SE3) was derived, which has an elegant matrix-vector form. These formulas can attain root-exponential convergence $O(\exp(-c\sqrt{n}))$. After a decade, the formulas (SE1) and (SE2) were improved as the formulas (DE1) and (DE2). In these two formulas, the tanh transformation is replaced with the DE transformation, and the convergence rate is drastically improved to $O(\exp(-cn/\log n))$. Motivated by this result, we improved the formula (SE3) as formula (DE3) in the present work by the same replacement of the variable transformation. By theoretical analysis, we have shown that the proposed formula (DE3) exhibits the same convergence property as the formulas (DE1) and (DE2). According to the numerical results, the formula (DE2) seems the most efficient, followed by the proposed formula (DE3), in all examples which satisfy the assumptions of Theorems 2.4–2.6. We note that the formula (DE3) exhibited the second-best performance, and it inherits the beautiful feature of the formula (SE3). Furthermore, as pointed out by Stenger [13, Eq. (4.6.10)], the basis function $\omega_{-M}^{\text{SE}}(x)$ can be replaced with $\text{sinc}((\psi^{-1}(x) + Mh)/h)$ in the case of indefinite integration (because indefinite integration approaches 0 as $x \rightarrow -1$). The same applies to the formula (DE3); $\omega_{-M}^{\text{DE}}(x)$ can be replaced with $\text{sinc}((\phi^{-1}(x) + Mh)/h)$. If the replacement were performed in the numerical implementation, the performance gap between the formulas (DE2) and (DE3) may be expected to narrow.

We consider the following as a future work. As described in Section 1, the formula (SE3) has already been applied to indefinite convolution [14] and in further application (see references in [14]). We are working to apply the formula (DE3) to those applications, and expect to report the results soon.

Acknowledgments

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A Rigorous bound of the discretization error of the Sinc indefinite integration

In this section, we consider the discretization error of the Sinc indefinite integration, namely,

$$\left| \int_{-\infty}^x f(t) dt - \sum_{k=-\infty}^{\infty} f(kh)J(k,h)(x) \right|,$$

and its bound. In fact, there exists a result for the bound [13, Lemma 3.6.4] and it has been used as a standard bound of the error, but we believe that this has not been rigorously proven. This is because the order of sum and integration was used in the proof, but this is difficult to justify under the given assumptions. Therefore, we present Theorem 4.3 instead, and provide a proof here.

First, we can derive the simple fact that functions in $B(\mathcal{D}_d)$ are integrable on the real line in the sense of the Riemann integral.

Proposition A.1. *If $f \in B(\mathcal{D}_d)$, then*

$$\left| \int_{-\infty}^{\infty} f(x) dx \right| \leq \frac{1}{2} \Lambda(f, d).$$

Proof. Let a be a positive real number, let ε be a real number with $0 < \varepsilon < d$, and let $d_\varepsilon = d - \varepsilon$. Furthermore, we define a contour $C_{a,\varepsilon}$ in \mathcal{D}_d by the rectangle

$$\{z = \pm a + iy \mid 0 \leq y \leq d_\varepsilon\} \cup \{z = x \mid -a \leq x \leq a\} \cup \{z = x + id_\varepsilon \mid -a \leq x \leq a\}$$

with counterclockwise direction (see Figure 9). Because f is analytic in \mathcal{D}_d , it follows from Cauchy's integral theorem that

$$\oint_{C_{a,\varepsilon}} f(z) dz = 0.$$

As $a \rightarrow \infty$, the integrals of f on the vertical parts of $C_{a,\varepsilon}$ converge to 0 owing to condition (23). Therefore we have

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} f(x + id_\varepsilon) dx.$$

Similarly, we can derive a variant of this relation in which the sign of d_ε is opposite. Therefore, we have

$$\int_{-\infty}^{\infty} f(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} (f(x + id_\varepsilon) + f(x - id_\varepsilon)) dx.$$

Taking the absolute values of both sides, we obtain the conclusion. \square

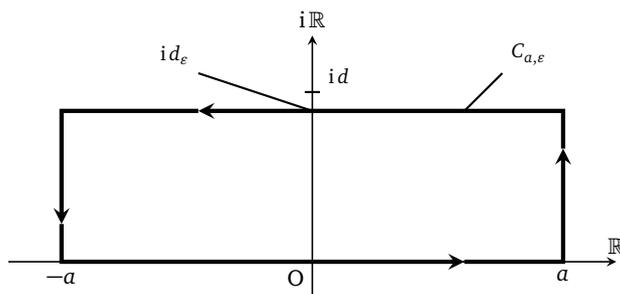


Figure 9: Contour $C_{a,\epsilon}$

Next, we present a well-known expression of the error of the Sinc approximation by a contour integral.

Proposition A.2 (Stenger [13, Eq. (3.1.6)]). *Let n be a positive integer, let ϵ be a real number with $0 < \epsilon < d$, and let $\Gamma_{n,\epsilon}$ be the contour given by*

$$\{z = \pm(n + 1/2)h + iy \mid -d_\epsilon \leq y \leq d_\epsilon\} \cup \{z = x \pm id_\epsilon \mid -(n + 1/2)h \leq x \leq (n + 1/2)h\}$$

with counterclockwise direction (see Figure 10), where $d_\epsilon = d - \epsilon$. When x is contained in the region surrounded by $\Gamma_{n,\epsilon}$ (i.e., n is sufficiently large),

$$f(x) - \sum_{k=-n}^n f(kh) \operatorname{sinc}\left(\frac{x - kh}{h}\right) = \frac{\sin(\pi x/h)}{2\pi i} \oint_{\Gamma_{n,\epsilon}} \frac{f(z)}{(z - x) \sin(\pi z/h)} dz \tag{26}$$

holds.

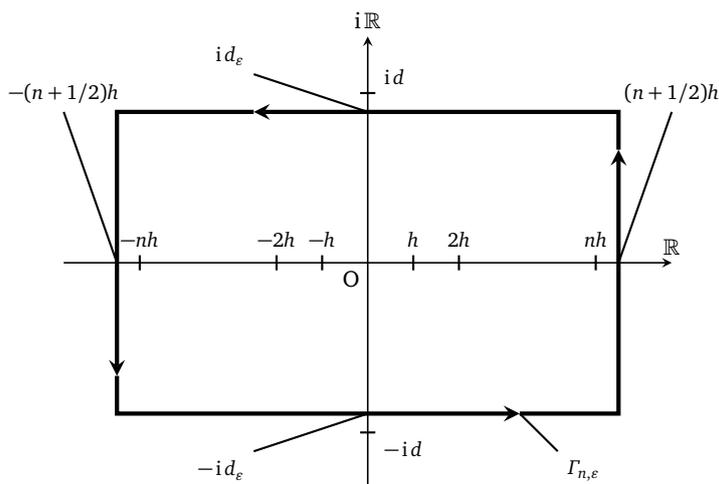


Figure 10: Contour $\Gamma_{n,\epsilon}$

To bound the discretization error of the Sinc indefinite integration, we integrate the right-hand side of equality (26) and bound it as shown in the proof of Theorem 4.3 below. In this bounding, the following lemma is key.

Lemma A.3. *Let x , y , s , and h be real numbers and assume that $h > 0$. Then, we have*

$$\left| \frac{1}{2\pi i} \int_{-\infty}^x \frac{\sin(\pi t/h)}{(s + iy) - t} dt \right| \leq \frac{3}{4\pi|y|} h \quad (y \neq 0), \tag{27}$$

$$\left| \frac{1}{2\pi i} \int_{-\infty}^{-(n+1/2)h} \frac{\sin(\pi t/h)}{(\pm(n + 1/2)h + iy) - t} dt \right| \leq \frac{1}{\pi} \left(2 + \frac{5|y|}{h} \right). \tag{28}$$

Furthermore, when $-nh \leq x \leq nh$, we have

$$\left| \frac{1}{2\pi i} \int_{-nh}^x \frac{\sin(\pi t/h)}{(s + iy) - t} dt \right| \leq \frac{3}{4\pi|y|} h \quad (y \neq 0), \tag{29}$$

$$\left| \frac{1}{2\pi i} \int_{-nh}^x \frac{\sin(\pi t/h)}{(\pm(n + 1/2)h + iy) - t} dt \right| \leq \frac{1}{\pi} \left(2 + \frac{5|y|}{h} \right). \tag{30}$$

Because this lemma can be proven similarly to existing results, we omit its proof. In fact, to obtain inequality (27), we need only to tighten inequality (3.6.24) of [13, Lemma 3.6.3] slightly. In addition, inequalities (28)–(30) are owing to Lemmas 4.11 and 4.12 of [15].

Based on the preparations above, we can show Theorem 4.3 as follows.

Proof of Theorem 4.3. First, we consider the truncated sum with $-n \leq k \leq n$ in the left-hand side of inequality (24). Then, by letting $n \rightarrow \infty$, we obtain the bound. To this end, we partition the integral with the truncated sum as

$$\begin{aligned} \int_{-\infty}^x \left\{ f(t) - \sum_{k=-n}^n f(kh) \operatorname{sinc}\left(\frac{t-kh}{h}\right) \right\} dt &= \int_{-nh}^x \left\{ f(t) - \sum_{k=-n}^n f(kh) \operatorname{sinc}\left(\frac{t-kh}{h}\right) \right\} dt \\ &\quad + \int_{-(n+1)h}^{-nh} \left\{ f(t) - \sum_{k=-n}^n f(kh) \operatorname{sinc}\left(\frac{t-kh}{h}\right) \right\} dt \\ &\quad + \int_{-\infty}^{-(n+1)h} \left\{ f(t) - \sum_{k=-n}^n f(kh) \operatorname{sinc}\left(\frac{t-kh}{h}\right) \right\} dt, \end{aligned}$$

and estimate each term. It suffices to consider the case in which n is sufficiently large that $-nh \leq x \leq nh$ holds.

We begin with the first term. By integration of equality (26), the first term is rewritten as

$$\int_{-nh}^x f(t) dt - \sum_{k=-n}^n f(kh) \int_{-nh}^x \operatorname{sinc}\left(\frac{t-kh}{h}\right) dt = \oint_{\Gamma_{n,\varepsilon}} \left\{ \int_{-nh}^x \frac{\sin(\pi t/h)}{2\pi i(z-t)} dt \right\} \frac{f(z)}{\sin(\pi z/h)} dz.$$

Based on the conditions (22) and (23) defining $B(\mathcal{D}_d)$, we can bound the integrals on the horizontal and vertical paths of $\Gamma_{n,\varepsilon}$ when $n \rightarrow \infty$ and $\varepsilon \rightarrow +0$. For the integral on the horizontal paths, we use inequality (29). In addition, for the integral on the vertical paths, we use inequalities (30) and (29) for the cases $|y| \leq h$ and $|y| > h$, respectively. As a result, we have

$$\begin{aligned} \lim_{\substack{n \rightarrow \infty \\ \varepsilon \rightarrow +0}} \left| \oint_{\Gamma_{n,\varepsilon}} \left\{ \int_{-nh}^x \frac{\sin(\pi t/h)}{2\pi i(z-t)} dt \right\} \frac{f(z)}{\sin(\pi z/h)} dz \right| &\leq \frac{3h}{4\pi d \sinh(\pi d/h)} \lim_{n \rightarrow \infty} \int_{-(n+1/2)h}^{(n+1/2)h} (|f(t+id)| + |f(t-id)|) dt \\ &\quad + \frac{1}{\pi} \left(2 + \frac{5h}{h}\right) \lim_{n \rightarrow \infty} \int_{|y| \leq h} (|f((n+1/2)h+iy)| + |f(-(n+1/2)h+iy)|) dy \\ &\quad + \frac{3h}{4\pi h} \lim_{n \rightarrow \infty} \int_{h < |y| \leq d} (|f((n+1/2)h+iy)| + |f(-(n+1/2)h+iy)|) dy \\ &\leq \frac{3h}{4\pi d \sinh(\pi d/h)} \lim_{n \rightarrow \infty} \int_{-(n+1/2)h}^{(n+1/2)h} (|f(t+id)| + |f(t-id)|) dt \\ &\quad + \lim_{n \rightarrow \infty} \frac{7}{\pi} \int_{-d}^d (|f((n+1/2)h+iy)| + |f(-(n+1/2)h+iy)|) dy \\ &= \frac{3h e^{-\pi d/h}}{2\pi d(1 - e^{-2\pi d/h})} \Lambda(f, d) + 0. \end{aligned}$$

Next, we bound the third term. By complex contour integral, we have

$$\int_{-\infty}^{-(n+1)h} f(t) dt - \sum_{k=-n}^n f(kh) J(k, h)(-(n+1)h) = \int_{-\infty}^{-(n+1)h} f(t) dt - \oint_{\Gamma_{n,\varepsilon}} \left\{ \int_{-\infty}^{-(n+1)h} \frac{\sin(\pi t/h)}{2\pi i(z-t)} dt \right\} \frac{f(z)}{\sin(\pi z/h)} dz.$$

Here, note that t is not contained in the region surrounded by $\Gamma_{n,\varepsilon}$. Because it follows from Proposition A.1 that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{-(n+1)h} f(t) dt = 0,$$

we need only to bound the part of the contour integral similarly to the first term. For the integral on the horizontal paths, we use the inequality (27). In addition, for the integral on the vertical paths, we use inequalities (28) and (27) for the cases $|y| \leq h$ and $|y| > h$, respectively. As a result, we have

$$\lim_{\substack{n \rightarrow \infty \\ \varepsilon \rightarrow +0}} \left| \oint_{\Gamma_{n,\varepsilon}} \left\{ \int_{-\infty}^{-(n+1)h} \frac{\sin(\pi t/h)}{2\pi i(z-t)} dt \right\} \frac{f(z)}{\sin(\pi z/h)} dz \right| \leq \frac{3h e^{-\pi d/h}}{2\pi d(1 - e^{-2\pi d/h})} \Lambda(f, d) + 0.$$

Finally, we bound the second term. We replace the contour $\Gamma_{n,\varepsilon}$ with $\Gamma_{n-1,\varepsilon}$. Noting that $-(n+1)h \leq t \leq -nh$, we have

$$\begin{aligned} \int_{-(n+1)h}^{-nh} \left\{ f(t) - \sum_{k=-n}^n f(kh) \operatorname{sinc}\left(\frac{t-kh}{h}\right) \right\} dt &= \int_{-(n+1)h}^{-nh} \left\{ f(t) - f(-nh) \operatorname{sinc}\left(\frac{t+nh}{h}\right) - f(nh) \operatorname{sinc}\left(\frac{t-nh}{h}\right) \right\} dt \\ &\quad - \int_{-(n+1)h}^{-nh} \left\{ \frac{\sin(\pi t/h)}{2\pi i} \oint_{\Gamma_{n-1,\varepsilon}} \frac{f(z)}{(z-t)\sin(\pi z/h)} dz \right\} dt. \end{aligned}$$

Then, by using $|\operatorname{sinc}((t - kh)/h)| \leq 1$, we have

$$\left| \int_{-(n+1)h}^{-nh} \left\{ f(t) - \sum_{k=-n}^n f(kh) \operatorname{sinc}\left(\frac{t - kh}{h}\right) \right\} dt \right| \leq h \left\{ \max_{-(n+1)h \leq t \leq -nh} |f(t)| + |f(-nh)| + |f(nh)| \right\} \\ + h \max_{-(n+1)h \leq t \leq -nh} \left\{ \frac{|\sin(\pi t/h)|}{|2\pi i|} \left| \oint_{\Gamma_{n-1, \varepsilon}} \frac{f(z)}{(z - t) \sin(\pi z/h)} dz \right| \right\}.$$

As $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$, the last term of the right-hand side is bounded as

$$h \lim_{\substack{n \rightarrow \infty \\ \varepsilon \rightarrow +0}} \frac{|\sin(\pi t/h)|}{|2\pi i|} \left| \oint_{\Gamma_{n-1, \varepsilon}} \frac{f(z)}{(z - t) \sin(\pi z/h)} dz \right| \leq \frac{h e^{-\pi d/h}}{\pi d(1 - e^{-2\pi d/h})} \Lambda(f, d).$$

The other terms tend to 0 as $n \rightarrow \infty$ because $\lim_{t \rightarrow \pm\infty} |f(t)| = 0$, which follows from Proposition A.1.

Combining all the bounds, we obtain inequality (24). \square

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