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# Explicit algebraic solution of Zolotarev's First Problem for low-degree polynomials, Part II

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**Abstract**

With recourse to [41], we consider three algorithms for explicitly solving, by algebraic means, Zolotarev's First Problem (ZFP) of 1868 which is described e.g. in [1, 5, 26, 27]. We avoid the application of elliptic functions by drawing first on three tentative forms $Z_{n,s,\alpha,\beta}$ $(1<\alpha<\beta)$ of the sought-for monic proper Zolotarev polynomial $Z_{n,s}$ $(n\geq 4, s>\tan^2(\pi/(2n)))$. In order to compute then the compatible $\alpha=\alpha_0$ and $\beta=\beta_0$, so that $Z_{n_0,s_0,\alpha_0,\beta_0}=Z_{n_0,s_0}$ will hold for a prescribed degree $n=n_0$ and prescribed intrinsic parameter $s=s_0$, we draw on three intertwined variants and deploy them exemplarily to the third tentative form (not considered in [41]). We conclude that our first tentative form constitutes, in conjunction with our third variant, a deterministic algebraic algorithm for solving ZFP, which is advantageous with respect to complexity reduction. Three related algebraic algorithms from literature for solving ZFP, [20, 28, 48], are examined, refined and exemplified. Further existing non-elliptic approaches to ZFP, including the one by means of parametrization of algebraic curves [44], are referenced and annotated. Explicit representations of $Z_{n,s}$ in the algebraic power form (unexampled if $n>7$) and novel characteristics, which facilitate the algebraic construction of $Z_{n,s}$, are provided and additionally stored, for $n\leq 13$, in an online ZFP-repository.

**Keywords:** Algebraic solution, Gröbner Basis, least deviation from zero, reduced relation curve, two fixed leading coefficients, Zolotarev's First Problem, Zolotarev polynomials.

**AMS 2010 AMS classification:** 41A10, 41A29, 41A50, 68W30.

## 1 Reviewing Introduction

### 1.1 Zolotarev's First Problem (ZFP)

A forerunner of Zolotarev's First Problem dates back to 1854 [8, p. 123], see also [9, Theorem 5]:

**Chebyshev's Extremal Problem (CEP).** Determine, for a given degree $n\geq 1$, among all real monic polynomials $P_n$, with $P_n(x)=\sum_{k=0}^{n-1}a_{k,n}x^k+x^n$, the one which deviates least from the zero-function on the unit interval $\boldsymbol{I}=[-1,1]$, measured in the uniform norm $\|.\|_\infty$.

Thus, the first leading coefficient of $P_n$ is assumed to be fixed ($a_{n,n}=1$). The solution of CEP is $P_n^*=2^{1-n}T_n$ with least deviation $L_n=\min_{(a_{0,n},\ldots,a_{n-1,n})}\|P_n-0\|_\infty=\|2^{1-n}T_n\|_\infty=2^{1-n}$, where

$$T_n(x)=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n}{n-k}\binom{n-k}{k}2^{n-2k-1}x^{n-2k},\text{ with } x\in\mathbb{R}\text{ and }\|T_n\|_\infty=1, \quad (1)$$

denotes the $n$-th Chebyshev polynomial of the first kind with respect to $\boldsymbol{I}$, see [1, 11, 46].

CEP emanated from Chebyshev's passion for linkage mechanisms which convert rotary to approximate straight-line motion [38, p. 102]. He posed to his student Zolotarev [60, p. 2] an extension of CEP which was later renamed after the latter:

**Zolotarev's First Problem (ZFP).** Determine, for a given degree $n\geq 2$ and for a given parameter $s\in\mathbb{R}\backslash\{0\}$, among all real monic polynomials $Q_{n,s}$ with $Q_{n,s}(x)=\sum_{k=0}^{n-2}a_{k,n}x^k+(-ns)x^{n-1}+x^n$, the one which deviates least from the zero-function on $\boldsymbol{I}$, measured in the uniform norm.

Thus, in ZFP not only the first but also the second leading coefficient of $Q_{n,s}$ is assumed to be fixed ($a_{n-1,n}=-ns$). Zolotarev's original assumption was $a_{n-1,n}=-\sigma$, which is still in use [35, p. 67] while some authors [20, p. 934] prefer to consider

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$a_{n-1,n} = \sigma$, where $\sigma \in \mathbb{R}$. However, the here considered form of $a_{n-1,n}$ is most commonly used [2, p. 16], [5]. We denote by $Q^*_{n,s} = Z_{n,s}$ with

$$Z_{n,s}(x) = \sum_{k=0}^{n-2} a^*_{k,n}(s)x^k + (-ns)x^{n-1} + x^n, \tag{2}$$

the solution of ZFP in the algebraic power form with optimal coefficients $a^*_{k,n}(s)$, and by $L_n(s)$ its (least) deviation from the zero-function on $\boldsymbol{I}$. We shall write $n_0$ respectively $s_0$ in place of $n$ and $s$ when a concrete number is considered.

At first sight, CEP might appear to be harder to solve than ZFP because the latter has one less coefficient to be optimized. The following Example is an eligible candidate for disproving that impression.

**Example 1.1.** Let $n_0 = 5$. The solution of CEP is then $T_5(x)/16 = \left(\frac{5}{16}\right)x + \left(\frac{-5}{4}\right)x^3 + x^5$, and the least deviation is $L_{n_0} = L_5 = \frac{1}{16}$. That is, three of the optimal coefficients are zero, and the remaining two, as well as $L_5$, are neat rational numbers.

On the other hand, ZFP, with $s_0 = \frac{1}{5}$ (say), requires to optimize $Q_{5,\frac{1}{5}}(x) = \sum_{k=0}^{3} a_{k,5}x^k + (-1)x^4 + x^5$. Using the approach (A) or (B) to ZFP as described below, the solution turns out to be $Z_{5,\frac{1}{5}}(x) = \sum_{k=0}^{3} a^*_{k,5}(\frac{1}{5})x^k + (-1)x^4 + x^5$, where the optimal coefficients $a^*_{k,5}(\frac{1}{5})$ and the least deviation $L_{n_0}(s_0) = L_5(\frac{1}{5})$ can be explicitly expressed in terms of bulky radicals, see Appendix 8.1. After chopping, a numerical decimal approximation to $Z_{5,\frac{1}{5}}$ and $L_5(\frac{1}{5})$ reads

$$\begin{gathered} Z_{5,\frac{1}{5}}(x) \approx -0.1065834340 + 0.4581775889x + 0.9557598788x^2 + (-1.4581775889)x^3 + (-1)x^4 + x^5, \\ L_5(\tfrac{1}{5}) \approx 0.1508235551. \end{gathered} \tag{3}$$

Thus, here the explicit solution of ZFP is much more involved than the explicit solution of CEP, and this is because the function given by $f(x) = -x^4 + x^5$, which is to be approximated by a polynomial of degree $\le 3$, is not symmetric, whereas the function given by $g(x) = x^5$, which is to be approximated by a polynomial of degree $\le 4$, is symmetric (to the origin). Figure 1 displays the graphs of $\frac{T_5}{16}$ and $Z_{5,\frac{1}{5}}$ on the interval $[-1, 1.5]$ (including the constants $\pm L_5$ and $\pm L_5(\frac{1}{5})$). □

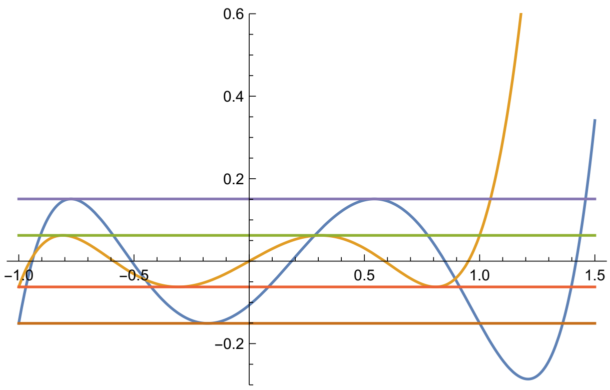


**Figure 1**

Zolotarev's solution of ZFP in 1868 [59], and in a reworked form in 1877 [60], splits into an improper and into a proper part. But first of all, he observed that $Z_{n,-s}(x) = (-1)^n Z_{n,s}(-x)$ holds, so that it henceforth suffices to consider $s > 0$. Zolotarev then showed that, if $0 < s \le \tau_n := \tan^2(\frac{\pi}{2n}) \in (0,1]$ ($\tau_n = 1$ only if $n = 2$), $Z_{n,s}$ can be represented algebraically by means of $T_n$ as

$$Z_{n,s}(x) = 2^{1-n}(1+s)^n T_n\left(\frac{x-s}{1+s}\right), \text{ with } L_n(s) = 2^{1-n}(1+s)^n, \tag{4}$$

which nowadays is called an improper Zolotarev polynomial (in the limiting case $s = 0$ it would reduce to the solution of CEP), see [1, p. 280], [2, p. 16], [5], [26, p. 406], [53]. Using Zolotarev's original assumption $a_{n-1,n} = -\sigma$, the condition $0 < s \le \tau_n$ has to be replaced by $0 < \sigma \le n\tau_n$ (misprinted e.g. in [12], [35, (5.7)]), see [13, p. 467].

In what follows, we focus on the solution of ZFP when $s > \tau_n$ holds. It is considered as *complicated* [26, p. 405], *unwieldy* [52, p. 118] or even as *mysterious* [53, p. 219], and is nowadays called *a proper* [15, 49, 57], or *hard-core* [26, 30, 47] *Zolotarev polynomial*, due to the fact that, for $s > \tau_n$, $Z_{n,s}$ was represented transcendentally by means of elliptic functions (without providing algebraic expressions for the optimal coefficients, although in the original problem statement [60, p. 2] the sought-for *remaining coefficients* are explicitly enumerated). That solution resembles the solution of CEP when describing it with the aid of circular (trigonometric) functions, i.e., $2^{1-n}T_n = 2^{1-n}\cos(n \arccos(x))$ for $x \in \boldsymbol{I}$. In contrast to the circular solution of CEP [11, p. 46], there does not exist a simple process for converting Zolotarev's elliptic solution of ZFP into an algebraic one. Even for the first reasonable degree $n_0 = 2$ (for which $Z_{2,s}(x) = a^*_{0,2}(s) + (-2s)x + x^2$ holds (with $a^*_{0,2}(s) = -1$) for all $s > \tau_2 = 1$, and which we henceforth leave aside) it turns out to be *unexpectedly complicated* and *very hard* to do so [5, p. 3 and Section 3], [28, p. 245]. On the other hand, an algebraic solution of ZFP for all $n$ does exist, see the editorial remark in [61, p. 361] where the paper [7] by Chebotarev is referenced.

Here we describe how to construct, for a given $s > \tau_n$, a proper $Z_{n,s}$ by algebraic means, and we provide precomputed data for the solution of ZFP if $n \le 13$. This is within the spirit of statements e.g. by Kaltofen [16, p. 8]: *One of my favorite open problems in Symbolic Computation ... solve [ZFP] for $n \ge 6$ on a computer* or by Peherstorfer [32, p. 143]: *There was and still is a demand for a description [of $Z_{n,s}$] without elliptic functions*.

In textbooks on Approximation Theory, we were able to find a non-elliptic solution to ZFP only for $n_0 = 3$, see [27, p. 156], [57, p. 98], and Remark 6 below. Experience shows that intricacies and complexity are typical features to encounter when dealing with algebraic solutions of ZFP if $s > \tau_n$. Bernstein [3, p. 156] concedes: *I soon recognized its algebraic difficulties which increase rapidly with the degree n of the polynomial, and it occurred to me to formulate the asymptotic problem*, and Malyshev [20, p. 932] states: *An algebraic solution [of ZFP] requires an amazing amount of calculations*.

From the Alternation (aka: Equal Ripple) Theorem of Approximation Theory the following properties of $Z_{n,s}$ can be condensed [1, p. 280], [2, p. 16], [5, 13], [26, p. 406]:

**Theorem 1.1 (Characterizing Properties of a proper Zolotarev Polynomial).** *$Z_{n,s}$ attains the values $\pm L_n(s) = \pm\|Z_{n,s}\|_\infty$ alternately at $n$ equioscillation points $-1 = z_0 < z_1 = z_1(n,s) < \ldots < z_{n-2} = z_{n-2}(n,s) < z_{n-1} = 1$ of $\boldsymbol{I}$, where $Z_{n,s}(-1) = (-1)^n L_n(s)$ may be assumed. Additionally, there exist three special points $\gamma = \gamma(n,s) < \alpha = \alpha(n,s) < \beta = \beta(n,s)$ such that $-Z_{n,s}(1) = -Z_{n,s}(\alpha) = Z_{n,s}(\beta) = L_n(s)$ holds, and $\gamma$ (with $1 < \gamma$) is a zero of the first derivative of $Z_{n,s}$ with respect to $x$, so that $Z_{n,s}$ possesses $n$ equioscillation points on $\boldsymbol{I}$ and two on $[\alpha,\beta]$ (i.e., the endpoints of that interval). The range of $\gamma$ and $\alpha$ is $(1,\infty)$, whereas the range of $\beta$ is $(\nu_n,\infty)$ with $\nu_n := 1 + 2\tau_n$. The four parameters $\alpha,\beta,\gamma,s$ are interrelated by the identity $\alpha + \beta = 2(\gamma + s)$.*

Thus there holds $|Z_{n,s}(x)| \le L_n(s)$ for $x \in \boldsymbol{I} \cup [\alpha,\beta]$ and $|Z_{n,s}(x)| > L_n(s)$ elsewhere, so that the set $\{x \in \mathbb{R} : |Z_{n,s}(x)| \le L_n(s)\}$, the inverse polynomial image of $Z_{n,s}$, is said to consist of two Jordan arcs ($\boldsymbol{I}$ and $[\alpha,\beta]$), see [36, 48].

We note in passing that in Example 1.1 the three special points of $Z_{5,\frac{1}{5}}$, that is, $\gamma = \gamma(n_0,s_0) = \gamma(5,\frac{1}{5}) < \alpha = \alpha(n_0,s_0) = \alpha(5,\frac{1}{5}) < \beta = \beta(n_0,s_0) = \beta(5,\frac{1}{5})$, can be expressed as

$$\begin{aligned}
\gamma &= \text{Root}[-185 + 560z - 18z^2 - 1080z^3 + 675z^4, 2] \approx 1.2102107032,\\
\alpha &= \text{Root}[-17 - 332z + 738z^2 - 540z^3 + 135z^4, 2] \approx 1.3606346018,\\
\beta &= \text{Root}[-269 + 260z - 702z^2 - 540z^3 + 675z^4, 2] \approx 1.4597868046.
\end{aligned} \tag{5}$$

Here, and in what follows, a root object $\text{Root}[P(z), l]$ with index $l$ denotes (in the traditional form) the $l$-th root of the polynomial $P$ in the variable $z$ and constitutes an exact representation for algebraic numbers (for details see the CAS *Mathematica*, which we are using to conduct computations). Since in (5) $P$ is a quartic polynomial, the three special points $\gamma, \alpha, \beta$ could alternatively, but less compactly, be expressed in terms of radicals. On the other hand, the least deviation $L_5(\frac{1}{5})$ which was given in terms of radicals, see Example 1.1 and Appendix 8.1, can be denoted more compactly in terms of a root object:

$$L_{n_0}(s_0) = L_5(\frac{1}{5}) = \text{Root}[-11943936 - 693026816z + 13578720768z^2 - 85074300000z^3 + 192216796875z^4, 2]. \tag{6}$$

The same holds for the optimal coefficients of $Z_{5,\frac{1}{5}}$, see also Section 4 below:

$$\begin{aligned}
a^*_{0,5}(\tfrac{1}{5}) &= \text{Root}[-1178141 + 1571039500z + 23548707018z^2 + 102160237500z^3 + 192216796875z^4, 1],\\
a^*_{1,5}(\tfrac{1}{5}) &= \text{Root}[-37 - 44z + 378z^2 - 540z^3 + 675z^4, 2],\\
a^*_{2,5}(\tfrac{1}{5}) &= \text{Root}[2697536 - 10535680z + 15437952z^2 - 10060200z^3 + 2460375z^4, 1],\\
a^*_{3,5}(\tfrac{1}{5}) &= \text{Root}[1600 + 5120z + 6048z^2 + 3240z^3 + 675z^4, 1].
\end{aligned} \tag{7}$$

Generally, a representation of $Z_{n,s}$ in terms of root object coefficients will assume a compact form if the intrinsic parameter $s > \tau_n$ is chosen to be an algebraic number.

It follows from Theorem 1.1 that $y = Z_{n,s}$ satisfies the Abel-Pell differential equation [1, p. 280], [2, p. 17], [5, p. 10], [22, p. 15], [51, p. 2486]:

$$(1 - x^2)(x - \alpha)(x - \beta)\left(\frac{y'(x)}{n(x - \gamma)}\right)^2 + (y(x))^2 = (L_n(s))^2. \tag{8}$$

Summarizing, we found in literature two frequent approaches to a constructive algebraic solution of ZFP (thus avoiding the use of elliptic functions):

(A) Determine a power form representation of the (proper) Zolotarev polynomial with re-parametrized and explicit closed-form expressions for the coefficients:

$$\widetilde{Z}_{n,t}(x) = \sum_{k=0}^{n-1} b_{k,n}(t) x^k + x^n, \tag{9}$$

where the parameter $t$ varies in some finite interval $I_n$ and $b_{k,n}$ is a continuous injective function of $t$ with $b_{n-1,n}(t) = -ns_n(t)$. For a prescribed $n_0$ and $s_0 > \tau_{n_0}$ one then has to solve the equation $s_{n_0}(t) = s_0$ to get the optimal $t_0 \in I_{n_0}$ so that $\widetilde{Z}_{n_0,t_0} = Z_{n_0,s_0}$ will hold. To the best of our knowledge, the highest known degree $n = n_0$ for which ZFP has been solved along the approach (A), is $n = 7$, and the delineated parametrization turns out to be rational (for $n < 5$) or radical (for $5 \le n \le 7$), see [44] and the references given therein.

(B) Create, by deploying Theorem 1.1 and (8), a tentative form $Z_{n,s,\alpha,\beta}$ of $Z_{n,s}$ which depends, additionally to $n$ and $s$, also on $\alpha$ and $\beta$ (the endpoints of the interval $[\alpha,\beta]$ in Theorem 1.1), and then determine algebraically, for a prescribed $n = n_0$ and $s = s_0 > \tau_{n_0}$, the compatible numbers $\alpha = \alpha_0 = \alpha(n_0,s_0)$ and $\beta = \beta_0 = \beta(n_0,s_0)$ so that $Z_{n_0,s_0,\alpha_0,\beta_0} = Z_{n_0,s_0}$ will hold. We shall consider three tentative forms of $Z_{n,s}$ and three variants for the determination of $\alpha_0$ and $\beta_0$. To the best of our knowledge, the highest known degree $n = n_0$ for which ZFP has been explicitly solved along the approach (B), is $n = 13$ (see Appendix 8.3 and [55]).

Since $s = s(\alpha,\beta)$, see Section 2.4 below, we shall henceforth denote a tentative form $Z_{n,s,\alpha,\beta}$ by $Z_{n,\alpha,\beta}$. We leave aside purely numerical approaches to ZFP, such as given e.g. in [14, 17, 19], and focus here on the approach (B).

In doing so it would suffice, in view of the approach (A), to consider $n > 7$. However, in the Examples given below we treat cases of $n \le 7$ to allow a cross-check between these two approaches and to narrow the bulkiness of the terms occurring. Because, when applying the approach (B) for $n > 7$, the Examples grow quite complex, see Appendix 8.3 and [55].

Three algebraic solution paths to ZFP from literature, which are related to approach (B), are considered in Sections 4 - 6 below. Further non-elliptic approaches to ZFP are referenced in Remark 6 below.

### 1.2 Three algebraic Algorithms for solving ZFP

In [41] we have proposed two solutions to ZFP by algebraic means along the approach (B). In view of the mentioned complexity, we have confined ourselves to the construction of (proper) Zolotarev polynomials of a degree $n \le 12$, backed by an online ZFP-repository [55] to which bulky precomputed data were outsourced. In the meantime, we have added more data to [55] in order to facilitate the practical calculation for the degree $n_0 = 13$, too. We briefly sketch the two two-staged algorithms from [41]:

Our 1st algebraic algorithm for solving ZFP sets up for $Z_{n,s}$ the ansatz $Z_{n,s}(x) = \sum_{k=0}^{n-2} a_{k,n}x^k + (-ns)x^{n-1} + x^n$. A recursive description is provided which expresses first $a_{n-2,n}$ in terms of $\alpha,\beta,s$; then expresses $a_{n-3,n}$ in terms of $\alpha,\beta,s$ and $a_{n-2,n} = a_{n,n-2}(\alpha,\beta,s)$; and so forth percolating down, until finally describing how $a_{0,n}$ can be recovered from previously determined coefficients by back substitution, see [41, Theorems 4.1, 5.1] for $n \in \{6,7\}$. In this way $Z_{n,s}(x)$ transforms into a tentative form (we keep the prescribed parameter $s$ in the second leading coefficient only)

$$R_{n,\alpha,\beta}(x) = \sum_{k=0}^{n-2} A_{k,n}(\alpha,\beta)x^k + (-ns)x^{n-1} + x^n. \tag{10}$$

This solves ZFP algebraically, provided $\alpha$ and $\beta$ are known for the given $n$ and $s$. In the second part of the algorithm, the determination of $\alpha = \alpha_0$ and $\beta = \beta_0$ in [41] was accomplished by means of Malyshev polynomials, see also Remark 7 below. A granulated, hands-on creation of (10) and of Malyshev polynomials, in a *Mathematica*-conformal syntax is given, for $n = 5$, in [55] where $R_{n,\alpha,\beta}$ is provided for $n \le 13$.

Our 2nd algebraic algorithm for solving ZFP along the approach (B) starts with a rough tentative form of $Z_{n,s}$ which depends on $\alpha,\beta$ and additionally on those equioscillation points $z_j = y_j$ from $\boldsymbol{I}\backslash\{-1,1\}$ where $Z_{n,s}(y_j) = -L_n(s)$ holds:

$$\begin{aligned} &(\text{if } n = 2m+1 \text{ is odd}) \quad S_{n,\alpha,\beta,y_j}(x) = (x-\alpha)(x^2-1)\prod_{j=1}^{m-1}(x-y_j)^2 - (\tfrac{1}{2})(\beta-\alpha)(\beta^2-1)\prod_{j=1}^{m-1}(\beta-y_j)^2 \\ &(\text{if } n = 2m+2 \text{ is even}) \quad S_{n,\alpha,\beta,y_j}(x) = (x-1)(x-\alpha)\prod_{j=1}^{m}(x-y_j)^2 - (\tfrac{1}{2})(\beta-1)(\beta-\alpha)\prod_{j=1}^{m}(\beta-y_j)^2, \end{aligned} \tag{11}$$

see Theorem 1.1 and [48, (36), (37)] (corrected is here a misprint for $n$ even).

Expressing then, for $n \ge 4$, the $y_j$ by means of $\alpha,\beta$ (using a result of Schiefermayr [48, Theorems 1(ii), 2(ii)] for which in [55] we provide a calculation rule in *Mathematica*-syntax) one obtains a refined tentative representation of $Z_{n,s}$ whose coefficients depend on $\alpha$ and $\beta$:

$$S_{n,\alpha,\beta}(x) = \sum_{k=0}^{n-1} C_{k,n}(\alpha,\beta)x^k + x^n, \text{ where } C_{n-1,n}(\alpha,\beta) = -ns \text{ must be in force.} \tag{12}$$

This solves ZFP algebraically, provided $\alpha$ and $\beta$ are known for the prescribed $n$ and $s$. In the second part of the algorithm, the determination of $\alpha = \alpha_0$ and $\beta = \beta_0$ in [41] was again accomplished by using Malyshev polynomials. For $n \in \{6,7\}$, $S_{n,\alpha,\beta}$ is given in [41, (4.15), (5.5)], and for $n \le 13$ in [55]. If $n_0 = 3$ (for this degree there is no $z_j = y_j$), one gets from (11) that $\alpha = 3s$, and using $\alpha + \beta = 2(\gamma + s)$, one furthermore gets $\beta = s + 2\sqrt{\frac{1}{3} + s^2}$ so that, for $s > \tau_3 = \frac{1}{3}$,

$$Z_{3,s}(x) = \sum_{k=0}^{1} a^*_{k,3}(s)x^k + (-3s)x^2 + x^3 = 2s + s^3 - \left(\frac{1}{3} + s^2\right)^{\frac{3}{2}} + (-1)x + (-3s)x^2 + x^3 \tag{13}$$

constitutes a desired form of (2) if $n_0 = 3$, see also [5, p. 4], [27, p. 156], [46, Exercise 2.4.39], [57, p. 98]. This special degree we henceforth leave aside, except in Appendix 8.2 below where we will retain it for the sake of brevity. Schiefermayr [48, p. 156] also provides a representation, quite similar to (11), which is based on those equioscillation points $z_j = x_j$ from $\boldsymbol{I}\backslash\{-1,1\}$ where $Z_{n,s}(x_j) = L_n(s)$ holds. Our *Mathematica*-based calculation rule for $z_j = y_j$ can be adapted for $z_j = x_j$, see [55].

In the present paper we consider an 3rd two-staged algebraic algorithm for solving ZFP which is inspired by an approach to ZFP by Sodin & Yuditskii [50, 51]: An alternative tentative form, $P_{n,\alpha,\beta}$, for $Z_{n,s}$ is set up which in contrast to the tentative form $S_{n,\alpha,\beta}$ in (12) works without recourse to the equioscillation points $z_j = y_j$, see Section 2.1 below for details.

In order to convert, in a second step, a tentative form $Z_{n,\alpha,\beta} \in \{P_{n,\alpha,\beta}, R_{n,\alpha,\beta}, S_{n,\alpha,\beta}\}$ of $Z_{n,s}$ into the desired final form (2) for a concretely prescribed $n = n_0 \geq 4$ and $s = s_0 > \tau_{n_0}$, all three algorithms require as input the compatible numbers $\alpha = \alpha_0$ and $\beta = \beta_0$ (with $1 < \alpha_0 < \beta_0$), so that $Z_{n_0,s_0} = Z_{n_0,\alpha_0,\beta_0}$ will hold (solution of ZFP along the approach (B)). Hence the key question is: How to determine $\alpha_0$ and $\beta_0$? We have already pointed to the variant via Malyshev polynomials. This variant will be examined here in more detail (Variant 1 below), and we will also provide two alternative variants (Variant 2 and Variant 3).

### 1.3 Overview

With recourse to [41], we have at our disposal altogether three algebraic algorithms to generate, in a first step, three tentative forms $Z_{n,\alpha,\beta} \in \{P_{n,\alpha,\beta}, R_{n,\alpha,\beta}, S_{n,\alpha,\beta}\}$ of $Z_{n,s}$ which still depend on $\alpha$ and $\beta$. For the conversion of any of them into the desired final explicit algebraic solution (2), when $n = n_0$ and $s = s_0 > \tau_{n_0}$ are prescribed, we provide three variants for the calculation of the sought-for compatible numbers $\alpha = \alpha_0$ and $\beta = \beta_0$. Although each variant is applicable to each of the three tentative forms $Z_{n,\alpha,\beta}$ (thus offering, in sum, nine solution paths to ZFP), we will focus on $Z_{n,\alpha,\beta} = P_{n,\alpha,\beta}$ to demonstrate how to deploy and to rate the three variants. This is accomplished in Section 2.

In Section 3 we provide, by means of resultants, some theoretical background which reveals how the algebraic terms, that occur when deploying the three variants, are intertwined.

In Section 4 we examine an 4th algebraic solution path to ZFP, as proposed by Malyshev [20, p. 935]. The there considered tentative form of $Z_{n,s}$ is basically identical with $P_{n,\alpha,\beta}$ and the determination of the two parameters (denoted there by $x$ and $y$) is basically identical with the here deployed Variant 1.

In Section 5 we examine an 5th algebraic solution path to ZFP, based on (12), as proposed by Schiefermayr [48, Section 4.2]. His determining equations for $\alpha_0$ and $\beta_0$ [48, Corollary 3] may produce non-unique results, contrary to [48, Remark 1 (i)]. Nevertheless, we translate [48, Corollary 3] into a *Mathematica*-conformal syntax to facilitate the application of our final version of Variant 3, which produces $\alpha_0$ and $\beta_0$ unambiguously.

In Section 6 we examine an 6th algebraic solution path to ZFP due to Peherstorfer [28] who exploits an orthogonal property of $T$-polynomials (of which Zolotarev polynomials are a special case). This path turns out to be related to the 3rd algorithm which builds on the tentative form $Z_{n,\alpha,\beta} = P_{n,\alpha,\beta}$.

Overall, we conclude that the 1st tentative form, $Z_{n,\alpha,\beta} = R_{n,\alpha,\beta}$, in conjunction with the final version of Variant 3 provides an advantageous algorithm, with respect to complexity reduction and uniqueness, for the algebraic solution of ZFP along the approach (B).

Section 7 includes additional information on non-elliptic approaches to ZFP, and on Malyshev polynomials. We outsource to the Appendices in Section 8, and also to the online ZFP-repository [55], some lengthy calculations and bulky algebraic terms related to ZFP. In particular, in [55] we provide, for $n \leq 13$, the three tentative forms of $Z_{n,s}$ as well as terms occurring in the three deployed variants, e.g., Malyshev polynomials, reduced relation curves, moments, and interrelationships among the four parameters in Theorem 1.1. These objects increase our arsenal for attacking ZFP by algebraic means, and we generate them algorithmically by means of *Mathematica*-functions which allow extensions to $n > 13$. Furthermore, explicit Examples of $Z_{n,s}$ in the algebraic power form (2) are provided for $n \leq 13$. To the best of our knowledge, these are novel for $n > 7$, see Appendix 8.3 and [55]. The stored objects in [55] are easily accessible and can be copied and pasted into a CAS for further processing. Our paper advances [41] and is of a surveying and unifying character, but it contains also elements of novelty; the style of writing is expository.

## 2 Explicit algebraic Solution of ZFP by the 3rd Algorithm with three Variants

### 2.1 The 3rd Algorithm in detail

The announced 3rd algebraic algorithm for solving ZFP builds on a tentative representation of (2) as a quotient of two determinants. Our starting point is a determinant-based representation of a monic polynomial (used e.g. in the Theory of Moments and Continued Fractions [50]):

$$P_{n,\alpha,\beta}(x) = \det M_0^{n+1\times n+1} / \det M_1^{n\times n} = \det \begin{pmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_n \\ \sigma_1 & \sigma_2 & \dots & \sigma_{n+1} \\ . & . & . & . \\ 1 & x & \dots & x^n \end{pmatrix} / \det M_1^{n\times n} \tag{14}$$

with det = determinant and Hankel matrix

$$M_1^{n\times n} = \begin{pmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_{n-1} \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \\ . & . & . & . \\ \sigma_{n-1} & \sigma_n & \dots & \sigma_{2n-2} \end{pmatrix}. \tag{15}$$

The moments $\sigma_k$, $k \geq 0$, are here assumed to depend on the (unknown) interval endpoints $\alpha$ and $\beta$ in Theorem 1.1, i.e., $\sigma_k = \sigma_k(\alpha,\beta)$. Such a link of (14) to Theorem 1.1, via (8) and exploitation of an orthogonal property, can be achieved by means of the Laurent series expansion

$$\sum_{k=0}^{\infty} \frac{\sigma_k}{x^{k+1}} = \sqrt{\frac{x-1}{(x+1)(x-\alpha)(x-\beta)}}, \tag{16}$$

see Sodin & Yuditskii [51, p. 2487]. We have computed $\sigma_k = \sigma_k(\alpha,\beta)$, which is in fact a bivariate polynomial in $\alpha$ and $\beta$, up to $k = 25$, and have stored the values in [55] so that one is able to set up $P_{n,\alpha,\beta}$ in the tentative form (14) for $n \leq 13$.

**Algorithmic Generation of Moments $\boldsymbol{\sigma_k = \sigma_k(\alpha,\beta)}$.** A calculation formula (in *Mathematica*-syntax) for $\sigma_k = \sigma_k(\alpha,\beta)$, here exemplarily for the first four instances, is

$$\text{Reverse[CoefficientList[Collect[}x^4\text{Normal[Series[}\sqrt{\frac{x-1}{(x+1)(x-\alpha)(x-\beta)}}, \{x, \text{Infinity}, 4\}]], x], x]]//\text{Simplify} \tag{17}$$

which yields the list

$$\begin{gathered}\{\sigma_0,\sigma_1,\sigma_2,\sigma_3\} = \{1, \tfrac{1}{2}(-2+\alpha+\beta), \tfrac{1}{8}(4-4\alpha+3\alpha^2-4\beta+2\alpha\beta+3\beta^2), \\ \tfrac{1}{16}(-8+4\alpha-6\alpha^2+5\alpha^3+4\beta-4\alpha\beta+3\alpha^2\beta-6\beta^2+3\alpha\beta^2+5\beta^3)\}.\end{gathered} \tag{18}$$

Replacing in (14) $\sigma_k$ by $\sigma_k(\alpha,\beta)$ as indicated in (18), and expanding $\det M_0^{n+1\times n+1}$ by minors of the last row, one eventually gets, after evaluation of the occurring determinants in (14), a power form representation of $P_{n,\alpha,\beta}$ with coefficients $B_{k,n}(\alpha,\beta)$ depending on the two parameters $\alpha$ and $\beta$, i.e.

$$P_{n,\alpha,\beta}(x) = \sum_{k=0}^{n-1} B_{k,n}(\alpha,\beta)x^k + x^n, \text{ where } B_{n-1,n}(\alpha,\beta) = -ns \text{ must be in force.} \tag{19}$$

With the goal to select and to insert, for a concretely prescribed $n \geq 4$ and $s > \tau_n$, the sought-for optimal values $\alpha = \alpha_0$ and $\beta = \beta_0$ into $Z_{n,\alpha,\beta}$ and thus solving ZFP, we consider three intertwined variants which constitute the second step in each of the three algorithms. We deploy these variants exemplarily to the tentative form $Z_{n,\alpha,\beta} = P_{n,\alpha,\beta}$, and will denote by $L_n(\alpha,\beta) = \|Z_{n,s}\|_\infty = |Z_{n,s}(1)|$ its (least) deviation, on $\boldsymbol{I}$, from the zero-function.

### 2.2 Variant 1 (Zeros of Malyshev polynomials)

Malyshev [21] introduced, for $2 \leq n \leq 5$ only, bivariate polynomials $F_{m(n),s}(\alpha)$ and $G_{m(n),s}(\beta)$, with degree $m(n)$ and parameter $s$ (originally denoted in [21] by $f_n(\alpha,\sigma)$ and $g_n(\beta,\sigma)$ which is misleading in so far as $m(n) = n$ only if $n = 4$) which we termed Malyshev polynomials in [41].

**Algorithmic Generation of Malyshev Polynomials.** The polynomials $F_{m(n),s}(\alpha)$ and $G_{m(n),s}(\beta)$ can be computed by Gröbner basis as a byproduct of the 1st algorithm, see [41, Sections 4.1, 5.1], [54, Lemma 2.4] and [55] (for $n = 5$); or by resultants, see [55]. We provide them, up to $n \leq 13$, in [55], see also Remark 7 below.

For a given $n_0$ and $s_0 > \tau_{n_0}$ the sought-for $\alpha_0$ and $\beta_0$ (with $1 < \alpha_0 < \beta_0$) are to be found, according to Malyshev [20, p. 936], [21], among the (finitely many real) zeros $> 1$ of $F_{m(n_0),s_0}(\alpha)$ respectively among the zeros $> \nu_{n_0}$ of $G_{m(n_0),s_0}(\beta)$. To reduce the search for $\alpha_0$ and $\beta_0$, Malyshev advises to make use of asymptotic approximations, see (76) below. However, this strategy for fixing the sought-for $\alpha_0$ and $\beta_0$ remains vague. In [21, p. 711] Malyshev moreover observes: *Examples show that [for a given $n = n_0$ and $s = s_0$] the desired $\alpha$ and $\beta$ [i.e. $\alpha_0$ and $\beta_0$] are the greatest real roots of the [Malyshev] polynomials in question.* If his observation were true in general, then one would have a deterministic strategy for finding $\alpha_0$ and $\beta_0$. However, this is not the case, as the setting $n_0 = 7$ and $s_0 = \frac{1}{10} > \tau_7$ reveals:

**Example 2.1.** The largest real zero of $F_{m(7),s_0}(\alpha) = F_{12,\frac{1}{10}}(\alpha)$ quantified by its index, is $\alpha_0 = \text{Root}[\text{F}_{12,\frac{1}{10}}(\alpha), 4] \approx 1.1986614376$, and the largest real zero of $G_{m(7),s_0}(\beta) = G_{12,\frac{1}{10}}(\beta)$ quantified by its index, is $\beta_0 = \text{Root}[G_{12,\frac{1}{10}}(\beta), 4] \approx 1.2952331978$. But the pair $(\alpha_0,\beta_0)$ does not produce $Z_{n_0,s_0} = Z_{7,\frac{1}{10}}$; rather, the said largest (simple) zero $\alpha_0$ and the second largest (simple) zero $\hat{\beta}_0 = \text{Root}[G_{12,\frac{1}{10}}(\beta), 3] \approx 1.2398104774$, when inserted into $P_{7,\alpha,\beta}$, do generate $Z_{7,\frac{1}{10}}$, see [41, Theorem 5.1].

Alternatively, this fact can be deduced from $s_7(\alpha_0,\hat{\beta}_0) = s_0 = \frac{1}{10}$ (whereas $s_7(\alpha_0,\beta_0) \neq \frac{1}{10}$ holds), or from $H^7_{m(7)}(\alpha_0,\hat{\beta}_0) = H^7_{12}(\alpha_0,\hat{\beta}_0) = 0$ (whereas $H^7_{12}(\alpha_0,\beta_0) \neq 0$ holds), see Section 2.4 below for the definitions of the analytical Formulae $s_n(\alpha,\beta)$ and $H^n_{m(n)}(\alpha,\beta)$ which enter into (31). □

But we can confirm that a weaker version (concerning the range of $s$) of Malyshev's observation holds true. We confine ourselves to the degrees $n \leq 13$ covered by the online ZFP-repository [55].

**Proposition 2.1 (Variant 1 (assuming $1 \le s$)).** *For a prescribed degree $n = n_0 \in \{4, \dots 13\}$, take for the sought-for unique $\alpha_0$ and $\beta_0$ (with $1 < \alpha_0 < \beta_0$) the indexed root objects $\alpha_0 = \mathrm{Root}[F_{m(n_0),s_0}(\alpha), l(n_0)]$ respectively $\beta_0 = \mathrm{Root}[G_{m(n_0),s_0}(\beta), l(n_0)]$, provided that the prescribed parameter $s = s_0$ satisfies $1 \le s_0$. Here $l(n_0)$ is the common index of the largest zero of $F_{m(n_0),s_0}(\alpha)$ and $G_{m(n_0),s_0}(\beta)$. It is given in Table 1 below, together with the total degree $m(n) = m(n_0)$ of $F_{m(n_0),s_0}(\alpha)$, $G_{m(n_0),s_0}(\beta)$.* $\square$

| $n_0$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|
| $l(n_0)$ | 2 | 4 | 2 | 6 | 4 | 6 | 4 | 10 | 4 | 12 |
| $m(n_0)$ | 4 | 6 | 8 | 12 | 16 | 18 | 24 | 30 | 32 | 42 |

**Table 1**

Incidentally, the finite sequence $\{l(n_0)\}_{n_0=4}^{13}$ in Table 1 turns out to be part of the infinite integer sequence A000010 (Euler's totient function) in the database OEIS (see `www.oeis.org`).

Variant 1 is based on the fact that, for $4 \le n \le 13$, the *Mathematica*-function Solve, i.e. here

$$\mathrm{Solve}[F_{m(n_0),s}(\alpha) == 0 \wedge s \ge 1 \wedge \alpha > 1, \{\alpha\}] \quad \text{and} \quad \mathrm{Solve}[G_{m(n_0),s}(\beta) == 0 \wedge s \ge 1 \wedge \beta > \nu_{n_0}, \{\beta\}], \tag{20}$$

returns several real zeros of $F_{m(n_0),s}(\alpha)$ respectively of $G_{m(n_0),s}(\beta)$ (in terms of root objects) of which the respective largest zero is indexed by $l(n_0)$, say. With the additional conditions $|\alpha - n_0 s| < 1$ and $|\beta - n_0 s| < 1$ the *Mathematica*-function Reduce, i.e. here

$$\begin{aligned} &\mathrm{Reduce}[F_{m(n_0),s}(\alpha) == 0 \wedge s \ge 1 \wedge \alpha > 1 \wedge \mathrm{Abs}[\alpha - n_0 s] < 1, \{\alpha\}] \\ &\text{and} \\ &\mathrm{Reduce}[G_{m(n_0),s}(\beta) == 0 \wedge s \ge 1 \wedge \beta > \nu_{n_0} \wedge \mathrm{Abs}[\beta - n_0 s] < 1, \{\beta\}], \end{aligned} \tag{21}$$

returns as the unique solution the largest real zero, i.e., $\alpha = \mathrm{Root}[F_{m(n_0),s}(\alpha), l(n_0)]$ and $\beta = \mathrm{Root}[G_{m(n_0),s}(\beta), l(n_0)]$, with identical index $l(n_0)$. According to [41, Remark 8.2], the additionally imposed conditions $|\alpha - n_0 s| < 1$ respectively $|\beta - n_0 s| < 1$ hold for all $n_0 \in \{6, \dots 12\}$ and $s \ge 1$; they hold for $n_0 \in \{4, 5, 13\}$ too, as we have verified by the same method as indicated in [41], see also Remark 7 below.

By analyzing the outcomes of

$$\begin{aligned} &\mathrm{Reduce}[F_{m(n_0),s}(\alpha) == 0 \wedge 1 > s > \tau_{n_0} \wedge \alpha > 1 \wedge \mathrm{Abs}[\alpha - n_0 s] < 1, \{s, \alpha\}] \\ &\text{and} \\ &\mathrm{Reduce}[G_{m(n_0),s}(\beta) == 0 \wedge 1 > s > \tau_{n_0} \wedge \beta > \nu_{n_0} \wedge \mathrm{Abs}[\beta - n_0 s] < 1, \{s, \beta\}], \end{aligned}$$

the following can be concluded:

**Proposition 2.2 (Addendum to Variant 1).** *The inequality $1 \le s = s_0$ in Variant 1 can be improved to $\delta(n_0) < s_0$ (if $\delta(n_0) = \tau_{n_0}$) or to $\delta(n_0) \le s_0$ (if $\delta(n_0) \ne \tau_{n_0}$)). Here $\delta(n_0) < 1$ varies with $n = n_0$ and is given, together with few exceptional cases of $s_0$, in Table 2 below. Table 2 implies the following overall improvement of the Variant 1: $\frac{9}{13} \le s = s_0$ for $n_0 \le 13$.*

| $n_0$ | $\delta(n_0)$ |
|---|---|
| 4 | $\delta(4) = \tau_4 = 3 - 2\sqrt{2}$ |
| 5 | $\delta(5) = \frac{1}{5} \ne \tau_5$ |
| 6 | $\delta(6) = \tau_6 = 7 - 4\sqrt{3}$, except for $s_0 = \frac{1}{3}$ |
| 7 | $\delta(7) = \frac{3}{7} \ne \tau_7$ |
| 8 | $\delta(8) = \tau_8 = 7 + 4\sqrt{2} - 2\sqrt{2(10 + 7\sqrt{2})}$ except for $s_0 \in \{\frac{1}{4}, \frac{2}{4}\}$ |
| 9 | $\delta(9) = \frac{5}{9} \ne \tau_9$ |
| 10 | $\delta(10) = \tau_{10} = 11 + 4\sqrt{5} - 2\sqrt{2(25 + 11\sqrt{5})}$ except for $s_0 \in \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}\}$ |
| 11 | $\delta(11) = \frac{7}{11} \ne \tau_{11}$ |
| 12 | $\delta(12) = \tau_{12} = 15 - 6\sqrt{6} - 2\sqrt{2(49 - 20\sqrt{6})}$ except for $s_0 \in \{\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}\}$ |
| 13 | $\delta(13) = \frac{9}{13} \ne \tau_{13}$ |

**Table 2**

The pattern in Table 2 suggests an obvious parity-dependent generalization for $n = n_0 > 13$. Observe that the Addendum to Variant 1 does not cover the marginal cases $\tau_{n_0} < s_0 < \delta(n_0)$ if $n_0 \in \{5, 7, 9, 11, 13\}$; for $n_0 = 7$ see Example 2.1 above. Neither it covers the exceptional cases $s_0$ if $n_0 \in \{6, 8, 10, 12\}$.

In order to close this second gap, construct $Z_{n_0,\alpha_0,\beta_0} = Z_{n_0,s_0}$ by taking, with $l(n_0)$ from Table 1, $\alpha_0 = \hat{\alpha}_0 = \mathrm{Root}[F_{m(n_0),s_0}(\alpha), l(n_0)+4]$ and $\beta_0 = \hat{\beta}_0 = \mathrm{Root}[G_{m(n_0),s_0}(\beta), l(n_0)+4]$, if $(n_0, s_0) \in \{(10, \frac{1}{5}), (12, \frac{1}{6}), (12, \frac{2}{6})\}$, respectively $\alpha_0 = \hat{\alpha}_0 = \mathrm{Root}[F_{m(n_0),s_0}(\alpha), l(n_0)+2]$ and $\beta_0 = \hat{\beta}_0 = \mathrm{Root}[G_{m(n_0),s_0}(\beta), l(n_0)+2]$ for the remaining exceptional pairs. To close the first said gap (the marginal cases if $n_0 \in \{5, 7, 9, 11, 13\}$) would require a fragmented case-by-case analysis. We skip it here and propose to apply in these cases the final version of Variant 3 below.

We now deploy the Variant 1 in order to convert $P_{n,\alpha,\beta}$ into (2) so that $P_{n_0,\alpha_0,\beta_0} = Z_{n_0,s_0}$ will hold.

**Example 2.2.** The goal is to solve ZFP algebraically by using the 3rd algorithm in the setting $n_0 = 6$ and $s_0 = 1 > \tau_6$. We set up the particular case of (14) for $n_0 = 6$, i.e., the tentative form $P_{6,\alpha,\beta} = \det M_0^{6+1\times 6+1} / \det M_1^{6\times 6}$. Here we replace $\sigma_k$ by $\sigma_k(\alpha,\beta)$, which we adopt, for $k \in \{0, \dots, 11\}$ from [55]. Finally, we replace $\alpha$ and $\beta$ by $\alpha_0$ respectively $\beta_0$ which we get as follows: We adopt $F_{m(6),s}(\alpha) = F_{8,s}(\alpha)$ and $G_{m(6),s}(\beta) = G_{8,s}(\beta)$ from [55], replace there $s$ by $s_0 = 1$, and form (see Table 1), with *Mathematica*, $\alpha_0 = \mathrm{Root}[F_{8,1}(\alpha), 2] \approx 6.0827162524$ and $\beta_0 = \mathrm{Root}[G_{8,1}(\beta), 2] \approx 6.0828088063$. This will eventually give, after expanding $P_{6,\alpha,\beta}(x)$ into the polynomial power form, the sought-for monic Zolotarev polynomial $P_{6,\alpha_0,\beta_0} = Z_{6,s_0} = Z_{6,1}$. Its optimal coefficients as well as its least deviation from zero, $\|Z_{6,1}\|_\infty = |Z_{6,1}(1)|$, can be explicitly expressed in terms of root objects, see Appendix 8.1 below. These quantities can be evaluated to arbitrary numerical precision $p$ by means of the *Mathematica*-function N (numerical value), that is here: $\mathrm{N}[\mathrm{Root}[P(z), l], p]$. Chopping them after the tenth digit after decimal point, a numerical approximation to $Z_{6,1}$ and to $|Z_{6,1}(1)|$ is given by

$$\begin{gathered} Z_{6,1}(x) = \textstyle\sum_{k=0}^{4} a_{k,6}^{*}(1)x^k + (-6)x^5 + x^6 \approx \\ -0.0620748101 + (-1.8673118711)x + 0.8103624015x^2 + 7.4897255420x^3 + (-1.7482875914)x^4 + (-6)x^5 + x^6, \\ L_6(\alpha_0,\beta_0) = |Z_{6,1}(1)| \approx 0.3775863290. \end{gathered} \tag{22}$$

This deviation of $Z_{6,1}$ from zero, on $\boldsymbol{I}$, is least among all polynomials of form $Q_{6,1}(x) = \sum_{k=0}^{4} a_{k,6}x^k + (-6)x^5 + x^6$, where $a_{k,6} \in \mathbb{R}$ is arbitrary. Figure 2 displays the graph of $Z_{6,1}$ on the interval $[-1, 1.1]$. □

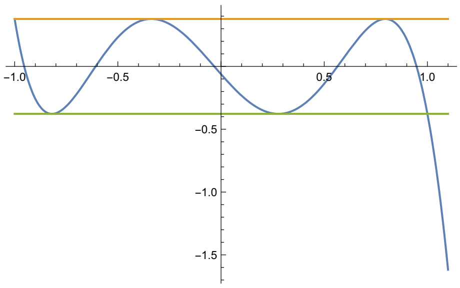


**Figure 2**

The preceding Example coincides with our solution of ZFP in [41, Example 4.2], which we have deduced recursively along our 1st algorithm (and Variant 1), and with our solution of ZFP in [42, Example 3.2], which we have deduced along the approach (A) deploying the representation (9) of proper Zolotarev polynomials. Solving ZFP in the setting $n_0 = 6$ and $s_0 = \frac{1}{8}$ by using (14) and Table 2, would give the same result as in [41, Example 4.6].

**Example 2.3.** Solving ZFP in the setting $n_0 = 6$ and $s_0 = \frac{1}{3}$ (exceptional pair, see Table 2) would give $\hat{\alpha}_0 = \mathrm{Root}[F_{8,\frac{1}{3}}(\alpha), 4] = \mathrm{Root}[-257923 - 33678z + 156979z^2 + 52988z^3 + 7187z^4 - 84750z^5 + 28125z^6, 2] \approx 2.2332896407$, $\hat{\beta}_0 = \mathrm{Root}[G_{8,\frac{1}{3}}(\beta), 4] = \mathrm{Root}[46493 - 397778z + 944755z^2 - 1098684z^3 + 676787z^4 - 215250z^5 + 28125z^6, 2] \approx 2.2388280242$, and with these values $P_{6,\hat{\alpha}_0,\hat{\beta}_0} = Z_{6,\frac{1}{3}}$ can be constructed, see also [41, p. 186]. For solving ZFP in the setting $n_0 = 7$ and $s_0 = \frac{1}{10}$ (not covered by Table 2), see Example 2.1 and Example 2.6 below. □

### 2.3 Variant 2 (Solution of determinant equations)

Variant 2 deploys two further determinants, $\det M_2^{n\times n}$ and $\det M_3^{n+1\times n+1}$, in order to convert $P_{n,\alpha,\beta}$ into $Z_{n_0,s_0}$ for a prescribed $n = n_0$ and $s = s_0$:

Since the coefficient of $P_{n,\alpha,\beta}$ at $x^{n-1}$ must be of form $B_{n-1,n}(\alpha,\beta) = -ns$, one gets, in view of (14), the determinant equation

$$s = s_n(\sigma_k) = \left(\frac{1}{n}\right) \det \begin{pmatrix} \sigma_0 & \sigma_1 & . & . & \sigma_{n-2} & \sigma_n \\ \sigma_1 & \sigma_2 & . & . & \sigma_{n-1} & \sigma_{n+1} \\ . & . & . & . & . & . \\ \sigma_{n-2} & \sigma_{n-1} & . & . & \sigma_{2n-4} & \sigma_{2n-2} \\ \sigma_{n-1} & \sigma_n & . & . & \sigma_{2n-3} & \sigma_{2n-1} \end{pmatrix} / \det M_1^{n\times n} = \left(\frac{1}{n}\right) \det M_2^{n\times n} / \det M_1^{n\times n}, \tag{23}$$

which is also given in [51, p. 2487] (deleted is here a misprinted minus sign).

A further determinant equation we adopt directly from [51, (4)]:

$$0=\det\begin{pmatrix}\sigma_0 & \sigma_1 & \sigma_2 & . & . & \sigma_{n-1} & \sigma_n\\ \sigma_1 & \sigma_2 & \sigma_3 & . & . & \sigma_n & \sigma_{n+1}\\ . & . & . & . & . & . & .\\ \sigma_{n-1} & \sigma_n & \sigma_{n+1} & . & . & \sigma_{2n-2} & \sigma_{2n-1}\\ 0 & \sigma_0 & \sigma_0+\sigma_1 & \sigma_0+\sigma_1+\sigma_2 & . & . & \sigma_0+\sigma_1+\sigma_2+\ldots+\sigma_{n-1}\end{pmatrix}=\det M_3^{n+1\times n+1}. \tag{24}$$

In both of these identities the right-hand sides depend via $\sigma_k=\sigma_k(\alpha,\beta)$ on $\alpha$ and $\beta$ and the left-hand sides are concrete real numbers if we identify $s=s_n(\sigma_k)$ with $s_0$ (assuming $s_0>\tau_n$). We now merge the identities (23) and (24) by deploying Reduce and using, as additional information, the known range of $\alpha$ and $\beta$:

**Proposition 2.3 (Variant 2).** *The sought-for $\alpha_0,\beta_0$ are among the solutions of*

$$\text{Reduce}[s_{n_0}(\sigma_k)-s_0==0\wedge\det M_3^{n_0+1\times n_0+1}==0\wedge 1<\alpha<\beta\wedge\nu_{n_0}<\beta,\{\alpha,\beta\}]. \tag{25}$$

*But note that (25) also may produce nuisance values (non-uniqueness).*

With $\alpha_0$ and $\beta_0$ at hand one can proceed analogously as before, that is, inserting them into (14) and simplifying the outcome (e.g., with the *Mathematica*-function FullSimplify).

A rationale for Variant 2 can be concluded from a statement by Malyshev [21, p. 711] when he surveyed the paper [51]: *The authors construct a system of algebraic equations $\widetilde{F}_n(\alpha,\beta)=0$, $s\widetilde{H}_n(\alpha,\beta)=\widetilde{G}_n(\alpha,\beta)$ from which the desired roots $1<\alpha<\beta$ [in the Abel-Pell differential equation] can be obtained for any fixed $s$* (we have added here the tilde and have set $\sigma=s$ to avoid confusion with our notation). One readily identifies this quoted system of algebraic equations with (24) and (23); furthermore, we point out that in [51] $\beta$ is assumed to be fixed.

To substantiate the delineated Variant 2, we set the goal to solve algebraically ZFP in the setting $n_0=4$ and $s_0=\text{Root}[11-88z+263z^2-318z^3+96z^4,1]=\frac{1}{192}(159+17\sqrt{33}-\sqrt{30(527+97\sqrt{33})})=0.3974272232\ldots>\tau_4$.

**Example 2.4.** With $n_0=4$ and $\sigma_k=\sigma_k(\alpha,\beta)$ adopted from [55], Formula (23) reads

$$s_4(\sigma_k)=\left(\frac{1}{4}\right)\det\begin{pmatrix}\sigma_0 & \sigma_1 & \sigma_2 & \sigma_4\\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_5\\ \sigma_2 & \sigma_3 & \sigma_4 & \sigma_6\\ \sigma_3 & \sigma_4 & \sigma_5 & \sigma_7\end{pmatrix}/\det\begin{pmatrix}\sigma_0 & \sigma_1 & \sigma_2 & \sigma_3\\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4\\ \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5\\ \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6\end{pmatrix}, \tag{26}$$

where its expanded form (in terms of $\alpha$ and $\beta$) is provided in Appendix 8.1 below.

Formula (24) reads, in expanded form,

$$\begin{gathered}0=\det M_3^{4+1\times4+1}=\\ =\frac{H_4^4(\alpha,\beta)H_4^4(-\alpha,-\beta)(-2+\alpha+\beta)(2+\alpha-\beta)(\alpha+\beta)(-16+16\alpha^2+\alpha^4+4\alpha^3\beta+16\beta^2-10\alpha^2\beta^2+4\alpha\beta^3+\beta^4)}{-33554432},\end{gathered} \tag{27}$$

with $H_4^4(\alpha,\beta)=6-8\alpha^2+8\alpha^3+\alpha^4+16\alpha\beta+8\alpha^2\beta-4\alpha^3\beta-8\beta^2-8\alpha\beta^2+6\alpha^2\beta^2-8\beta^3-4\alpha\beta^3+\beta^4$. Executing (25) with the prescribed $s_0$ and using (26) - (27), yields the unique optimal values

$$\begin{array}{lllll}\alpha & = & \alpha_0 & = & \text{Root}[-503+83z+715z^2-531z^3+108z^4,2] & \approx & 1.8034303689,\\ \beta & = & \beta_0 & = & \text{Root}[1543+667z+13z^2-603z^3+108z^4,1] & \approx & 1.9444055070.\end{array} \tag{28}$$

Inserting them now into the tentative form (14) produces (with $n_0=4$ and $\sigma_k=\sigma_k(\alpha_0,\beta_0)$), after simplification, the desired solution for the particular ZFP:

$$\begin{gathered}Z_{4,s_0}(x)=P_{4,\alpha_0,\beta_0}(x)=\det M_0^{4+1\times4+1}/\det M_1^{4\times4}=\\ =\text{Root}[-101+481z-207z^2+79z^3+4z^4,2]+\text{Root}[391072-1811744z+2620930z^2-1528713z^3+314928z^4,2]x+\\ \text{Root}[-864-1116z-420z^2-63z^3+4z^4,1]x^2+\text{Root}[352+704z+526z^2+159z^3+12z^4,2]x^3+x^4=\end{gathered}$$

$$\begin{gathered}=\textstyle\sum_{k=0}^{2}a_{k,4}^{*}(s_0)x^k+(-4s_0)x^3+x^4\quad\left(\text{with }-4s_0=\frac{1}{48}\left(-159-17\sqrt{33}+\sqrt{30(527+97\sqrt{33})}\right)\right)\\ \approx 0.2308733204+1.1564149581x+(-1.2308733204)x^2+(-1.5897088931)x^3+x^4,\\ |Z_{4,s_0}(1)|=L_{n_0}(\alpha_0,\beta_0)=L_4(s_0)=\text{Root}[442368-176832z-861584z^2-2644083z^3+314928z^4,1]\approx 0.4332939350.\end{gathered} \tag{29}$$

$L_4(s_0)$ is least, on $\boldsymbol{I}$, among all polynomials of form $Q_{4,s_0}(x)=\sum_{k=0}^{2}a_{k,4}x^k+(-4s_0)x^3+x^4$, where $a_{k,4}\in\mathbb{R}$ is arbitrary. Figure 3 displays the graph of $Z_{4,s_0}$ on the interval $[-1,2.04]$ and indicates the interval $[\alpha_0,\beta_0]$. □

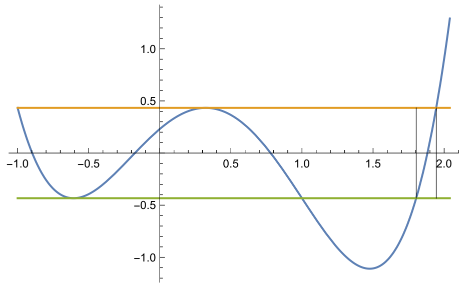


**Figure 3**

*Remark* 1. The polynomial $-Z_{4,s_0}/L_4(s_0)$, see (29), has uniform norm 1 on $\boldsymbol{I}$ and coincides with the explicit solution, for $n=4$, of an extremal problem posed by Schur in 1919, see [13], [35, Section 5d], [39]. For explicit solutions of Schur's problem in terms of $Z_{n,s}$, if $n \in \{5,6,7\}$, see [40, 43, 45]. Another problem, for which the Zolotarev polynomials $Z_{n,s}/L_n(s)$ turn out as extremizers, is the characterization of the supremum in the pointwise inequality of V. A. Markov, see e.g. [49]. Incidentally, V. A. Markov was the first to give a representation (9) of $Z_{n,s}$, if $n=4$, see [37], [39, p. 160], [49, p. 242].

Note that e.g. in the setting $n_0=5$ and $s_0=1$, the Variant 2 would produce two solutions, $(\alpha_0,\beta_0)$ and $(\alpha_1,\beta_1)$, with $\alpha_0 \approx 5$, $\alpha_1 \approx 2$, i.e., non-uniqueness. Since $s \geq 1$, one deduces from $|\alpha - ns| < 1$ (see Variant 1) that $(\alpha_0,\beta_0)$ would be the sought-for optimal pair.

*Remark* 2. We observe the comparatively bulky expressions on the right-hand sides of (26) and (27), although the polynomial degree $n_0=4$ is quite low. Actually, the numerator and denominator of (23) increase with $n$ as $\mathcal{O}(n^2)$, see [21, p. 711], and for $n_0=13$ the corresponding right-hand side in (23) might result in a bulky term which stretches over more than one printed page. Therefore, we are looking for more concise representations of the right-hand sides in (23) and (24). In the next Section we will introduce such substitutes to the extent that, when taking the setting of Example 2.4, the right-hand side in (26), as given in Appendix 8.1, will be dramatically reduced to the quotient in (30) below, and the right-hand side in (27) will shrink to the first factor in the numerator, i.e., to $H_4^4(\alpha,\beta)$. Executing (25) with these terms, that is,

$$\mathrm{Reduce}[\frac{4(1+\beta)+(\alpha-\beta)(\alpha+2\alpha^2+5\alpha\beta+\beta(3+5\beta))}{4(4+(\alpha-\beta)(\alpha+3\beta))}-s_0==0 \wedge H_4^4(\alpha,\beta)==0 \wedge 1<\alpha<\beta \wedge \nu_4<\beta, \{\alpha,\beta\}], \tag{30}$$

indeed produces the same result $(\alpha_0,\beta_0)$ as given in (28). Therefore, the Variant 2 is to be considered inferior to (31) below.

## 2.4 Variant 3 (Common points on reduced algebraic curves)

Setting in the ansatz $S_{n,\alpha,\beta}$, see (12), $\frac{C_{n-1,n}(\alpha,\beta)}{-n}=s$ we now write $s=s_n(\alpha,\beta)$ (or simply $s=s(\alpha,\beta)$ if there is no ambiguity) to better distinguish it from $s=s_n(\sigma_k)$ in (23).

**Algorithmic Generation of Moments $s_n(\alpha,\beta)$.** The terms $s_n(\alpha,\beta)$ can be generated by inserting $n\geq 4$ into equations (54) respectively (57) below, and then solving for the variable $s$. We have stored the results in [55] for $4 \leq n \leq 13$.

The special cases $s_6(\alpha,\beta)$ and $s_7(\alpha,\beta)$ are given in [41, pp. 195-196], $s_5(\alpha,\beta)$ in Appendix 8.1 below and $s_4(\alpha,\beta)$ is the quotient in (30). The terms $s_n(\alpha,\beta)$ are, for $n\geq 4$, bivariate rational functions and will serve as downsized substitutes for the right-hand side in (23), i.e., for $s_n(\sigma_k)$. For a prescribed $n=n_0$ and $s=s_0$ we obtain the condition $s_{n_0}(\alpha,\beta)-s_0=0$ which defines a planar algebraic curve.

To derive downsized substitutes for the right-hand side in (24), i.e., for $\det M_3^{n+1\times n+1}$, we choose, in view of (44) below, the bivariate polynomials $H^n_{m(n)}(\alpha,\beta)$ of total degree $m(n)$ which describe reduced relation curves $H^n_{m(n)}(\alpha,\beta)=0$ (with respect to $\alpha$ and $\beta$) associated to ZFP, see [15, 44, 54] for details. For a concretely prescribed $n=n_0$, the planar curve $H^{n_0}_{m(n_0)}(\alpha,\beta)=0$ contains all compatible pairs $(\alpha,\beta)=(\alpha(n_0,s),\beta(n_0,s))=(\alpha(s),\beta(s))$ for $Z_{n_0,\alpha,\beta}$ which are generated when $s$ varies in $(\tau_{n_0},\infty)$, see Theorem 1.1, and it even contains the limiting pair $(\alpha,\beta)=(1,\nu_{n_0})$.

**Algorithmic Generation of $H^n_{m(n)}(\alpha,\beta)$.** These terms can be generated as those factors with degree $m(n)$, and coefficient 1 at the monomial $\alpha^{m(n)}$, which arise on the left-hand side of equations (53) respectively (56) below, when $n\geq 4$ is inserted there. $H^n_{m(n)}$ is stored, for $n\leq 13$, in [55]; for the degrees see Table 1.

The special cases $H^6_{m(6)}=H^6_8$ and $H^7_{m(7)}=H^7_{12}$ are given in [41, p. 195, p. 197] and $H^7_{12}$ and $H^4_{m(4)}=H^4_4$ have already emerged in Sections 2.2 and 2.3 above; see also Section 3 below. It is remarkable that the degrees $m(n)$ are not strictly increasing with $n$ because $m(14)=m(15)=48$, see [54, Table 2].

To identify points $(\alpha_0,\beta_0)$ which lie on both said planar curves, and satisfy $1<\alpha_0<\beta_0$ as well as $\nu_{n_0}<\beta_0$, we continue with

**Proposition 2.4 (Preliminary version of Variant 3).** *The sought-for $\alpha_0,\beta_0$ are among the solutions of*

$$\mathrm{Reduce}[s_{n_0}(\alpha,\beta)-s_0==0 \wedge H^{n_0}_{m(n_0)}(\alpha,\beta)==0 \wedge 1<\alpha<\beta \wedge \nu_{n_0}<\beta, \{\alpha,\beta\}]. \tag{31}$$

*But note that (31) also may produce nuisance values.*

Given the setting of Example 2.4, (31) becomes identical with (30). For $n = n_0 \in \{6, 7\}$, (31) is identical with [41, (7.5), (7.10)]. For a visualization of (31), if $n = n_0 = 5$, see Figure 6 (contour plot) in Appendix 8.1 below.

The first respectively second equation occurring in (31) correspond, for $n \leq 13$, to equations (33), (35) respectively to equations (32), (34) of Corollary 3 in [48]. It describes how to generate $\alpha_0$ and $\beta_0$ for a given $n_0$ and $s_0 > \tau_{n_0}$, and was deduced by different means in [48], see Section 5 below. The non-uniqueness of (31) is in contrast to [48, Remark 1(i)].

To exemplify (31), we are going to solve ZFP algebraically in the setting $n_0 = 7$ and $s_0 = 2 > \tau_7 = \mathrm{Root}[-1 + 21z - 35z^2 + 7z^3, 1] \approx 0.0520950836$, so that one can draw a comparison to the Example given in [44, Remark 4.3]. Note that we pursued there, for $n_0 = 7$, the particular approach (A) to ZFP via the form (9). We want to know the compatible $\alpha_0$ and $\beta_0$.

**Example 2.5.** With $n_0 = 7$ and $s_0 = 2$, (31) reads

$$\mathrm{Reduce}[s_7(\alpha,\beta) - 2 == 0 \wedge H^7_{12}(\alpha,\beta) == 0 \wedge 1 < \alpha < \beta \wedge \nu_7 < \beta, \{\alpha,\beta\}]. \tag{32}$$

Before executing (32), one has to replace $s_7(\alpha,\beta)$ and $H^7_{12}(\alpha,\beta)$ by the corresponding terms as provided in [55]. As a result, one obtains two root objects for $\alpha$, $\alpha \approx 4$ and $\alpha \approx 14$, and two associated root objects for $\beta$, i.e., non-uniqueness. Since $s \geq 1$, one deduces from $|\alpha - ns| < 1$ (see Variant 1) that $\alpha \approx 14$ must be the optimal one, so that we set

$$\begin{gathered}\alpha = \alpha_0 = \\ \mathrm{Root}[1596640261888 + 1382185254912z - 2943026759680z^2 + 1110163950336z^3 + 433952450240z^4 - \\ 579218113600z^5 + 268614534928z^6 - 73851260224z^7 + 13191160848z^8 - \\ 1542461200z^9 + 113418000z^{10} - 4725000z^{11} + 84375z^{12}, 6] \approx 14.0356687900, \\ \beta = \beta_0 = \\ \mathrm{Root}[614128742778112 - 378933592095744z - 107722313416704z^2 + 365630011835648z^3 - 320300886075520z^4 + \\ 150022279642816z^5 - 39835757959760z^6 + 5296218682048z^7 - 1384872496z^8 - \\ 114339324400z^9 + 17574291000z^{10} - 1157625000z^{11} + 28940625z^{12}, 6] \approx 14.0356689052.\end{gathered} \tag{33}$$

□

We note in passing that one will get the identical values for $\alpha_0$ and $\beta_0$ e.g. by choosing $\alpha_0 = \mathrm{Root}[F_{m(7),2}(\alpha), 6]$ and $\beta_0 = \mathrm{Root}[G_{m(7),2}(\beta), 6]$, see Variant 1 above, or by inserting the parameter value $t = t_0$, which is given in [44, Remark 4.3], into the Formulae for $\alpha = \alpha(t)$ and $\beta = \beta(t)$ which are given in [44, (2.10)].

To obtain uniqueness for the Variant 3, we introduce the following

**Proposition 2.5 (Final version of Variant 3).** *For a prescribed degree $n_0 \in \{4, \dots, 13\}$, and $s = s_0 > \tau_{n_0}$, the sought-for $\alpha_0$ and $\beta_0$ can be determined uniquely by means of*

$$\mathrm{Reduce}[s_{n_0}(\alpha,\beta) - s_0 == 0 \wedge \beta == \beta(\alpha, \lambda_{n_0}) \wedge 1 < \alpha, \{\alpha,\beta\}, \mathrm{Reals}, \mathrm{Backsubstitution} \to \mathrm{True}], \tag{34}$$

*where $\beta(\alpha, \lambda_{n_0})$ is the parametric root object with lowest index $\lambda_{n_0}$ resulting from*

$$\mathrm{Reduce}[H^{n_0}_{m(n_0)}(\alpha,\beta) == 0 \wedge 1 < \alpha < \beta \wedge \nu_{n_0} < \beta, \{\beta\}]. \tag{35}$$

*The unique index $\lambda_{n_0}$ is displayed in Table 3.*

| $n_0$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_{n_0}$ | 1 | 3 | 1 | 4 | 1 | 4 | 1 | 6 | 1 | 7 |

**Table 3**

**Example 2.6.** Let again $n_0 = 7$ and $s_0 = 2$. Executing (35) yields three possible solutions for $\beta$ consisting of parametric root objects (which are related to $H^7_{12}(\alpha,\beta)$ and depend on $\alpha$) with respective indexes 4, 5 or 6, see [55] for details. Hence $\lambda_7 = 4$. Inserting the corresponding parametric root object, $\beta(\alpha, 4)$, into (34), as well as $s_0 = 2$ and the known $s_7(\alpha,\beta)$, and executing, yields the unique solution $\alpha_0$ and $\beta_0$ as given in (33).

Let now $n = 7$ and $s_0 = \frac{1}{10}$, see Example 2.1, so that here again $\beta(\alpha, \lambda_{n_0}) = \beta(\alpha, 4)$ holds. Executing (34) with $\beta(\alpha, 4)$, $s_0 = \frac{1}{10}$ and with the known $s_7(\alpha,\beta)$ yields the expected unique solution $\alpha_0 = \mathrm{Root}[F_{12,\frac{1}{10}}(\alpha), 4]$, $\beta_0 = \mathrm{Root}[G_{12,\frac{1}{10}}(\beta), 3]$. □

The uniqueness of $\alpha_0$ and $\beta_0$, when computed according to Proposition 2.5, follows from $\lim_{\alpha\to 1} \beta(\alpha, \lambda_{n_0}) = \nu_{n_0}$ and $\lim_{\alpha\to 1} \beta(\alpha, \lambda_{n_0} + k) > \kappa > \nu_{n_0}$ where $k \geq 1$ is an integer and $\kappa$ is a constant; this property implies, according to Theorem 1.1, that the limiting pair $(\alpha, \beta) = (1, \nu_{n_0})$ which solves $H^{n_0}_{m(n_0)}(\alpha,\beta) = 0$ can only be reached if $\beta = \beta(\alpha, \lambda_{n_0})$ is chosen. Note that the curve $H^{n_0}_{m(n_0)}(\alpha,\beta) = 0$ does not intersect itself when $\beta > \alpha > 1$ and therefore the appropriate branch can be uniformly determined by the unique index $\lambda_{n_0}$ given in Table 3.

**Example 2.7.** In continuation of Example 2.5 and Example 2.6, one then sets up the particular case of the tentative form (14) for $n_0 = 7$ and proceeds as in Example 2.2, that is, replacing $\sigma_k$ by $\sigma_k(\alpha, \beta)$, which can be adopted for $k \in \{0, 1, \ldots, 13\}$ from [55], and replacing $\alpha$ and $\beta$ by $\alpha_0$ respectively $\beta_0$ as given in (33). However, one may run, when using a not enough powerful hardware, into time constraints due to the complexity of (14), when $n = n_0 = 7$. As a way out one may determine the coefficients of $P_{7,\alpha_0,\beta_0}$ at first numerically (by applying to them the *Mathematica*-function N with high precision (e.g., using $p = 3000$)), and then convert the outcomes, with the aid of the *Mathematica*-function RootApproximant, to root objects Root$[P(z), l]$, which, by definition, are approximating the coefficients of $P_{7,\alpha_0,\beta_0}$ well (the $P$'s would turn out to be polynomials of degree 12). Actually, the so approximated coefficients coincide here with the true optimal coefficients of $P_{7,\alpha_0,\beta_0} = Z_{7,2}$, as a comparison with the expanded version of the polynomial, given in [44, Remark 4.3], reveals. They are given, together with the deviation from zero, $|Z_{7,2}(1)|$, in Appendix 8.1.

Chopping them after the tenth digit after decimal point, a numerical approximation to (2) and to its norm, for $n_0 = 7$ and $s_0 = 2$, is thus given by

$$\begin{gathered} Z_{7,2}(x) = \textstyle\sum_{k=0}^{5} a^*_{k,7}(2)x^k + (-14)x^6 + x^7 \approx \\ 0.4369440905 + (-0.1873410678)x + (-7.8705484829)x^2 \\ +1.1870230011x^3 + 20.9955470665x^4 + (-1.9996819333)x^5 + (-14)x^6 + x^7, \\ |Z_{7,2}(1)| \approx 0.4380573257. \end{gathered} \tag{36}$$

The deviation $|Z_{7,2}(1)|$ of $Z_{7,2}$ from zero, on $\boldsymbol{I}$, is least among all polynomials of form $Q_{7,2}(x) = \sum_{k=0}^{5} a_{k,7}x^k + (-14)x^6 + x^7$, where $a_{k,7} \in \mathbb{R}$ is arbitrary. Figure 4 displays the graph of $Z_{7,2}$ on the interval $[-1, 1.1]$. □

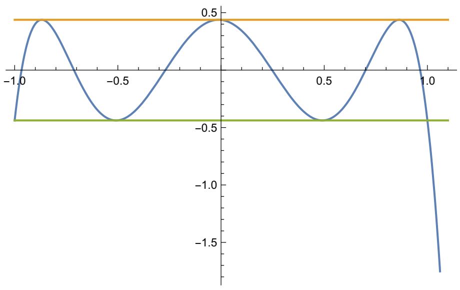


**Figure 4**

*Remark* 3. The Variants in (25) and (31) are, due to possible non-uniqueness, inferior to the Variant in (34).

*Remark* 4. The preceding Examples indicate that the 3rd algorithm for solving ZFP, although it is of theoretical interest, is not well suited for the practical algebraic construction of $Z_{n,s}$ if the coefficients $a^*_{k,n}(s)$ are required to be expressed non-numerically, even when the degree $n$ is comparatively low. With either of the three variants, the determinant-based tentative form (14) of $Z_{n,s}$ must be held responsible for possibly encountered computing time constraints because basically (14) corresponds to Cramer's rule for the solution of a system of linear equations, and this rule is known to be computationally inefficient, see also the related Remark 4 in [4, p. 23] which references to [20, 51]. To compare the complexity of the three tentative forms $P_{n,\alpha,\beta}$, $R_{n,\alpha,\beta}$, $S_{n,\alpha,\beta}$ we computed, exemplarily for the degree $n = n_0 = 4$, their constant terms $A_{0,n}(\alpha, \beta)$, $B_{0,n}(\alpha, \beta)$ and $C_{0,n}(\alpha, \beta)$. The results are given in Appendix 8.1 below, and they evidently support the assumption that $R_{n,\alpha,\beta}$ is the least complex one and $P_{n,\alpha,\beta}$ is the most complex one among the three tentative forms. Analogous computations which we have conducted for several $n = n_0 > 4$ confirm this assumption.

**Proposition 2.6 (Conclusion).** *Based on the preceding discussion (in particular Remarks 2, 3, 4) we propose to solve (for $s > \tau_n$ and $n \le 13$) ZFP algebraically by deploying the 1st algorithm consisting of the tentative form $R_{n,\alpha,\beta}$ in conjunction with Proposition 2.5.*

In closing this Section, we point out how the parameters $\alpha$ and $\beta$ in Theorem 1.1 are related to each other via the parameter $s$, here exemplarily for $4 \le n \le 7$.

**Algorithmic Generation of $\boldsymbol{\alpha = \alpha_n(\beta, s)}$ and $\boldsymbol{\beta = \beta_n(\alpha, s)}$.** These function equations can be generated by Gröbner basis computation as a byproduct of the 1st algorithm, see [55] for $n = 5$. For $n = 4$ they read as follows:

$$\alpha = \alpha(\beta, s) = \frac{-27 + 130s - 192s^2 + 96s^3 - 35\beta - 44s\beta + 120s^2\beta + 27\beta^2 - 126s\beta^2 + 27\beta^3}{8(1 + 2s + 9s^2)}, \tag{37}$$

$$\beta = \beta(\alpha, s) = \frac{27 + 130s + 192s^2 + 96s^3 - 35\alpha + 44s\alpha + 120s^2\alpha - 27\alpha^2 - 126s\alpha^2 + 27\alpha^3}{8(1 - 2s + 9s^2)}. \tag{38}$$

For $n \geq 6$ one can proceed by analogy, and for $n = 6, 7$ the Formulae for $\alpha = \alpha_n(\beta, s)$ and $\beta = \beta_n(\alpha, s)$ (or simply $\alpha = \alpha(\beta, s)$ and $\beta = \beta(\alpha, s)$ if there is no ambiguity) are stored in [55].

*Remark* 5. With $\alpha = \alpha_n(\beta, s)$, $\beta = \beta_n(\alpha, s)$, $s = s_n(\alpha, \beta)$, $H^n_{m(n)}(\alpha, \beta) = 0$, $F_{m(n),s}(\alpha) = 0$, $G_{m(n),s}(\beta) = 0$, one has at hand interrelated Formulae for validating whether, for a given $n = n_0$ and admissible $s = s_0$, a compatible pair $(\alpha_0, \beta_0)$ has been found for the construction of $Z_{n_0,\alpha_0,\beta_0} = Z_{n_0,s_0}$ (necessary conditions); note that then also $\gamma = \gamma_n$ will be known. Furthermore, we point to Formulae [35, (5.11)], [41, (8.3)], [48, (30), (31)] which interrelate equioscillation points of $Z_{n,s}$ with the parameters $\alpha, \beta, s$. All these Formulae add to the stated properties of $Z_{n,s}$, see Theorem 1.1.

**Example 2.8.** In continuation of Example 1.1, where the setting was $n = n_0 = 5$ and $s = s_0 = \frac{1}{5}$, one readily verifies by insertion that the pair $(\alpha, \beta) = (\alpha_0, \beta_0)$, as given in (5), is indeed a compatible one for $Z_{5,\frac{1}{5}}$ since there holds: $\alpha_5(\beta_0, s_0) = \alpha_0$, $\beta_5(\alpha_0, s_0) = \beta_0$, $s_5(\alpha_0, \beta_0) = s_0$, $H^5_6(\alpha_0, \beta_0) = 0$, $F_{6,s_0}(\alpha_0) = 0$, $G_{6,s_0}(\beta_0) = 0$. □

## 3 Intertwining of the three Variants by means of Resultants

To find possible intersection points of two planar curves, use can be made of resultants which are implemented in *Mathematica* as Resultant. We now reveal, for $4 \leq n \leq 13$, how the terms $F_{m(n),s}, G_{m(n),s}, H^n_{m(n)}, \det M_3^{n+1\times n+1}, s_n(\alpha, \beta)$ occurring in the preceding three variants, see Sections 2.2, 2.3, 2.4, are intertwined.

$$H^n_{m(n)}(\alpha, \beta) \,|\, \text{Resultant}[F_{m(n),s}(\alpha), G_{m(n),s}(\beta), s], \tag{39}$$

i.e., the resultant (with respect to $s$) of the pair of Malyshev polynomials $F_{m(n),s}, G_{m(n),s}$ is divided by the polynomial $H^n_{m(n)}$ which defines the reduced relation curves (with respect to $\alpha$ and $\beta$); this property has already been noticed in [54, Remark 3.2].

Writing $s_n(\alpha, \beta) - s = \frac{\text{Num}_n(\alpha,\beta,s)}{\text{Denom}_n(\alpha,\beta)}$ and $\det M_3^{n+1\times n+1} = \det M_3^{n+1\times n+1}(\alpha, \beta) = \frac{\text{Num}_{n+1}(\alpha,\beta)}{d_{n+1}}$ with ($d_{n+1} \neq 0$ constant), we furthermore get that

$$\begin{aligned}
G_{m(n),s}(\beta) \,&|\, \text{Resultant}[\text{Num}_n(\alpha, \beta, s), \text{Num}_{n+1}(\alpha, \beta), \alpha], && (40)\\
F_{m(n),s}(\alpha) \,&|\, \text{Resultant}[\text{Num}_n(\alpha, \beta, s), \text{Num}_{n+1}(\alpha, \beta), \beta], && (41)\\
G_{m(n),s}(\beta) \,&|\, \text{Resultant}[\text{Num}_n(\alpha, \beta, s), H^n_{m(n)}(\alpha, \beta), \alpha], && (42)\\
F_{m(n),s}(\alpha) \,&|\, \text{Resultant}[\text{Num}_n(\alpha, \beta, s), H^n_{m(n)}(\alpha, \beta), \beta], && (43)\\
H^n_{m(n)} \,&|\, \det M_3^{n+1\times n+1}. && (44)
\end{aligned}$$

**Example 3.1.** For $n_0 = 4$, the said terms occurring in the three variants can be adopted from (27), (30), [55] and using these terms, the statements (39) - (43) are readily checked with *Mathematica*. As for (39) one obtains

$$\text{Resultant}[F_{4,s}(\alpha), G_{4,s}(\beta), s] = 4294967296 \cdot H^4_4(\alpha, \beta) \cdot P_{12}(\alpha, \beta), \tag{45}$$

with a bivariate polynomial $P_{12}$ of total degree 12, and concerning (44) see (27). □

Checking (39) - (44) for $5 \leq n \leq 13$ can be accomplished similarly, with recourse to [55].

## 4 On an algebraic Solution Path to ZFP due to Malyshev

Malyshev [20, p. 934] was unaware of the paper by Sodin & Yuditskii [51] (and of the paper by Peherstorfer [28]), when he proposed an algebraic algorithm for solving ZFP, see also [21].

It will turn out that Malyshev's algorithm basically builds on a tentative representation of (2) which is equivalent to (14) as used in the 3rd algorithm. Furthermore, it will turn out that Malyshev's determination of the two variable parameters is basically equivalent to Variant 1.

In place of $Q_{n,s}(x)$, see Section 1.1, Malyshev [20] considers polynomials in the variable $t$ of form $\widetilde{Q}_{n,\sigma}(t) = t^n + \sigma t^{n-1} + \sum_{k=2}^{n} a_k t^{n-k}$ where $\sigma \in \mathbb{R}$ (originally, $\sigma$ is denoted by $z$ in [20]). In place of the moments $\sigma_k = \sigma_k(\alpha, \beta)$, see Section 2.1, he uses bivariate polynomials $\lambda_k = \lambda_k(x, y)$ (not to be confused with $\lambda_{n_0}$ from Proposition 2.5), which are coefficients in the series expansion

$$\sqrt{R_*} = \sqrt{(1-\tau^2)(1-\tau^2+2x\tau+y\tau^2)} = 1 + x\tau + \sum_{k=0}^{\infty} \lambda_k \tau^{k+2}. \tag{46}$$

The *Mathematica*-function Series[$\sqrt{\text{R}_*}$, $\{\tau, 0, 7\}$]//Simplify yields the first six instances of $\lambda_k$:

$$\begin{aligned}
&\lambda_0(x,y) = \tfrac{1}{2}(-2 - x^2 + y),\ \lambda_1(x,y) = \tfrac{1}{2}(x^3 - xy),\ \lambda_2(x,y) = \tfrac{1}{8}(-5x^4 - y^2 + x^2(-4 + 6y)),\\
&\lambda_3(x,y) = \tfrac{1}{8}(7x^5 + x^3(8 - 10y) + xy(-4 + 3y)),\\
&\lambda_4(x,y) = \tfrac{1}{16}(-21x^6 + (-2 + y)y^2 + 5x^4(-6 + 7y) + x^2(-8 + 24y - 15y^2)),\\
&\lambda_5(x,y) = \tfrac{1}{16}(33x^7 - 7x^5(-8 + 9y) + xy(-8 + 12y - 5y^2) + x^3(24 - 60y + 35y^2)).
\end{aligned} \tag{47}$$

Malyshev then defines the Hankel determinant

$$\Delta_{n-1}(x,y)=\det\begin{pmatrix}\lambda_1 & \lambda_2 & \dots & \lambda_{n-2}\\ \lambda_2 & \lambda_3 & \dots & \lambda_{n-1}\\ . & . & \dots & .\\ \lambda_{n-2} & \lambda_{n-1} & \dots & \lambda_{2n-5}\end{pmatrix},\ \text{setting}\ \Delta_1(x,y)=1, \tag{48}$$

and with its aid he asserts for the coefficients of the least-deviating $\widetilde{Q}_{n,\sigma}$ a tentative representation (which we compare to the tentative coefficients of $P_{n,\alpha,\beta}$):

$$\begin{aligned} a_2 &= \tfrac{\phi_2(x,y)}{\Delta_{n-1}(x,y)} && \text{(corresponds to } B_{n-2,n}(\alpha,\beta) \text{ if } \sigma=-ns)\\ \dots & \\ a_n &= \tfrac{\phi_n(x,y)}{\Delta_{n-1}(x,y)} && \text{(corresponds to } B_{0,n}(\alpha,\beta) \text{ if } \sigma=-ns)\end{aligned} \tag{49}$$

without, however, defining exactly the meaning of, and exemplifying, $\phi_2(x,y),\dots,\phi_n(x,y)$. But the similarity of (49) to the representation (14), when $\det M_0^{n+1\times n+1}$ is expanded by minors of the last row, becomes obvious. To be more specific, we compare the Hankel determinant $\Delta_{n-1}(x,y)$ in the denominator of (49) to the Hankel determinant $\det M_1^{n\times n}=\det M_1^{n\times n}(\alpha,\beta)$ in the denominator of (14) and we deduce that they differ, when the parameters $x$ and $y$ are properly expressed by means of the parameters $\alpha$ and $\beta$, only by a scalar factor, and this is the least deviation of $P_{n,\alpha,\beta}$ from zero:

$$\det M_1^{n\times n}(\alpha,\beta)=\Delta_{n-1}(x,y)L_n(\alpha,\beta)\ \text{with}\ x=x(\alpha,\beta)\ \text{and}\ y=y(\alpha,\beta). \tag{50}$$

This implies that the numerator $\phi(x,y)$ of the coefficient $a_j$, $(2\le j\le n)$, is equal to the quotient $\frac{\text{Numerator}_{n-j}}{L_n(\alpha,\beta)}$, where $\frac{\text{Numerator}_{n-j}}{\det M_1^{n\times n}}=B_{n-j,n}(\alpha,\beta)$ is that coefficient which corresponds to $a_j$. A verification of (50) for the degree $n_0=3$ is given in Appendix 8.2.

To determine, for a given $n=n_0$ and $\sigma=\sigma_0$, the optimal parameters $x=x_0$ and $y=y_0$, Malyshev [20, p. 936] introduces, for $2\le n\le 5$, bivariate polynomials $U_{m(n),\sigma}(x)$ and $V_{m(n),\sigma}(y)$ and he seeks $x_0$ and $y_0$ among the (positive) solutions of the polynomial equations $U_{m(n),\sigma}(x)=0$ and $V_{m(n),\sigma}(y)=0$. $U_{m(n),\sigma}$ and $V_{m(n),\sigma}$ correspond to the Malyshev polynomials $F_{m(n),s}$ and $G_{m(n),s}$ which he later introduced in [21], see Remark 7 below, so that his proceeding can be viewed as being basically equivalent to deploying the Variant 1.

In order to exemplify his own algebraic solution (49) of ZFP, Malyshev [20] considers the special degree $n_0=5$ and he prescribes $\sigma_0=1$, which in our notation corresponds to $s_0=-\frac{1}{5}$, so that he actually proceeds to calculate the least deviating $\widetilde{Q}_{5,1}(t)=Z_{5,-\frac{1}{5}}(x)=-Z_{5,\frac{1}{5}}(-x)$. In view of Example 1.1 and (6) - (7) we know $Z_{5,\frac{1}{5}}$ and hence we know the exact expressions, by radicals as well as by root objects, for the optimal coefficients and for the least deviation $L_5(\frac{-1}{5})$ of $Z_{5,-\frac{1}{5}}$. Malyshev [20, p. 937], on the other hand, provides these coefficients (denoted by $a_2,\dots,a_5$) in a biased numerical form and sketches the graph of $Z_{5,-\frac{1}{5}}$ from which the value $L_5(\frac{-1}{5})=L_5(\frac{1}{5})$, see (3) and (6), can be roughly read (compare Figure 1 above to the mirrored Figure 7 in [20]). Also, for the numerical form of the optimal parameters $x=x_0$ and $y=y_0$ as given in [20, p. 937] we can now subtend an exact expression in view of (5) and (80) below:

$$\begin{aligned} x_0 &= \tfrac{\alpha(5,\frac{1}{5})+\beta(5,\frac{1}{5})}{2}=\text{Root}[-288+416z+792z^2-1620z^3+675z^4,2]\approx 1.4102107032,\\ y_0 &= \alpha(5,\tfrac{1}{5})\beta(5,\tfrac{1}{5})+1=\text{Root}[695808-1723904z+1573344z^2-631800z^3+91125z^4,2]\approx 2.9862364377.\end{aligned} \tag{51}$$

Observe that these values are among the positive zeros of $U_{m(5),1}(x)$ and $V_{m(5),1}(y)$.

Shadrin [49, p. 243] compares Malyshev's algebraic approach to ZFP with the algebraic approaches published in [28, 51] and he comments: *Recently, the interest in an explicit algebraic solution of the Zolotarev problem [i.e., ZFP] was revived in the papers by Peherstorfer, Sodin-Yuditsky and Malyshev, but it is only Malyshev who demonstrates how his theory can be applied to some explicit constructions for particular n.*

We note in passing that Malyshev [20] was not the first to provide an algebraic solution of ZFP for $n_0=5$ and particular value(s) of $s=s_0$: To the best of our knowledge, it was Collins [10], see also [6], [40, Section 5].

For $n_0\ge 6$ however, explicit (non-numerical) algebraic solutions to ZFP are rare in literature: see [41, 42, 44] (and the present paper) for $n_0\in\{6,7\}$; partially [28, Section 5] for $n_0=6$ and [35, Section 7] for $n_0\le 7$; for $n_0>7$, see Appendix 8.3 below and [55].

## 5 On an algebraic Solution Path to ZFP due to Schiefermayr

Yet another algebraic algorithm for solving ZFP was proposed by Schiefermayr [48, Section 4.2]. As indicated in Section 1.2 above, his suggested tentative form of $Z_{n,s}$ is closely related to the one in the 2nd algorithm, $S_{n,\alpha,\beta}$. To the key question stated at the end of Section 1.2 above, he provides an answer in Corollary 3 of [48], which is related to (31) and likewise may produce nuisance values for $\alpha$ and $\beta$. Nevertheless, we translate that answer into the *Mathematica*-syntax to facilitate tedious determinant evaluations as well as the computation of the moments $s_n(\alpha,\beta)$ and the reduced relation curves $H^n_{m(n)}(\alpha,\beta)=0$.

If one defines, for $n = 2m+1 \geq 5$ odd:

(52)

```
so[α_,β_,k_]=1/2(−α^k−β^k−1−(−1)^k);
sso[α_,β_,k_]=1/2(α^k+β^k+1+(−1)^k);
Fo[α_,β_,0]=1;
Fo[α_,β_,k_?Negative]=0;
Fo[α_,β_,k_]:=(−1)^k/k!Det[Table[If[j>i,If[j==i+1,i,0],so[α,β,i−j+1]],
{i,k},{j,k}]];
FFo[α_,β_,k_?Negative]=0;
FFo[α_,β_,0]=1;
FFo[α_,β_,k_]:=(−1)^k/k!Det[Table[If[j>i,If[j==i+1,i,0],sso[α,β,i−j+1]],
{i,k},{j,k}]];
Psi2[3]=1;
Psi3[3]=0;
Psi2[n_]:=Module[{m=(n−1)/2},Det[Table[FFo[α,β,m+i−j+1],{i,m−1},{j,m−1}]]];
Psi3[n_]:=Module[{m=(n−1)/2},Det[Table[If[j==1,−FFo[α,β,m+i+1],FFo[α,β,m+i−j+1]],{i,m−1},{j,m−1}]]];
Phio[k_,n_]:=Module[{m=(n−1)/2},Det[Table[If[j==k,−Fo[α,β,m+i−1],Fo[α,β,m+i−j−1]],{i,m+1},{j,m+1}]]];

lhsequation32odd[n_]:=Factor[Sum[β^j Phio[(n+1)/2−j,n],{j,0,(n+1)/2}]]
lhsequation33odd[n_]:=Factor[−2 Psi3[n]+(α−n s)Psi2[n]]
```

then the proposed system of determinant equations [48, (32),(33)] for the computation of the endpoints $\alpha = \alpha_0$ and $\beta = \beta_0$ (when $n$ is odd) reads, here exemplarily for $n = 5$:

$$\text{lhsequation32odd}[5] == 0, \text{yielding} \tag{53}$$

$$\frac{64-80\alpha^2+44\alpha^4+\alpha^6+96\alpha\beta-16\alpha^3\beta+6\alpha^5\beta-16\beta^2+104\alpha^2\beta^2-29\alpha^4\beta^2-80\alpha\beta^3+36\alpha^3\beta^3-52\beta^4-9\alpha^2\beta^4-10\alpha\beta^5+5\beta^6}{1024} = 0$$

$$\left(\text{i.e., } \tfrac{1}{1024} H^5_{m(5)}(\alpha,\beta) = \tfrac{1}{1024} H^5_6(\alpha,\beta) = 0\right),$$

$$\text{lhsequation33odd}[5] == 0, \text{yielding} \tag{54}$$

$$\frac{1}{64}(-9\alpha^4 + 8\alpha^3\beta + 6\alpha^2\beta^2 + 24\alpha^2 - 5\beta^4 + 8\beta^2 + 20\alpha^3 s - 20\alpha^2\beta s - 20\alpha\beta^2 s - 80\alpha s + 20\beta^3 s - 80\beta s - 16) = 0.$$

Likewise, if one defines, for $n = 2m+2 \geq 4$ even:

(55)

```
se[α_,β_,k_]=1/2(−α^k−β^k+1−(−1)^k);
sse[α_,β_,k_]=1/2(α^k−β^k+1+(−1)^k);
Fe[α_,β_,k_?Negative]=0;
Fe[α_,β_,0]=1;
Fe[α_,β_,k_]:=(−1)^k/k!Det[Table[If[j>i,If[j==i+1,i,0],se[α,β,i−j+1]],
{i,k},{j,k}]];
FFe[α_,β_,0]=1;
FFe[α_,β_,k_?Negative]=0;
FFe[α_,β_,k_]:=(−1)^k/k!Det[Table[If[j>i,If[j==i+1,i,0],sse[α,β,i−j+1]],{i,k},{j,k}]];
Psi0[n_]:=Module[{m=(n−2)/2},Det[Table[FFe[α,β,m+i−j+1],{i,m},{j,m}]]];
Psi1[n_]:=Module[{m=(n−2)/2},Det[Table[If[j==1,−FFe[α,β,m+1+i],FFe[α,β,m+i−j+1]],{i,m},{j,m}]]];
Phie[k_,n_]:=Module[{m=(n−2)/2},Det[Table[If[j==k,−Fe[α,β,m+i],Fe[α,β,m+i−j]],
{i,m+1},{j,m+1}]]];

lhsequation34even[n_]:=Factor[Sum[α^j Phie[n/2−j,n],{j,0,n/2}]];
lhsequation35even[n_]:=Factor[−2 Psi1[n]+(α+1−n s)Psi0[n]]
```

then the proposed system of determinant equations [48, (34), (35)] for the computation of the endpoints $\alpha = \alpha_0$ and $\beta = \beta_0$ (when $n$ is even) reads, here exemplarily for $n = 4$:

$$\text{lhsequation34even}[4] == 0, \text{yielding} \tag{56}$$

$$\tfrac{1}{64}(-\alpha^4 + 4\alpha^3\beta - 8\alpha^3 - 6\alpha^2\beta^2 - 8\alpha^2\beta + 8\alpha^2 + 4\alpha\beta^3 + 8\alpha\beta^2 - 16\alpha\beta - \beta^4 + 8\beta^3 + 8\beta^2 - 16) = 0$$

$$\left(\text{i.e., } -\tfrac{1}{64} H^4_{m(4)}(\alpha,\beta) = -\tfrac{1}{64} H^4_4(\alpha,\beta) = 0\right),$$

$$\text{lhsequation35even}[4] == 0, \text{yielding} \tag{57}$$

$$\frac{1}{8}(-2\alpha^3 - 3\alpha^2\beta - \alpha^2 - 2\alpha\beta + 5\beta^3 + 3\beta^2 - 4\beta + 4\alpha^2 s + 8\alpha\beta s - 12\beta^2 s + 16s - 4) = 0.$$

We examine first the equations given in [48, (32)] (for $n$ odd) and [48, (34)] (for $n$ even): They arise from Theorems 1 (i), 2 (i) in [48] when $d$ is replaced there by $\beta$, and $a$ by $\alpha$. Evaluating now (53), (56), i.e., the equations [48, (32), (34)], one finds that the left-hand sides contain as a factor the polynomial $H^n_{m(n)}$. More precisely: For $n \in N_1 = \{4,5,7,8,11,13\}$ the left-hand sides are of form $c_n H^n_{m(n)}(\alpha,\beta)$ where $c_n \neq 0$ is a constant (e.g., $c_4 = \frac{-1}{64}$). Hence for $n \in N_1$ the equations in [48, (32), (34)] are equivalent to $H^n_{m(n)}(\alpha,\beta) = 0$. But for $n \in N_2 = \{6,9,10,12\}$ the left-hand sides are of form

$$\text{lhsequation34even}[6] = c_6 \cdot H_1^2(\alpha,\beta) \cdot H_8^6(\alpha,\beta) \text{ with } H_1^2(\alpha,\beta) = (2+\alpha-\beta) \tag{58}$$

$$\text{lhsequation32odd}[9] = c_9 \cdot H_2^3(\alpha,\beta) \cdot H_{18}^9(\alpha,\beta) \text{ with } H_2^3(\alpha,\beta) = (4+\alpha^2+2\alpha\beta-3\beta^2) \tag{59}$$

$$\text{lhsequation34even}[10] = c_{10} \cdot H_1^2(\alpha,\beta) \cdot H_{24}^{10}(\alpha,\beta) \tag{60}$$

$$\text{lhsequation34even}[12] = c_{12} \cdot H_4^4(\alpha,\beta) \cdot H_{32}^{12}(\alpha,\beta) \text{ with } H_4^4(\alpha,\beta) \text{ as given in (27).} \tag{61}$$

Thus for $n \in N_2$ the equations in [48, (32), (34)] are equivalent to, respectively,

$$H_1^2(\alpha,\beta)\cdot H_8^6(\alpha,\beta)=0, H_2^3(\alpha,\beta)\cdot H_{18}^9(\alpha,\beta)=0, H_1^2(\alpha,\beta)\cdot H_{24}^{10}(\alpha,\beta)=0, H_4^4(\alpha,\beta)\cdot H_{32}^{12}(\alpha,\beta)=0. \tag{62}$$

We refer to [54, Theorem 3.4] for a deeper analysis, which even covers degrees $n > 13$, of the two kinds of factorization occurring in the left-hand sides of (53) and (56) when $n \geq 4$. Table 2 in [54] displays the total degrees $m(n)$ of the polynomials $H^n_{m(n)}(\alpha,\beta)$, see also Table 1 above, and of the composite polynomials in (62). Moreover, as pointed out in [54, Remark 3.5], both kinds of polynomials correspond to the four-variate polynomials $p(a,b,c,d)$ whose zeros $\{a,b,c,d\}$ are so-called $\mathbb{T}_n$-tuples (see [48, pp. 149] for details), here specialized to $p(\alpha,1,-1,\beta)$ according to the inverse polynomial image of $Z_{n,s}$. In [48, Section 4.1] the polynomials $p(a,b,c,d)$ are given, due to their bulkiness, only for $n \in \{2,3,4\}$. Specializing there $(a,b,c,d) = (\alpha,1,-1,\beta)$, one gets back the polynomials $H_1^2(\alpha,\beta)$, $H_2^3(\alpha,\beta)$ and $H_4^4(\alpha,\beta)$ , but for $n=6$ one would get back the composite form $H_1^2(\alpha,\beta)\cdot H_8^6(\alpha,\beta)$.

We draw here another parallel, with a paper by Lazard [18], see also Remark 6 (iv) below: Investigating ZFP by methods of Algebraic Geometry, the author considers algebraic curves, denoted by $V_2'$ and $V_2$, whose degrees for $n \in N_2$ coincide with the degrees of $H^n_{m(n)}$ respectively with the degrees of the composite polynomials in (62), see [18, Proposition 7]. However, in [18] only their total degrees are provided, for particular values of $n \leq 12$, but not their explicit defining polynomials. Now e.g. the polynomials $H^n_{m(n)}$ can be generated algorithmically, see (53) and (56), even if $n > 13$.

Next, we examine equations [48, (33), (35)] which are deduced as follows:

In view of Theorem 1.1 and (11), the monic polynomials $S_{n,\alpha,\beta,y_j} + L_n(s)$ possess $m-1$ double zeros $y_j$ and additionally the zeros $\pm 1$ and the zero $\alpha$ (if $n = 2m+1$), respectively $m$ double zeros $y_j$ and additionally the zeros 1 and $\alpha$ (if $n = 2m+2$), where $n \geq 4$. By Vieta's theorem, the sum of these zeros equals the negative second leading coefficient, which is $ns$, and this yields the equations [48, (30), (31)]. The $y_j$ in turn are the zeros of the polynomials given in [48, Theorems 1 (ii), 2 (ii)]. By Vieta's theorem, applied to these polynomials, the sum of the $y_j$ equals the number $-\det\mathbb{F}_1/\det\mathbb{F}$ (in the notation of [48]) and inserting it into equations [48, (30), (31)] finally yields the equations [48, (33), (35)], which correspond to our equations (54) and (57) when $n \geq 4$.

Evaluating now (54) and (57), one sees that the left-hand sides of the said equations are of form $c_n J_n(\alpha,\beta,s)$, with $c_n \neq 0$ a constant, and with certain trivariate polynomials $J_n$ for $n \in N_1 \cup N_2$. Hence the equations in [48, (33), (35)] are equivalent to $-J_n(\alpha,\beta,s) = 0$. Invoking $s_n(\alpha,\beta)$ and setting, as in Section 3, $s_n(\alpha,\beta) - s = \frac{\text{Num}_n(\alpha,\beta,s)}{\text{Denom}_n(\alpha,\beta)}$, the condition $s_n(\alpha,\beta) - s = 0$ becomes equivalent to $\text{Num}_n(\alpha,\beta,s) = 0$, but on the other hand, $\text{Num}_n(\alpha,\beta,s) = -J_n(\alpha,\beta,s)$ holds, as can be readily verified by comparing $\text{Num}_n(\alpha,\beta,s)$ to the values generated by Formulae (54), (57) when $n \geq 4$. Hence the equations [48, (33), (35)] are equivalent to $s_n(\alpha,\beta) - s = 0$. In view of (31) we conclude that Schiefermayr's proposal [48, Corollary 3] how to determine $\alpha = \alpha_0$ and $\beta = \beta_0$ for a concretely prescribed $n = n_0$ and $s = s_0 > \tau_{n_0}$ is thus identical with (31), and hence lacks uniqueness, provided that $n \in N_1$. For $n = n_0 \in N_2$, however, that proposal amounts to a modification of (31):

$$\text{Reduce}[s_{n_0}(\alpha,\beta) - s_0 == 0 \wedge \text{lhs}_{n_0}(\alpha,\beta) == 0 \wedge 1 < \alpha < \beta \wedge \nu_{n_0} < \beta, \{\alpha,\beta\}], \tag{63}$$

where $\text{lhs}_{n_0}(\alpha,\beta)$ is an appropriate left-hand side in (62).

We observe that for $n_0 \in N_2$ the (first) extra factor at $H^n_{m(n)}(\alpha,\beta)$ in (62) may affect the set of solutions for $\alpha$ and $\beta$ to the extent that nuisance values are generated. In such a case the determination of $\alpha_0$ and $\beta_0$ according to [48, Corollary 3] would again not be unique (so that Remark 1(i) in [48] would need to be adapted).

More specifically, we strive to determine $\alpha = \alpha_0$ and $\beta = \beta_0$ by means of Corollary 3 of [48] for the setting $n_0 = 6 \in N_2$ and $s_0 = 2$ (see also [41, p. 195]); the extra factor is $H_1^2(\alpha,\beta) = 2+\alpha-\beta$:

**Example 5.1.** With $n_0 = 6$ and $s_0 = 2$, Formula (63) reads

$$\text{Reduce}[s_6(\alpha,\beta) - 2 == 0 \wedge (2+\alpha-\beta)\cdot H_8^6(\alpha,\beta) == 0 \wedge 1 < \alpha < \beta \wedge \nu_6 < \beta, \{\alpha,\beta\}], \tag{64}$$

where $s_6(\alpha,\beta)$ and $H_8^6(\alpha,\beta)$ are given in [41, 55]. Equivalently, (64) can be stated as

$$\text{Reduce}[\text{lhsequation34even}[6] == 0 \wedge \text{lhsequation35even}[6] == 0 \wedge s == 2 \wedge 1 < \alpha < \beta \wedge \nu_6 < \beta, \{\alpha,\beta\}]. \tag{65}$$

Executing (64) or (65) with *Mathematica* yields the nuisance values

$$\alpha = \alpha_0 = 3 \text{ and } \beta = \beta_0 = 5 \tag{66}$$

as well as the alternative values (which can be cross-checked by deploying e.g. the Variant 1):

$$\begin{aligned} \alpha = \alpha_0 &= \mathrm{Root}[-458570907 - 42254808z + 244930588z^2 - 129874632z^3 + 39870030z^4 - \\ &\quad 8679464z^5 + 1247868z^6 - 99000z^7 + 3125z^8, 2] \approx 12.0415915958, \\ \beta = \beta_0 &= \mathrm{Root}[177361957 - 16506088z - 10658852z^2 - 37145848z^3 + 28468366z^4 - \\ &\quad 8498200z^5 + 1294268z^6 - 101000z^7 + 3125z^8, 2] \approx 12.0415975617. \end{aligned} \tag{67}$$

Inserting (66) into any tentative form $Z_{6,\alpha,\beta}$ of $Z_{6,2}$ produces a monic polynomial, $q_6$, with the desired second leading coefficient $-n_0 s_0 = -12$:

$$q_6(x) = 11 + 36x + (-57)x^2 + (-40)x^3 + 45x^4 + (-12)x^5 + x^6 = (-1 + (-4)x + x^2)(-11 + 8x + 14x^2 + (-8)x^3 + x^4). \tag{68}$$

However, $q_6$ is not a proper sextic Zolotarev polynomial as characterized in Theorem 1.1 since it exhibits less than six equioscillation points on $\mathbf{I}$ (actually, $q_6$ coincides with [41, (4.5)] if choosing there $s = 2$, and $q_6(x)/q_6(1)$ coincides with [42, (48)] if choosing there $t = 2$).

Whereas the insertion of (67) into $Z_{6,\alpha,\beta}$ does produce the sextic Zolotarev polynomial $Z_{6,2}$ in the power form (2) with coefficients $a^*_{k,6}(2)$ expressed by means of root objects. We provide them, together with the deviation from zero, $|Z_{6,2}(1)|$, in [55] and give here only their chopped numerical representations:

$$Z_{6,2}(x) \approx -0.0623920549 + (-3.7461095036)x + 0.8119595277x^2 + 14.9948096730x^3 + (-1.7495674727)x^4 + (-12)x^5 + x^6, \tag{69}$$

$$|Z_{6,2}(1)| \approx 0.7512998305. \tag{70}$$

Figure 5 displays the graphs of $q_6$ and $Z_{6,2}$ (with four respectively six equioscillation points) on $\boldsymbol{I}$. □

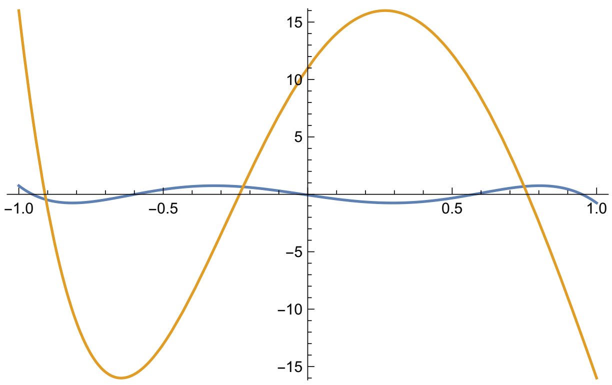


**Figure 5**

## 6 On an algebraic Solution Path to ZFP due to Peherstorfer

Peherstorfer [28, p. 259, p. 262] considers Zolotarev polynomials as special instances (with exactly 2 deviation points on $[b, 1]$) of the so-called (monic) $T$-polynomials of degree $n$, $\mathcal{T}_n$, on sets $E = [-1, a] \cup [b, 1]$, where $-1 < a < b < 1$, see also [34]. To ensure the existence of such polynomials, he exploits their orthogonality, with respect to a certain weight-function, to the set $\mathbb{P}_n$ of all polynomials of degree $\le n$, see [28, Theorem 4] (replace there the misprinted $\tau_n$ by $\mathcal{T}_n$). It turns out that the existence of $\mathcal{T}_n$ is equivalent to the periodicity of the recursion coefficients in the three-term recurrence relation of orthogonal polynomials [28, p. 259], [29]. This fact enables him to calculate Zolotarev polynomials on $E$ *for at least small* $n$ [28, p. 245], and in [28, Section 5] he provides a description how to proceed so for $2 \le n \le 6$. We are going to trace Peherstorfer's algebraic solution path to ZFP exemplarily, for $n = n_0 = 4$, and it will become obvious that his approach corresponds to the third algorithm which is based on the tentative form $Z_{4,\alpha,\beta} = P_{4,\alpha,\beta}$, inspired by [51]. Actually, in [31, p. 194] Peherstorfer himself refers to parallels between the papers [28] and [51].

For convenience, we reconsider Example 2.4 above with the particular prescribed $s = s_0$ as given there. The goal is to recover, from prescribed parameters $a = a_0$, $b = b_0$ (respectively $c = c_0$, $d = d_0$), following Peherstorfer's solution path [28], the particular Zolotarev polynomial $Z_{4,s_0}$ as given in (29). To set the stage, we transform $Z_{4,s_0}$, whose inverse polynomial image is $\boldsymbol{I} \cup [\alpha, \beta]$ (with $\alpha = \alpha_0$ and $\beta = \beta_0$ from (28)), to a compressed monic Zolotarev polynomial, whose inverse polynomial image is $E = [-1, a] \cup [b, 1]$, with

$$a = a_0 = \sqrt{\frac{11}{20}(-17 + 3\sqrt{33})} \approx 0.3585085309, \; b = b_0 = \sqrt{\frac{1}{20}(-403 + 73\sqrt{33})} \approx 0.9042420361. \tag{71}$$

This is accomplished by forming first $Z_{4,s_0}(\tilde{l}(x))$, where $\tilde{l}(x) = ((1+\beta_0)/2)x + (-1+\beta_0)/2$, and then dividing $Z_{4,s_0}(\tilde{l}(x))$ by its leading coefficient, which is $\left(\frac{1+\beta_0}{2}\right)^4$. The result is

$$\mathcal{Z}_4(x) = \frac{1}{400}(3221 - 555\sqrt{33}) + \frac{(-8)x}{\sqrt{775+135\sqrt{33}}} + \frac{3}{10}(-55+9\sqrt{33})x^2 + \frac{8x^3}{\sqrt{775+135\sqrt{33}}} + x^4. \tag{72}$$

To be compliant with [28, p. 247, p. 275], one then has to set $c = \frac{b+a}{2}$, and $d = \frac{b-a}{2}$ (i.e., $a = c - d$ and $b = c + d$) so that the inverse polynomial image of (72), $E$, turns to $\widetilde{E} = [-1, c-d] \cup [c+d, 1]$ with

$$c = c_0 = \sqrt{\frac{1}{20}(-285 + 51\sqrt{33})} \approx 0.6313752835,\ d = d_0 = \sqrt{\frac{1}{10}(-5+\sqrt{33})} \approx 0.2728667525. \tag{73}$$

After this preparation we restart and proceed reversely in three steps, guided by [28].

Step 1. Prescribe $c_0$ and $d_0$ (we choose here deliberately the values in (73)) and form the two lists of periodic (cyclic) recursion coefficients, for $n = 4$, as given in [28, p. 276] (we correct there $2d$ to $2c$ in the denominators of the first list):

$$\alpha 4 = (d, c - \frac{1-d^2}{2c}, -d, c - \frac{1-d^2}{2c}),\ \lambda 4 = (\frac{1-d^2+2dc}{4}, \frac{1-d^2+2dc}{2}, \frac{1-d^2-2dc}{4}, \frac{1-d^2-2dc}{4}). \tag{74}$$

The recursive computational steps for generating the sought-for polynomial we provide in the *Mathematica*-syntax:

```
α4={d,c−(1−d^2)/(2c),−d,c−(1−d^2)/(2c)};
λ4={(1−d^2+2d c)/4,(1−d^2+2d c)/2,(1−d^2−2d c)/4,(1−d^2−2d c)/4};
c0=Sqrt[1/20(−285+51Sqrt[33])];d0=Sqrt[1/10(−5+Sqrt[33])];
q40=1;q4m1=0;
q41=q40(x−α4[[1]])−λ4[[1]]q4m1/.{c→c0,d→d0};
q42=Expand[(x−α4[[2]])q41−λ4[[2]]q40/.{c→c0,d→d0}];
q43=Expand[(x−α4[[3]])q42−λ4[[3]]q41/.{c→c0,d→d0}];
q44=Expand[(x−α4[[4]])q43−λ4[[4]]q42/.{c→c0,d→d0}]//RootReduce//FullSimplify
Out[1]= 1/400 (3221−555 Sqrt[33])−(8 x)/Sqrt[775+135 Sqrt[33]]+
3/10 (−55+9 Sqrt[33]) x^2+(8 x^3)/Sqrt[775+135 Sqrt[33]]+x^4
```

Thus one gets $q_{44} = \mathcal{Z}_4(x)$, see (72), as expected.

Step 2. Identify $\mathcal{Z}_4(x)$ with a $T$-polynomial on $[-1, a] \cup [b, 1]$ with $a = a_0$ and $b = b_0$, see (71).

Step 3. Transform $\mathcal{Z}_4$ linearly to a Zolotarev polynomial whose inverse polynomial image is $\boldsymbol{I} \cup [\alpha, \beta]$, where $\alpha = \alpha_0$ and $\beta = \beta_0$ are from (28). This eventually recovers $Z_{4,s_0}$ (take $\mathcal{Z}_4(k(x))$ with $k(x) = \frac{1}{20}(10 + \sqrt{55(-17+3\sqrt{33})})x + \frac{1}{20}(-10 + \sqrt{55(-17+3\sqrt{33})})$ and divide by the leading coefficient).

It is insightful to consider the intermediate results $1 = q_{40}, q_{41}, q_{42}, q_{43}, q_{44}$ of the computation in terms of determinants, compare with (29): In the two determinants of (14) replace, for $n = 4$, $\sigma_k$ by $\sigma_k(\alpha, \beta)$ as indicated in (18); furthermore, in the first of these determinants replace $x$ by the above defined $\tilde{l}(x)$, where $\beta = \beta_0$ according to (28). Then there holds

$$q_{4j} = \left(\frac{1+\beta_0}{2}\right)^j (\det M_0^{j+1 \times j+1} / \det M_1^{j \times j}) \text{ for } j = 1,2,3,4, \text{ with } q_{44} = \mathcal{Z}_4(x). \tag{75}$$

This reveals the mentioned parallels between the papers [28] and [51].

**Example 6.1.** Prescribing e.g. $c_0 = 24/35$ and $d_0 = 1/7$, Peherstorfer's algebraic solution path will yield, for $n_0 = 4$, the Zolotarev polynomial $Z_{4,\tilde{s}_0}$ on $\boldsymbol{I} \cup [\alpha_0 = 37/27, \beta_0 = 43/27]$, where $\tilde{s}_0 = 5/18$. The details are left to the reader. □

## 7 Remarks (6-8)

*Remark* 6. Additionally to the here considered solution paths to ZFP, in literature there exist several further (non-elliptic) approaches to ZFP which remain a fruitful subject for future investigation:

(i) A. A. Markov [22, p. 15] starts off with the Abel-Pell differential equation (8), from which Zolotarev [59, 60] had deduced his elliptic solution of ZFP. By announcing: *Without relying on E. I. Zolotarev's formulas, we show how it is possible to reduce our problem to three algebraic equations*, Markov continues to transform (8) to the task of solving a system of linear first-order differential equations. With respect to (i), Sodin & Yuditskii [51, p. 2489] remark: *A. A. Markov noted that it is desirable to find an algebraic solution to this problem [i.e., ZFP when $s > \tau_n$] and sketched a method that could be used to find such a solution, without carrying the computation to completion.*

(ii) Voronovskaja's suggestion for solving ZFP (on $[0,1]$) [57, p. 96] requires to solve a non-linear system of differential equations. But curiously, when it came down to construct $Z_{3,s}/L_3(s)$ (on $[0,1]$) Voronovskaja [57, pp. 97-98] applied an ad hoc-method rather than the method suggested by herself. However, Paszkowski [27, p. 156] did apply a Voronovskaja-type method and solved ZFP for $n = n_0 = 3$ (on $\boldsymbol{I}$). With respect to (i) and (ii) Shadrin [49, p. 243] remarks: *But as far as we know, nobody (including A. Markov and Voronovskaya themselves) has ever tried to apply these methods for constructing $Z_n$ [i.e. $Z_{n,s}$] for any particular $n$.* But note the mentioned instance [27, p. 156]; also note that a polynomial solution of the differential equation (8) needs not be a proper $Z_{n,s}$, as Examples show.

(iii) By deploying methods of Complex Analysis, Meiman [23, 24, 25] addressed a generalization of ZFP when several leading coefficients $a_{n,n}, a_{n-1,n}, \dots, a_{n-j,n}$ of a polynomial are prescribed, $j \geq 1$. Peherstorfer in [33, p. 3] comments on Meiman's papers and in particular remarks: *For an explicit representation of the polynomial $\widetilde{Z}_n$ explicit expressions for the endpoints $\alpha_{j,n}, \beta_{j,n}$ and the zeros of the derivative $\gamma_{j,n}$ would be needed. To find such explicit expressions is extremely unlikely ... . In [20,21,22] [i.e., [23, 24, 25]] no way of solution is offered to this fundamental open question.*

Note that when the said generalization is adjusted to ZFP (i.e., $j = 1$), then $\widetilde{Z}_n$ (notation as used in [33]) becomes identical with $Z_{n,s}/L_n(s)$ (using our notation) and $\alpha_{j,n}$ coincides with $\alpha$, $\beta_{j,n}$ coincides with $\beta$, $\gamma_{j,n}$ coincides with $\gamma$; and Proposition 2.5 then offers, for $n \leq 13$ and $s > \tau_n$, a way of solution to determine $\alpha$ and $\beta$, and hence $\gamma$.

(iv) Lazard [18, Section 4.2] develops an algebraic solution strategy for solving ZFP for the degrees $n \leq 12$ without, however, providing a solution formula or a concrete Example, see e.g. the remark by Grasegger & Vo [15, p. 173]: *There is no explicit expression printed there*. But the paper [18] contains theoretical results to which we have already referenced to in [41, p. 182] and in the present paper, see Section 5.

(v) Peherstorfer & Schiefermayr [35, Section 7] compute $T$–polynomials of degree $n \leq 7$ on two intervals with the aid of Gröbner basis. Key is a system of equations [35, (7.3)] (correct there the second upper index of summation to $n-2$) which involves the unknown deviation points of a $T$-polynomial and hence, as a special case, the unknown equioscillation points of $Z_{n,s}$. We provide in [55] a worked-out Example for computing $Z_{n,s}$ along [35, Section 7]. Note that explicit Formulae for the equioscillation points of $Z_{n,s}$ of form (9) are given for $n \in \{4,5,6,7\}$ in [39, 15, 42, 44] respectively.

(vi) Vlček & Unbehauen [56] consider Zolotarev polynomials in their original elliptic form [59, 60], transformed to the intervals $[-1,a] \cup [b,1]$. Deploying the apparatus of Complex Analysis (e.g., elliptic and theta functions and conformal mappings) they derive from the Abel-Pell differential equations linear differential equations which are key for deducing two recurrent Formulae for the coefficients of a proper Zolotarev polynomial of form $\sum_{m=0}^{n} b(m)\omega^m$ or $\sum_{m=0}^{n} a(m)T_m(\omega)$, see [56, Table IV and V]. They call these forms algebraic forms of a Zolotarev polynomial, but e.g. the starting values (which in our notation would correspond to $\alpha_0$ and $\beta_0$) involve the modulus of elliptic functions. In [55] we provide a worked-out Example showing how to derive the coefficients $b(m)$ in the first form.

(vii) Suppose the goal is to determine algebraically $Z_{n,s}$ under the assumption that the right interval-endpoint $\beta = \beta_0 > \nu_n$ of $[\alpha,\beta]$ is prescribed. According to a result of Erdős & Szegő [13, Lemma 1], fixing $\beta$ implies that $\alpha > 1$ and $\gamma > 1$ are uniquely determined so that according to Theorem 1.1 then also $s$ would be uniquely determined and hence, in that setting, cannot be freely chosen. If one finds a way to determine the unique $\alpha = \alpha_0$ which corresponds to the fixed $\beta = \beta_0$, then $s = s_0$ can be determined by using $s = s_n(\alpha,\beta)$, see Section 2.4, and hence also $\gamma = \gamma_0 = \frac{\alpha_0+\beta_0}{2} - s_0$ will be known. Sodin & Yuditskii [51, (4), (4')] propose even two methods how to determine $\alpha_0$. Formula [51, (4)] is (24), with $\beta = \beta_0$ fixed, and can be simplified to $H^n_{m(n)}(\alpha,\beta_0) = 0$, see (44).

In a follow-up study we intend to investigate the mirrored Zolotarev's problem, mentioned in [30, p. 298], when the left interval-endpoint $\alpha = \alpha_0 > 1$ is prescribed in advance, and also the related problem, when $\gamma = \gamma_0 > 1$ is prescribed in advance, in lieu of the parameter $s$.

*Remark* 7. Concerning the origin of the Malyshev polynomials, Malyshev states in [21, p. 711]: *For a practical solution, in [6] [i.e., here [20]] the present author separated arguments, found useful branches, and reduced the initial algebraic system to $F_{m(n),s}(\alpha) = 0$, $G_{m(n),s}(\beta) = 0$ [using here our notation].* However, his statement is imprecise in so far as the polynomials $F_{m(n),s}, G_{m(n),s}$ do not occur in [20]; rather, the statement evidently applies to some different pair of polynomials $U_{m(n),\sigma}(x)$ and $V_{m(n),\sigma}(y)$, as given, for $2 \leq n \leq 5$, in [20, p. 936] using Malyshev's $x,y$-notation, see Section 4 above. So actually, the Malyshev polynomials in [21] emerge unanticipated.

They can be derived from the polynomials $U_{m(n),\sigma}$ and $V_{m(n),\sigma}$ (see [20]) and vice versa, but we do not dwell on this here.

As to the mentioned *useful branches* (of $x$ and $y$) Malyshev [20, p. 936] explicates: *To choose $x$ and $y$ properly, we must use the branch distinguished at infinity by the asymptotics $x \sim z$ and $y \sim z^2$ [with $z = \sigma = ns$].* In view of (51), the two said asymptotics thus translate as

$$\frac{\alpha+\beta}{2} \sim ns \, (\text{which in view of Theorem 1.1 implies } \gamma \sim (n-1)s) \text{ and } \alpha\beta + 1 \sim n^2 s^2. \tag{76}$$

The first (asymptotic) approximation is basically known since $\lim_{ns\to\infty} |\alpha - ns| = \lim_{ns\to\infty} |\beta - ns| = 0$, for $n \geq 3$, see [35, (5.12)]; recall the related conditions $|\alpha - ns| < 1$ and $|\beta - ns| < 1$ for $n \geq 4$ and $s \geq 1$ which we utilized to verify the Variant 1 above.

The second approximation in (76) seems to be new. The goodness of these approximations becomes obvious when one replaces the variables $\alpha$ and $\beta$ by appropriate concrete values taken from the Examples given in this paper. For the nuisance values $\alpha = \alpha_0 = 3$ and $\beta = \beta_0 = 5$ in (66) however, where $n_0 = 6$ and $s_0 = 2$ holds, (76) would result in the poor approximations $4 \sim 12$ and $16 \sim 144$, which immediately indicate an exceptional situation. We refer to Remark 2.5 for a non-asymptotic procedure for validating, in a given setting, whether a compatible point $(\alpha_0, \beta_0)$ has been found for the construction of $Z_{n_0,\alpha_0,\beta_0} = Z_{n_0,s_0}$. Another useful procedure, now for validating the computed least deviation of $Z_{n_0,s_0}$ from zero, on $\boldsymbol{I}$, $L = L_{n_0}(s_0)$, is to verify whether Bernstein's [3, p. 156] lower and upper bounds for $L$ are met, see also [32]. These bounds read, with $n = n_0$, $s = s_0$:

$$\begin{aligned} &2^{1-n}(ns + \sqrt{1+n^2s^2})(1 - (1 + \sqrt{1+n^2s^2})(ns + \sqrt{1+n^2s^2})^{-n}) < L, \\ &L < 2^{1-n}(ns + \sqrt{1+n^2s^2})(1 + (1 + \sqrt{1+n^2s^2})(ns + \sqrt{1+n^2s^2})^{-n}), \end{aligned} \tag{77}$$

see also [42, Section 4.2]

*Remark* 8. The reason why we have restricted our consideration in [55] to polynomials of low degrees $n \leq 13$ is for the sake of convenience but not mathematically inherent. In principle, with a powerful computer hardware, the tentative forms $Z_{n,\alpha,\beta}$ as well as the compatible numbers $\alpha = \alpha_0$ and $\beta = \beta_0$ can be computed beyond $n = 13$. For a related discussion see [18, Section 6].

The situation turns out to be different if one tries to raise the degree from $n = n_0 = 7$ to $n = n_0 = 8$ in the parametric representation (9) of Zolotarev polynomials: Here the obstacle is not the computer's performance, rather, one is faced with the problem of finding a radical parametrization of an algebraic curve of genus 5 which is non-hyperelliptic, see [44, Remark 4.8].

# 8 Appendices (1-3)

## 8.1 Appendix 1: Some bulky terms

**Cf. Example 1.1, $n = n_0 = 5$ and $s = s_0 = \frac{1}{5}$, exemplarily only one coefficient and the least deviation**

$$a^*_{3,5}\left(\frac{1}{5}\right) = \frac{1}{15}\left(-2\sqrt{-\left(\frac{A}{2}\right)^{1/3} + 11\left(\frac{2}{A}\right)^{1/3} + 10\sqrt{\frac{1}{B}} - 6} + 2\sqrt{B} - 18\right)$$

with

$$A = (151 + 75\sqrt{5}) \text{ and } B = \left(-3 - 11\left(\frac{2}{A}\right)^{1/3} + \left(\frac{A}{2}\right)^{1/3}\right);$$

$$L_5\left(\frac{1}{5}\right) = \frac{4}{84375}\left(\sqrt{62156312508\left(\frac{2}{C}\right)^{1/3} - (864C)^{1/3} + 1396207000\left(\frac{2}{D}\right) + 417704} - D + 2334\right)$$

with

$$C = 852402263891947 + 506449999328125\sqrt{5} \text{ and } D = \sqrt{208852 - 62156312508\left(\frac{2}{C}\right)^{1/3} + (864C)^{1/3}}.$$

**Cf. Example 2.2, $n = n_0 = 6$ and $s = s_0 = 1$**

$$\begin{aligned} a^*_{0,6}(1) = \mathrm{Root}[&3241459654490570809 + 4978376753382289128 8z - 3891062708229915069 2z^2 + 514525869372346186 4z^3 \\ &+ 1504602912709166710z^4 - 266755810397021656z^5 + 13486749433271644z^6 - 160761765625000z^7 \\ &+ 3814697265625z^8, 2], \end{aligned}$$

$$\begin{aligned} a^*_{1,6}(1) = \mathrm{Root}[&-10219890423697259157573 12 + 24466910759834783285145 6z + 25610150700205345810713 6z^2 \\ &- 8363826175126581538790 4z^3 - 668004338538650453097 6z^4 - 40687803012886272362 88z^5 \\ &+ 661923785943217011888z^6 - 2576413916015625000 0z^7 + 298023223876953125z^8, 1], \end{aligned}$$

$$\begin{aligned} a^*_{2,6}(1) = \mathrm{Root}[&-5317434243658104380775 + 4551246405611892070920z + 2175766930032706940796z^2 \\ &+ 352617906344731462584z^3 + 28678246167819701142z^4 + 1296099455313654072z^5 \\ &+ 33482656598971644z^6 + 494941453125000z^7 + 3814697265625z^8, 2], \end{aligned}$$

$$\begin{aligned} a^*_{3,6}(1) = \mathrm{Root}[&-43243251333529600 + 45342031458467840z - 18508977003249664z^2 + 1396579762397184z^3 \\ &+ 291712710213120z^4 - 45181918308864z^5 + 1425866885312z^6 + 28962500000z^7 + 1220703125z^8, 2], \end{aligned}$$

$$\begin{aligned} a^*_{4,6}(1) = \mathrm{Root}[&-24974796693375 + 518755962600z + 4763147078844z^2 - 1611134445672z^3 + 249132993894z^4 \\ &- 21634573608z^5 + 1098618876z^6 - 31095000z^7 + 390625z^8, 1], \end{aligned}$$

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and trivially $a^*_{5,6}(1) = -6$ and $a^*_{6,6}(1) = 1$;

$|Z_{6,1}(1)| = L_{n_0}(\alpha_0, \beta_0) = L_{n_0}(s_0) = L_6(1) =$
$\mathrm{Root}[879609302220800000 - 628419273785606144000z + 5462615245658028769282z^2 + 286041375809223838924 8z^3$
$+ 92597016584027268710 4z^4 - 20398803444359170621 44z^5 + 47147281019786026188 8z^6 + 43881015625000000 00z^7$
$+ 298023223876953125z^8, 2].$

## Cf. Example 2.4, $n = n_0 = 4$ and $s = s_0 = \mathrm{Root}[11 - 88z + 263z^2 - 318z^3 + 96z^4, 1]$

$s_4(\sigma_k) = (\alpha+\beta)(\alpha^{12} - 12\alpha^{11}\beta + 22\alpha^{11} + 66\alpha^{10}\beta^2 + 74\alpha^{10}\beta - 96\alpha^{10} - 220\alpha^9\beta^3 - 366\alpha^9\beta^2 + 416\alpha^9\beta + 280\alpha^9 + 495\alpha^8\beta^4 - 50\alpha^8\beta^3$
$- 992\alpha^8\beta^2 - 680\alpha^8\beta + 1392\alpha^8 - 792\alpha^7\beta^5 + 1276\alpha^7\beta^4 + 2432\alpha^7\beta^3 - 800\alpha^7\beta^2 - 384\alpha^7\beta - 1600\alpha^7 + 924\alpha^6\beta^6 - 956\alpha^6\beta^5 - 5056\alpha^6\beta^4$
$+ 3680\alpha^6\beta^3 + 7232\alpha^6\beta^2 - 2752\alpha^6\beta - 3072\alpha^6 - 792\alpha^5\beta^7 - 956\alpha^5\beta^6 + 6592\alpha^5\beta^5 - 2480\alpha^5\beta^4 - 14976\alpha^5\beta^3 + 5568\alpha^5\beta^2 + 7168\alpha^5\beta$
$- 256\alpha^5 + 495\alpha^4\beta^8 + 1276\alpha^4\beta^7 - 5056\alpha^4\beta^6 - 2480\alpha^4\beta^5 + 13472\alpha^4\beta^4 - 1216\alpha^4\beta^3 - 17408\alpha^4\beta^2 + 4864\alpha^4\beta + 5888\alpha^4 - 220\alpha^3\beta^9$
$- 50\alpha^3\beta^8 + 2432\alpha^3\beta^7 + 3680\alpha^3\beta^6 - 14976\alpha^3\beta^5 - 1216\alpha^3\beta^4 + 26624\alpha^3\beta^3 - 4608\alpha^3\beta^2 - 15360\alpha^3\beta + 3584\alpha^3 + 66\alpha^2\beta^{10} - 366\alpha^2\beta^9$
$- 992\alpha^2\beta^8 - 800\alpha^2\beta^7 + 7232\alpha^2\beta^6 + 5568\alpha^2\beta^5 - 17408\alpha^2\beta^4 - 4608\alpha^2\beta^3 + 18944\alpha^2\beta^2 + 512\alpha^2\beta - 8192\alpha^2 - 12\alpha\beta^{11} + 74\alpha\beta^{10}$
$+ 416\alpha\beta^9 - 680\alpha\beta^8 - 384\alpha\beta^7 - 2752\alpha\beta^6 + 7168\alpha\beta^5 + 4864\alpha\beta^4 - 15360\alpha\beta^3 + 512\alpha\beta^2 + 8192\alpha\beta - 2048\alpha + \beta^{12} + 22\beta^{11}$
$- 96\beta^{10} + 280\beta^9 + 1392\beta^8 - 1600\beta^7 - 3072\beta^6 - 256\beta^5 + 5888\beta^4 + 3584\beta^3 - 8192\beta^2 - 2048\beta + 4096)/$
$(2(\alpha^{12} - 12\alpha^{11}\beta + 24\alpha^{11} + 66\alpha^{10}\beta^2 + 104\alpha^{10}\beta - 104\alpha^{10} - 220\alpha^9\beta^3 - 376\alpha^9\beta^2 + 464\alpha^9\beta + 352\alpha^9 + 495\alpha^8\beta^4$
$- 520\alpha^8\beta^3 - 1096\alpha^8\beta^2 - 416\alpha^8\beta + 2032\alpha^8 - 792\alpha^7\beta^5 + 2416\alpha^7\beta^4 + 2496\alpha^7\beta^3 - 3200\alpha^7\beta^2 + 1152\alpha^7\beta - 1280\alpha^7 + 924\alpha^6\beta^6 - 1648\alpha^6\beta^5$
$- 4944\alpha^6\beta^4 + 8576\alpha^6\beta^3 + 9792\alpha^6\beta^2 - 7936\alpha^6\beta - 3840\alpha^6 - 792\alpha^5\beta^7 - 1648\alpha^5\beta^6 + 6368\alpha^5\beta^5 - 5312\alpha^5\beta^4 - 16512\alpha^5\beta^3 + 15104\alpha^5\beta^2$
$+ 8704\alpha^5\beta - 5120\alpha^5 + 495\alpha^4\beta^8 + 2416\alpha^4\beta^7 - 4944\alpha^4\beta^6 - 5312\alpha^4\beta^5 + 7072\alpha^4\beta^4 - 5888\alpha^4\beta^3 - 16640\alpha^4\beta^2 + 15360\alpha^4\beta + 7936\alpha^4$
$- 220\alpha^3\beta^9 - 520\alpha^3\beta^8 + 2496\alpha^3\beta^7 + 8576\alpha^3\beta^6 - 16512\alpha^3\beta^5 - 5888\alpha^3\beta^4 + 23552\alpha^3\beta^3 - 10240\alpha^3\beta^2 - 15360\alpha^3\beta + 14336\alpha^3$
$+ 66\alpha^2\beta^{10} - 376\alpha^2\beta^9 - 1096\alpha^2\beta^8 - 3200\alpha^2\beta^7 + 9792\alpha^2\beta^6 + 15104\alpha^2\beta^5 - 16640\alpha^2\beta^4 - 10240\alpha^2\beta^3 + 14848\alpha^2\beta^2 + 2048\alpha^2\beta$
$- 10240\alpha^2 - 12\alpha\beta^{11} + 104\alpha\beta^{10} + 464\alpha\beta^9 - 416\alpha\beta^8 + 1152\alpha\beta^7 - 7936\alpha\beta^6 + 8704\alpha\beta^5 + 15360\alpha\beta^4 - 15360\alpha\beta^3 + 2048\alpha\beta^2 + 4096\alpha\beta$
$- 8192\alpha + \beta^{12} + 24\beta^{11} - 104\beta^{10} + 352\beta^9 + 2032\beta^8 - 1280\beta^7 - 3840\beta^6 - 5120\beta^5 + 7936\beta^4 + 14336\beta^3 - 10240\beta^2 - 8192\beta + 4096)).$

### Cf. Example 2.7, $n = n_0 = 7$ and $s = s_0 = 2$

$$
\begin{aligned}
a^*_{0,7}(2) = \mathrm{Root}[&-1484280356032053583936860770516272885904527151240542918083981072863 \\
&+2942430910419341594590489563882834989240741139691643801796128582400z \\
&+990702952999654482961856713547320436591436783963493681796252001536z^2 \\
&+111156640145202985093920555109221314608071021564066788184951296640z^3 \\
&+5029934767716946074972760335931555719981297539950855134542833280z^4 \\
&+76676842477502699186749970119505894790110584131172458925822976z^5 \\
&+380168432196348318989427980556432282104428523774730521028352z^6 \\
&+3023645209840317332650316144014958977547678106005324177408z^7 \\
&+86422599316830298042747986095564865000641559962328735744z^8 \\
&+64690128173272265780796604050769444381665344000000000z^9 \\
&-7369864047740819304421640851367574369000000000000000z^{10} \\
&-2069910477015363183634253950342500000000000000000z^{11} \\
&+2147237938037359201129101562500000000000000000z^{12}, 4],
\end{aligned}
$$

$$
\begin{aligned}
a^*_{1,7}(2) = \mathrm{Root}[&22810679487187328966585804251376533512291 + 120119127469976475065324606507180488643328z \\
&-8738808881071380874974187481822371065216z^2 + 111780778822587207540255654349526782368z^3 \\
&+3420844544096061629663144080146768640z^4 - 49721232314192671401318715474803200z^5 \\
&-484981323444127877641786510503680z^6 - 151853013085226028361694818304z^7 \\
&+19470582126262293894633644032z^8 + 129175376354192978007040000z^9 \\
&+369554511385036800000000z^{10} + 507037536000000000000z^{11} + 270000000000000000z^{12}, 4],
\end{aligned}
$$

$$
\begin{aligned}
a^*_{2,7}(2) = \mathrm{Root}[&89094118010250569175485559904244080625130342266203 \\
&+32550922732001563643513794380880046769674850307092z \\
&-4034041478184383342951763234302956769368970423844z^2 \\
&+145247495082282903686507084690867494900998210672z^3 \\
&-2867439970259572842278770672695413757386355408z^4 \\
&+36731046013067960413189377739010358753869120z^5 \\
&-329748813518410821751377540750846966360048z^6 + 2128385464589646184398020848444854113408z^7 \\
&-9667658483643080962712414494809380288z^8 + 288943895659262224459145561500000000z^9 \\
&-498860515347880806562500000000z^{10} + 36360852910839843750000000000z^{11} \\
&+150169372558593750000000z^{12}, 2],
\end{aligned}
$$

$$
\begin{aligned}
a^*_{3,7}(2) = \mathrm{Root}[&-13551111365952692501857237668605796891807 + 9756945736662425735755686477601833950016z \\
&+1366397401707440099034290610919580953440z^2 + 26276706798092864773391352918893982624z^3 \\
&+80209590112387254303276057280098432z^4 - 855547113007416953291132499158016z^5 \\
&+461891579059016183740491969280z^6 + 12191109389858819507612366848z^7 \\
&-254502375997283101147869184z^8 + 187021383292109460480000z^9 + 3803648636838400000000z^{10} \\
&-12219168000000000000z^{11} + 10000000000000000z^{12}, 4],
\end{aligned}
$$

$$
\begin{aligned}
a^*_{4,7}(2) = \mathrm{Root}[&29735191037722607216799896605065469083 + 1121222282638149892012162522069028628z \\
&-26067643684587580913490910031668026z^2 - 2476775522979664216410167133954832z^3 \\
&-69188213867718108279513169218635z^4 - 1074980170552174625660885781832z^5 \\
&-10514733506034275574861770672z^6 - 66987675604282657464459408z^7 \\
&-274045904308245457854823z^8 - 666995041801777461500z^9 - 740115106646343750z^{10} \\
&+217674843750000z^{11} + 961083984375z^{12}, 5],
\end{aligned}
$$

$$
\begin{aligned}
a^*_{5,7}(2) = \mathrm{Root}[&38318777589714630392958077259 + 14331665480146967400892063200z - 2155949123251738666384843128z^2 \\
&+122446707330048177939186744z^3 - 3632836782987165835961712z^4 + 54804480026424548730464z^5 \\
&-92132037486244489776z^6 - 13147546858271652736z^7 + 281658291489807232z^8 - 3058367690761600z^9 \\
&+19490724288000z^{10} - 69552000000z^{11} + 108000000z^{12}, 2],
\end{aligned}
$$

and trivially $a^*_{6,7}(2) = -14$ and $a^*_{7,7}(2) = 1$;

$|Z_{7,2}(1)| = L_{n_0}(\alpha_0, \beta_0) = L_{n_0}(s_0) = L_7(2) =$

$\mathrm{Root}[395191926864256830918804125613406379982884491734296089086914062 5$
$- 612364685256339296552370489166753384695596705844175435207500000 0z$
$- 781356759203395463723027091234846734465116188043986259510488000 0z^2$
$+ 2903891779040594499179875304492805699796474258762485069343626816z^3$
$- 3955720059106648564472683840763314806720571313072794657806528 00z^4$
$+ 271466679623771184961691019435717051070062099487321783519662 08z^5$
$- 101698405775535396572582820259110275457317637992227779057382 4z^6$
$+ 2137278569233401448599836725077967251650320642572613013094 4z^7$
$- 2688152158047224235888950272905562689911142618136328642 56z^8$
$+ 2170046417801724173868455615179337122099546144000000000 0z^9$
$- 884695949794151422215743405056748909400000000000000000 0z^{10}$
$+ 154152423932980895150211279184500000000000000000000000 z^{11}$
$+ 214723793803735920112910156250000000000000000000000 z^{12}, 3].$

**Cf. Remark 4 (Constant Terms $B_{0,4}(\alpha,\beta), A_{0,4}(\alpha,\beta), C_{0,4}(\alpha,\beta)$ of, respectively, $P_{4,\alpha,\beta}, R_{4,\alpha,\beta}, S_{4,\alpha,\beta}$)**

Coefficient $B_{0,4}(\alpha,\beta) =$

$($

$\alpha^{16} + (16\beta - 32)\alpha^{15} + \left(-200\beta^2 + 160\beta - 160\right)\alpha^{14} + \left(816\beta^3 + 352\beta^2 - 1216\beta - 768\right)\alpha^{13} + \left(-1380\beta^4 - 5088\beta^3 - 736\beta^2 - 768\beta + 6592\right)\alpha^{12}$
$+ \left(-496\beta^5 + 19040\beta^4 + 8320\beta^3 - 24064\beta^2 - 2816\beta + 512\right)\alpha^{11} + \left(7304\beta^6 - 38368\beta^5 + 11360\beta^4 + 123392\beta^3 - 34944\beta^2 - 45568\beta - 15872\right)\alpha^{10}$
$+ \left(-16720\beta^7 + 44000\beta^6 - 38720\beta^5 - 175360\beta^4 + 33024\beta^3 + 173568\beta^2 - 60416\beta + 57344\right)\alpha^9$
$+ \left(21318\beta^8 - 20064\beta^7 - 10464\beta^6 + 54016\beta^5 - 50624\beta^4 - 202240\beta^3 - 42496\beta^2 + 221184\beta + 50688\right)\alpha^8$
$+ \left(-16720\beta^9 - 20064\beta^8 + 63232\beta^7 + 23552\beta^6 + 297472\beta^5 + 13312\beta^4 - 364544\beta^3 + 131072\beta^2 + 143360\beta - 253952\right)\alpha^7$
$+ \left(7304\beta^{10} + 44000\beta^9 - 10464\beta^8 + 23552\beta^7 - 497408\beta^6 + 60416\beta^5 + 1106944\beta^4 - 196608\beta^3 - 333824\beta^2 - 106496\beta - 90112\right)\alpha^6$
$+ \left(-496\beta^{11} - 38368\beta^{10} - 38720\beta^9 + 54016\beta^8 + 297472\beta^7 + 60416\beta^6 - 1247232\beta^5 - 212992\beta^4 + 1331200\beta^3 - 253952\beta^2 - 278528\beta + 327680\right)\alpha^5$
$+ \left(-1380\beta^{12} + 19040\beta^{11} + 11360\beta^{10} - 175360\beta^9 - 50624\beta^8 + 13312\beta^7 + 1106944\beta^6 - 212992\beta^5 - 2382848\beta^4 + 614400\beta^3\right.$
$\left. +1138688\beta^2 - 196608\beta + 114688\right)\alpha^4 + \left(816\beta^{13} - 5088\beta^{12} + 8320\beta^{11} + 123392\beta^{10} + 33024\beta^9 - 202240\beta^8 - 364544\beta^7 - 196608\beta^6\right.$
$\left. +1331200\beta^5 + 614400\beta^4 - 1540096\beta^3 - 131072\beta^2 + 458752\beta - 131072\right)\alpha^3$
$+ \left(-200\beta^{14} + 352\beta^{13} - 736\beta^{12} - 24064\beta^{11} - 34944\beta^{10} + 173568\beta^9 - 42496\beta^8 + 131072\beta^7 - 333824\beta^6 - 253952\beta^5 + 1138688\beta^4\right.$
$\left. - 131072\beta^3 - 622592\beta^2 + 131072\beta - 131072\right)\alpha^2 + \left(16\beta^{15} + 160\beta^{14} - 1216\beta^{13} - 768\beta^{12} - 2816\beta^{11} - 45568\beta^{10} - 60416\beta^9 + 221184\beta^8\right.$
$\left. + 143360\beta^7 - 106496\beta^6 - 278528\beta^5 - 196608\beta^4 + 458752\beta^3 + 131072\beta^2 - 262144\beta\right)\alpha$
$+ \beta^{16} - 32\beta^{15} - 160\beta^{14} - 768\beta^{13} + 6592\beta^{12} + 512\beta^{11} - 15872\beta^{10} + 57344\beta^9 + 50688\beta^8 - 253952\beta^7 - 90112\beta^6 + 327680\beta^5$
$+ 114688\beta^4 - 131072\beta^3 - 131072\beta^2 + 65536$

$)/$

$(128($

$\alpha^{12} + \alpha^{11}(24 - 12\beta) + \alpha^{10}\left(66\beta^2 + 104\beta - 104\right) + \alpha^9\left(-220\beta^3 - 376\beta^2 + 464\beta + 352\right) + \alpha^8\left(495\beta^4 - 520\beta^3 - 1096\beta^2 - 416\beta + 2032\right)$
$+ \alpha^7\left(-792\beta^5 + 2416\beta^4 + 2496\beta^3 - 3200\beta^2 + 1152\beta - 1280\right) + \alpha^6\left(924\beta^6 - 1648\beta^5 - 4944\beta^4 + 8576\beta^3 + 9792\beta^2 - 7936\beta - 3840\right)$
$+ \alpha^5\left(-792\beta^7 - 1648\beta^6 + 6368\beta^5 - 5312\beta^4 - 16512\beta^3 + 15104\beta^2 + 8704\beta - 5120\right)$
$+ \alpha^4\left(495\beta^8 + 2416\beta^7 - 4944\beta^6 - 5312\beta^5 + 7072\beta^4 - 5888\beta^3 - 16640\beta^2 + 15360\beta + 7936\right)$
$+ \alpha^3\left(-220\beta^9 - 520\beta^8 + 2496\beta^7 + 8576\beta^6 - 16512\beta^5 - 5888\beta^4 + 23552\beta^3 - 10240\beta^2 - 15360\beta + 14336\right)$
$+ \alpha^2\left(66\beta^{10} - 376\beta^9 - 1096\beta^8 - 3200\beta^7 + 9792\beta^6 + 15104\beta^5 - 16640\beta^4 - 10240\beta^3 + 14848\beta^2 + 2048\beta - 10240\right)$
$+ \alpha\left(-12\beta^{11} + 104\beta^{10} + 464\beta^9 - 416\beta^8 + 1152\beta^7 - 7936\beta^6 + 8704\beta^5 + 15360\beta^4 - 15360\beta^3 + 2048\beta^2 + 4096\beta - 8192\right)$
$+ \beta^{12} + 24\beta^{11} - 104\beta^{10} + 352\beta^9 + 2032\beta^8 - 1280\beta^7 - 3840\beta^6 - 5120\beta^5 + 7936\beta^4 + 14336\beta^3 - 10240\beta^2 - 8192\beta + 4096$

$)$;

Coefficient $A_{0,4}(\alpha,\beta) = -8s^2 + \frac{1}{4}(\alpha - \beta)^2 + s(\alpha + \beta)$;

Coefficient $C_{0,4}(\alpha,\beta) = \dfrac{1}{8(4 + \alpha^2 + 2\alpha\beta - 3\beta^2)^2}$

$(\alpha^7(\beta + 1) + \alpha^6\beta(7 - 3\beta) + \alpha^5\left(\beta^3 + 13\beta^2 - 8\beta - 8\right) + \alpha^4\beta\left(5\beta^3 - 13\beta^2 + 8\beta - 8\right) +$
$\alpha^3\left(-5\beta^5 + 3\beta^4 + 16\beta^3 - 48\beta^2 + 16\beta + 16\right) - \alpha^2\beta\left(\beta^5 + 59\beta^4 + 16\beta^3 - 144\beta^2 - 16\beta + 80\right) +$
$\alpha\beta^2\left(3\beta^5 + 47\beta^4 - 8\beta^3 - 72\beta^2 - 16\beta + 48\right) - (\beta - 1)\beta^3\left(\beta^2 - 4\right)^2).$

**Cf. Proposition 2.4 (Visualization of Formula (31))**

$s_5(\alpha,\beta) = \dfrac{16+9\alpha^4-8\alpha^3\beta-8\beta^2+5\beta^4-6\alpha^2(4+\beta^2)}{20(\alpha^3-\alpha^2\beta+\beta(-4+\beta^2)-\alpha(4+\beta^2))}; s_0 = \dfrac{41}{25\sqrt{145}};$

$H^5_{m(5)}(\alpha,\beta) = H^5_6(\alpha,\beta) = 64-80\alpha^2+44\alpha^4+\alpha^6+96\alpha\beta-16\alpha^3\beta+6\alpha^5\beta-16\beta^2+104\alpha^2\beta^2-29\alpha^4\beta^2-80\alpha\beta^3$
$+36\alpha^3\beta^3-52\beta^4-9\alpha^2\beta^4-10\alpha\beta^5+5\beta^6.$

Result of Formula (31) is $(\alpha_0,\beta_0) = \left(\dfrac{67}{5\sqrt{145}}, \dfrac{77}{5\sqrt{145}}\right) \approx (1.1128094300, 1.2789003897)$.

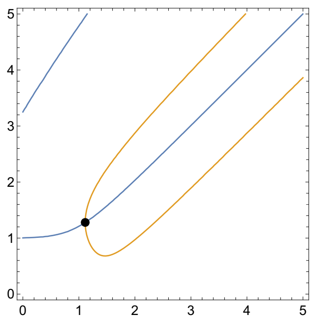


**Figure 6:** $(\alpha_0,\beta_0)$ as an intersection point of two algebraic curves

### 8.2 Appendix 2: Verification of Identity (50) for $n = n_0 = 3$

One readily deduces the following two equations:

$$\det M_1^{3\times3} = \det M_1^{3\times3}(\alpha,\beta) = \\ (1/1024)(\alpha^6+\alpha^5(12-6\beta)+\alpha^4\left(15\beta^2+28\beta-28\right)+\alpha^3\left(-20\beta^3-40\beta^2+48\beta+32\right)+\alpha^2\left(15\beta^4-40\beta^3-40\beta^2-32\beta+112\right)+ \\ \alpha\left(-6\beta^5+28\beta^4+48\beta^3-32\beta^2+32\beta-64\right)+\beta^6+12\beta^5-28\beta^4+32\beta^3+112\beta^2-64\beta-64) \tag{78}$$

and

$$L_3(\alpha,\beta) = -P_{3,\alpha,\beta}(1) = -\det M_0^{3+1\times3+1}(\alpha,\beta,1)/\det M_1^{3\times3} = \\ (\alpha^9+\alpha^8(9\beta-36)+\alpha^7\left(-60\beta^2+96\beta-96\right)+\alpha^6\left(116\beta^3+144\beta^2-864\beta+768\right)+\alpha^5\left(-66\beta^4-864\beta^3+96\beta^2+2304\beta-864\right)+ \\ \alpha^4\left(-66\beta^5+1320\beta^4+864\beta^3-3840\beta^2+288\beta+1920\right)+\alpha^3\left(116\beta^6-864\beta^5+864\beta^4+1536\beta^3+576\beta^2-4608\beta+2048\right)+ \\ \alpha^2\left(-60\beta^7+144\beta^6+96\beta^5-3840\beta^4+576\beta^3+5376\beta^2+3072\beta-6144\right)+\alpha\left(9\beta^8+96\beta^7-864\beta^6+2304\beta^5+288\beta^4-4608\beta^3+3072\beta^2-768\right)+ \\ \beta^9-36\beta^8-96\beta^7+768\beta^6-864\beta^5+1920\beta^4+2048\beta^3-6144\beta^2-768\beta+3072)/ \\ (32(\alpha^6+\alpha^5(12-6\beta)+\alpha^4\left(15\beta^2+28\beta-28\right)+\alpha^3\left(-20\beta^3-40\beta^2+48\beta+32\right)+\alpha^2\left(15\beta^4-40\beta^3-40\beta^2-32\beta+112\right)+ \\ \alpha\left(-6\beta^5+28\beta^4+48\beta^3-32\beta^2+32\beta-64\right)+\beta^6+12\beta^5-28\beta^4+32\beta^3+112\beta^2-64\beta-64)). \tag{79}$$

Setting

$$x = x(\alpha,\beta) = -(\alpha+\beta)/2, \text{ and } y = y(\alpha,\beta) = \alpha\beta+1 \tag{80}$$

one gets

$$\lambda_1(x,y) = \Delta_{3-1}(x,y) = \Delta_2(x(\alpha,\beta),y(\alpha,\beta)) = -\frac{1}{16}(-2+\alpha-\beta)(2+\alpha-\beta)(\alpha+\beta) \tag{81}$$

and

$$\Delta_2(x(\alpha,\beta),y(\alpha,\beta))\cdot L_3(\alpha,\beta)/\det M_1^{3\times3}(\alpha,\beta) = \\ -2(\alpha-\beta-2)(\alpha-\beta+2)(\alpha+\beta)(\alpha^9+9\alpha^8\beta-36\alpha^8-60\alpha^7\beta^2+96\alpha^7\beta-96\alpha^7+116\alpha^6\beta^3+144\alpha^6\beta^2-864\alpha^6\beta+768\alpha^6-66\alpha^5\beta^4-864\alpha^5\beta^3+96\alpha^5\beta^2+ \\ 2304\alpha^5\beta-864\alpha^5-66\alpha^4\beta^5+1320\alpha^4\beta^4+864\alpha^4\beta^3-3840\alpha^4\beta^2+288\alpha^4\beta+1920\alpha^4+116\alpha^3\beta^6-864\alpha^3\beta^5+864\alpha^3\beta^4+1536\alpha^3\beta^3+ \\ 576\alpha^3\beta^2-4608\alpha^3\beta+2048\alpha^3-60\alpha^2\beta^7+144\alpha^2\beta^6+96\alpha^2\beta^5-3840\alpha^2\beta^4+576\alpha^2\beta^3+5376\alpha^2\beta^2+3072\alpha^2\beta-6144\alpha^2+9\alpha\beta^8+96\alpha\beta^7- \\ 864\alpha\beta^6+2304\alpha\beta^5+288\alpha\beta^4-4608\alpha\beta^3+3072\alpha\beta^2-768\alpha+\beta^9-36\beta^8-96\beta^7+768\beta^6-864\beta^5+1920\beta^4+2048\beta^3-6144\beta^2-768\beta+3072)/ \\ \left(\alpha^6-6\alpha^5\beta+12\alpha^5+15\alpha^4\beta^2+28\alpha^4\beta-28\alpha^4-20\alpha^3\beta^3-40\alpha^3\beta^2+48\alpha^3\beta+32\alpha^3+15\alpha^2\beta^4-\right. \\ \left.40\alpha^2\beta^3-40\alpha^2\beta^2-32\alpha^2\beta+112\alpha^2-6\alpha\beta^5+28\alpha\beta^4+48\alpha\beta^3-32\alpha\beta^2+32\alpha\beta-64\alpha+\beta^6+12\beta^5-28\beta^4+32\beta^3+112\beta^2-64\beta-64\right)^2. \tag{82}$$

The goal is to show that (82) is identically 1 when inserting any pair $(\alpha,\beta)$ which constitutes a solution of (34) for $n = n_0 = 3$, i.e., when inserting $\alpha = \alpha_0 = 3s$ and $\beta = \beta_0 = \frac{\alpha+2\sqrt{3+\alpha^2}}{3} = s+2\sqrt{\frac{1}{3}+s^2}$, where $s > \frac{1}{3}$ is arbitrary, see (13). Applying FullSimplify to the so modified right-hand side in (82) indeed verifies the claim (50).

### 8.3 Appendix 3: Algebraic Solutions of ZFP if $8 \le n \le 13$, by Example

For illustration and for testing purposes we provide here certain proper Zolotarev polynomials $Z_{n,s}$ of degree $8 \le n \le 13$ (not exemplified above), and their graphs on $\boldsymbol{I}$. The corresponding second leading coefficients are chosen to be $-ns$ with $s \in \{3,4,5,6,7,8\}$ in ascending order. Use was made of the tentative form $R_{n,\alpha,\beta}$ (emerging from the 1st algorithm). The corresponding compatible values $\alpha = \alpha_0$ and $\beta = \beta_0$ as well as the explicit coefficients of $Z_{n_0,s_0} = R_{n_0,\alpha_0,\beta_0}$ are given in [55]. Due to their bulkiness, these coefficients are given here in a (numerical) chopped form. For the setting $n_0 = 8$ and $s_0 = 3$ additionally one exact coefficient, $a^*_{1,8}(3)$, is given here exemplarily in the explicit form as a root object, the remaining coefficients $a^*_{k,8}(3)$ are given in that form in [55]. To the best of our knowledge, Examples of explicit algebraic solutions of ZFP in the algebraic power form, for $n > 7$, are not provided elsewhere in literature.

**Chopped Coefficients and Graphs of $Z_{n,s}$ for $n = n_0 = 8, 9$ and $s = s_0 = 3, 4$:**

$$Z_{8,3}(x) \approx 0.0156182271 + 2.6241871151x + (-0.3905369292)x^2 + (-20.9967478963)x^3 + 1.6248102892x^4 + 41.9973980914x^5 + (-2.2498915871)x^6 + (-24)x^7 + x^8.$$

$$Z_{9,4}(x) \approx -0.2811957779 + 0.0624909624x + 8.9990238977x^2 + (-0.6874427580)x^3 + (-44.9973969212)x^4 + 2.1249035888x^5 + 71.9982645583x^6 + (-2.4999517932)x^7 + (-36)x^8 + x^9.$$


**(a)** Graph of $Z_{8,3}$ **(b)** Graph of $Z_{9,4}$

**Figure 7**

**Exact Coefficient $a^*_{1,8}(3)$ of $Z_{8,3}$:**

$$\begin{aligned}
&\mathrm{Root}[150711142311900910658184877222914389803206639915704727172851562 5z^{16}\\
&-3410159794333448028368589283139529860737900152067725219726562500000000z^{15}\\
&+56503151047559994593863092572001964445963038479399682298562421875000000000z^{14}\\
&-829848593363851861945348171240346513079450005104063974516492084730800000000000z^{13}\\
&+1629508079200374260164993173091811187006683238642197986905159796319548619750400000 0z^{12}\\
&-12991774357935765455523070919568435712235248727359243086264462438615587716383477124 3008z^{11}\\
&+47599236575485622145455665784512289774425034927030309374150579058174034674541396154 4368128z^{10}\\
&-10094347612882917717182798778937454402829516979677579332165391524188842897566648084940 59225088z^9\\
&+13993938810284169640537645577642839178114594037976596364437683246752830307526645597270 90469961728z^8\\
&-12874916652776378819989795184411870719929881252849652168834371420824493491513229496190 56783931736064z^7\\
&+68110556511428571617007352407092845510723744195236103867211500854374772038095144102577 08202488538071 04z^6\\
&-94541922812533316404835721367381311469165117150068438757911453259953046214417914999801 643017546638557184z^5\\
&-76925917920072700697651243364005125291687413481803016986883599426468918748372938824997 3287801864453011 53792z^4\\
&+14284047094867553042838147677353957251661690631089037980772937611285352616584492008279 38699460178216863 1902208z^3\\
&+57189372740494853122180946052129336644229396053296946942995963792356157567889263572044 26090208829895337 903652864z^2\\
&+38671309476919122192330677137438583672591655140030718411814660912397624169927459225125 9557840809327905079690264576z\\
&-10544446388807879250829626766302520478174420792461212223354380480249477008795458571354 43781676364644316162054684672, 2]\\
&\approx 2.6241871151.
\end{aligned}$$

**Chopped Coefficients and Graphs of $Z_{n,s}$ for $n = n_0 = 10, \ldots, 13$ and $s = s_0 = 5, \ldots, 8$:**

$$Z_{10,5}(x) \approx -0.0039058594 + (-1.7576758183)x + 0.1601464868x^2 + 23.4364065077x^3 + (-1.0937093842)x^4 + (-84.3728129686)x^5 + 2.6874437618x^6 + 112.4987502499x^7 + (-2.7499750049)x^8 + (-50)x^9 + x^{10}.$$

$$Z_{11,6}(x) \approx 0.1288988530 + (-0.0195303533)x + (-6.4450757923)x^2 + 0.3320213871x^3 + 51.5613164441x^4 + (-1.6249695143)x^5 + (-144.3731062915)x^6 + 3.3124641342x^7 + 164.9990531389x^8 + (-2.9999856536)x^9 + (-66)x^{10} + x^{11}.$$

$$Z_{12,7}(x) \approx 0.0009765279 + 0.9023175946x + (-0.0595688940)x^2 + (-18.0465262581)x^3 + 0.6054590626x^4 + 101.0612445201x^5 + (-2.2968511967)x^6 + (-230.9983260180)x^7 + 3.9999756431x^8 + 230.9992560051x^9 + (-3.2499911429)x^{10} + (-84)x^{11} + x^{12}.$$

$$Z_{13,8}(x) \approx 0.0507800763 + 0.0058592621x + 3.6561913159x^2 + (-0.1425762064)x^3 + (-42.6557805248)x^4 + 1.0117093604x^5 + 181.9986854643x^6 + (-3.1249808597)x^7 + (-350.9984976690)x^8 + 4.7499826654x^9 + 311.9993990662x^{10} + (-3.4999942217)x^{11} + (-104)x^{12} + x^{13}.$$

**(a)** Graph of $Z_{10,5}$ **(b)** Graph of $Z_{11,6}$

**Figure 8**

**(a)** Graph of $Z_{12,7}$ **(b)** Graph of $Z_{13,8}$

**Figure 9**

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