PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special Issue dedicated to Robert Schaback on the occasion of his 75th birthday, Volume 15 · 2022 · Pages 78–92

Compounded Product Integration rules on $(0,+\infty)$

Domenico Mezzanotte[a,c] · Donatella Occorsio[a,b,c]

Communicated by Francesco Marchetti

Abstract

The paper deals with the approximation of integrals of the type

$$\mathcal{I}(f,y) =: \int_0^{+\infty} f(x)k(x,y)\rho(x)\,dx, \quad \rho(x) := e^{-x}x^{\gamma},$$

where f is a sufficiently smooth function and the kernel k collects criticisms of many different types (highly oscillating, weakly singular, "nearly" singular, etc.). We propose an extended product rule based on the approximation of f by an extended Lagrange process at Laguerre zeros. We prove that the rule is stable and convergent with order of the best polynomial approximation in suitable function spaces. Furthermore, by combining the stated rule with a related product formula, we define a pattern that allows a significant saving in number of function evaluations. We give details on the construction of the coefficients of the rule for some selected kernels. Finally, some numerical tests are proposed to show the efficiency of the compounded quadrature scheme.

1 Introduction

The present paper deals with the approximation of integrals of the type

$$\mathcal{I}(f,y) =: \int_0^{+\infty} f(x)k(x,y)\rho(x)\,dx, \quad y \in \mathrm{S} \tag{1}$$

where f is a sufficiently smooth function, $\rho(x) := e^{-x}x^{\gamma}$ is a Laguerre weight and the kernel function k is defined in $(0,+\infty)\times \mathrm{S}$ being S a proper range for y. Usually the kernel k collects criticism in the integrand of many different types. Examples of problematic kernels are for instance $k(x,y) = \sin(yx)$, $y \in \mathbb{R}$, with $|y|$ "large" or $k(x,y) = |x-y|^{\lambda}$, $y > 0$, with $\lambda > -1$. The first one highly oscillates, while the second one is weakly singular in y for negative λ. It's well known that in both cases, Gauss-Laguerre rules and their possible variants provide unsatisfying performances, reason why integration formulas of product type are mainly used [6, 12, 18]. The efficient computation of integrals (1) is needed in many contexts (see e.g. [22, 23] and the references therein). In particular we recall the employment of such rules in the numerical treatment of integral (and systems of integral) equations (see e.g. [14, 13, 9, 8, 10, 11, 27].) For instance, the Marchenko system in [3] is connected to inverse and direct scattering problems extensively treated in [32, 5].

In [18], a truncated product formula essentially based on the zeros of Laguerre polynomials $\{p_m(w_\alpha,x)(4m-x)\}_m$, being $w_\alpha(x) = e^{-x}x^{\alpha}$, was considered and studied. The authors proved that the rule, to which here we will refer as *Ordinary Product Rule*, is stable and convergent in suitable spaces of locally continuous functions over $(0,+\infty)$, endowed with weighted norm. In the present paper first we construct and study a truncated *Extended Product Rule*, based on the zeros of the extended polynomials $\{p_m(w_\alpha,x)p_{m+1}(w_\alpha,x)(4m-x)\}_m$. We will prove that under suitable assumptions, both rules, Ordinary and Extended, are stable and convergent with the same rate. As second step, following an idea in [26, 28], we propose an algorithm obtained combining both rules. Once the Ordinary rule has been computed, its function samples are "recycled" in order to achieve the Extended one. By doing so, this $2m$ order rule is obtained with only m new samples of the function f. A repeated application of this scheme for increasing values of m defines an operative pattern sparing one third of function evaluations with respect to those needed by a sequence based on the only Ordinary rule. Such save can turn useful in some numerical methods for Fredholm Integral Equations, where the number of samples needed in the quadrature corresponds to the dimension of the final linear system (see e.g. [29, 21, 26]). An additional strong reduction of function evaluations is also realized by means of the "truncation" techniques [15], applied to both the ordinary and the extended formulas [26].

Although the basic idea follows that in [26], the mixed scheme we introduce here is new, being the extended rule introduced there based on two different but connected families of orthogonal Laguerre polynomials. Moreover, also the conditions assuring

[a]Dipartimento di Matematica ed Informatica, Università degli Studi della Basilicata, Viale dell'Ateneo Lucano 10, 85100 Potenza, Italy
[b]C.N.R. National Research Council of Italy, IAC Institute for Applied Computing "Mauro Picone",
Via P. Castellino, 111, 80131 Napoli, Italy
[c]INDAM-GNCS member, "Research ITalian network on Approximation (RITA)" member, TAA-UMI group member

stability and convergence proved there are different from those we establish here, and this can turn useful in all the cases where w_α and u are fixed and the quadrature scheme in [26] doesn't converge.

In the paper we provide also details about the effective computation of the coefficients of both the product rules, crucial to their success. We point out that in case of ordinary rules the coefficients are usually expressed in terms of the so-called *ordinary modified moments* (see e.g. [30, 24, 1, 2]), while those of the extended rule require the computation of the *generalized modified moments* (see (17)). Here, we revisit some recurrence relations to compute ordinary modified moments (see [20]), being such relations depending on the kernel we are dealing with, and provide a unique recurrence relation for computing the generalized modified moments, to be used once the ordinary moments are known.

The paper is organized as follows: in the next section some preliminary results and notations are collected. Section 3 contains the ordinary rule and some computational details to implement it. The extended product rule and the mixed quadrature scheme, their stability and convergence in weighted spaces of continuous functions are stated in Section 4. Some numerical tests which support the theoretical estimate are proposed in Section 5. Finally, Section 6 contains the proofs of the main results.

2 Preliminaries

In the sequel we will use $\mathcal{C}$ in order to denote a positive constant, which may have different values at different occurrences, and we write $\mathcal{C} \neq \mathcal{C}(m, f, \ldots)$ to mean that $\mathcal{C} > 0$ is independent of $m, f, \ldots$.
$\mathbb{P}_m$ will denote the space of all algebraic polynomials of degree at most m. For any bivariate function $g(x, y)$ we will denote by g_y the function of the only variable x and by g_x the function of the only variable y.

2.1 Function Spaces

For the weight u

$$u(x) := x^\gamma (1+x)^\delta e^{-x}, \qquad x \geq 0, \quad \gamma, \delta \geq 0,$$

let C_u be the following space of functions

$$C_u = \left\{ f \ : \ fu \in C^0((0,+\infty)), \ \lim_{x\to+\infty} f(x)u(x) = 0, \ \lim_{x\to 0^+} f(x)u(x) = 0, \ if \ \gamma > 0, \right\} \tag{2}$$

equipped with the norm

$$\|f\|_{C_u} := \|fu\|_\infty = \max_{x\geq 0} |f(x)|u(x).$$

In the case $\gamma = \delta = 0$ we set $C_u = C^0$.
We point out that the limit conditions in (2) are necessary to assure that

$$\lim_{m\to\infty} E_m(f)_u = 0, \quad \forall f \in C_u,$$

being

$$E_m(f)_u := \inf_{P\in\mathbb{P}_m} \|f - P\|_{C_u}$$

the error of best polynomial approximation of $f \in C_u$ (see for instance [9]).
For smoother functions, we will consider the Sobolev-type spaces

$$W_r(u) = \left\{ f \in C_u \ : \ f^{(r-1)} \in AC(\mathrm{R}^+), \ \|f^{(r)}\varphi^r u\|_\infty < \infty \right\}, \quad r \in \mathbb{N},$$

where $\varphi(x) = \sqrt{x}$ and $AC(\mathrm{R}^+)$ denotes the set of all absolutely continuous functions in $(0, +\infty)$, equipped with the norm

$$\|f\|_{W_r(u)} := \|f\|_{C_u} + \|f^{(r)}\varphi^r u\|_\infty.$$

We recall that for any function f in $W_r(u)$, the following estimate holds [4]:

$$E_m(f)_u \leq \mathcal{C} \frac{\|f\|_{W_r(u)}}{(\sqrt{m})^r}, \quad \mathcal{C} \neq \mathcal{C}(m, f). \tag{3}$$

For $1 \leq p < \infty$, let $L^p(\mathbb{R}^+)$ be the space of measurable functions f, equipped with the norm

$$\|f\|_{L^p(\mathbb{R}^+)} = \left(\int_0^{+\infty} |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Moreover, let $L^\infty(\mathbb{R}^+) =: C^0$.
In the end, setting $\log^+ f(x) = \log(\max(1, f(x)))$, we denote by $L \log^+ L$ the set of all measurable function f defined in $(0, +\infty)$ such that

$$\int_0^{+\infty} |f(x)|(1 + \log^+ |f(x)|) \, dx < +\infty.$$

2.2 Orthonormal polynomials and Lagrange interpolation

For a given Laguerre weight

$$w(x) := w_\alpha(x) = x^\alpha e^{-x}, \quad \alpha > -1,$$

let $\{p_m := p_m(w)\}_{m\in\mathbb{N}}$ be the corresponding sequence of orthonormal polynomials having positive leading coefficients, and let $\{x_k := x_{m,k}(w) \;:\; k = 1,\dots,m\}$ be the zeros of $p_m(w)$. Moreover, we denote by

$$\lambda_{m,k} := \lambda_{m,k}(w) = \left(\sum_{i=0}^{m-1} p_i^2(x_k)\right)^{-1}, \quad k = 1,\dots,m,$$

the Christoffel numbers of order m related to w. Now, for a fixed $0 < \theta < 1$, we set

$$j = j(m) \in \mathbb{N}: \; x_j = \min_k \{x_k \;:\; x_k \geq 4m\theta, \quad k = 1,\dots,m\}. \tag{4}$$

Denoted by $\{y_k\}_{k=1}^{m+1}$ the zeros of p_{m+1}, since the zeros of p_{m+1} interlace those of p_m, for j defined in (4) we have

$$\frac{\mathcal{C}}{m} < y_1 < x_1 < y_2 < \ldots < x_{j-1} < y_j < x_j < 4m\theta.$$

Let $\mathcal{L}^*_{2m+2}(w, f)$ be the *truncated extended Lagrange polynomial* interpolating a given f at the zeros of $p_m(x)p_{m+1}(x)(4m-x)$, i.e.

$$\mathcal{L}^*_{2m+2}(w, f; x) = \sum_{k=1}^{j} \left(\ell_{2m+2,k}(x) f(x_k) + \hat{\ell}_{2m+2,k}(x) f(y_k)\right), \tag{5}$$

where

$$\ell_{2m+2,k}(x) = \begin{cases} \lambda_{m,k} p_{m+1}(x) \dfrac{(4m-x)}{p_{m+1}(x_k)(4m-x_k)} \displaystyle\sum_{i=0}^{m-1} p_i(x) p_i(x_k), & k = 1,\dots,m, \\[2ex] \dfrac{p_m(x)p_{m+1}(x)}{p_m(4m)p_{m+1}(4m)}, & k = m+1, \end{cases}$$

$$\hat{\ell}_{2m+2,k}(x) = \lambda_{m+1,k} p_m(x) \frac{(4m-x)}{p_m(y_k)(4m-y_k)} \sum_{i=0}^{m} p_i(x) p_i(y_k), \quad k = 1,\dots,m+1.$$

Setting $z_{2i} := x_i$, $i = 1 : m$, $z_{2i+1} := y_{i+1}$, $i = 0, 1, 2, \dots m$, we denote by

$$\{z_i\}_{i=1}^{2m+1} \text{ the zeros of } \; Q_{2m+1} := p_m p_{m+1}. \tag{6}$$

Remark 1. For a fixed $0 < \theta < 1$ and with j defined as in (4), $\mathcal{L}^*_{2m+2}(w)$ projects C_u onto $\tilde{\mathcal{P}}^*_{2m+1}$, being

$$\tilde{\mathcal{P}}^*_{2m+1} = \left\{q \in \mathbb{P}_{2m+1} : q(z_i) = q(4m) = 0, \quad z_i > z_{2j}\right\} \subset \mathbb{P}_{2m+1},$$

(see [25, 7]).

3 Ordinary Product integration rules

For the integral in (1), in [18] the following truncated product integration rule was introduced and studied

$$\mathcal{I}(f, y) = \int_0^{+\infty} f(x) k(x, y) \rho(x)\, dx = \mathcal{I}_m(f, y) + e_m(f, y), \quad \mathcal{I}_m(f, y) = \sum_{k=1}^{j} C_k(y) f(x_k), \tag{7}$$

where

$$C_k(y) = \lambda_{m,k} \sum_{i=0}^{j-1} p_i(x_k) \widetilde{M}_i(y), \qquad \widetilde{M}_i(y) = \int_0^{+\infty} p_i(x)(4m-x) k(x, y) \rho(x)\, dx, \;\; i = 0, 1, \dots, j-1, \tag{8}$$

and $e_m(f, y)$ is the quadrature error. In [18] it was proved that the rule preserves polynomials $P \in \mathcal{P}^*_m$ being

$$\mathcal{P}^*_m := \left\{q \in \mathbb{P}_m : q(x_i) = q(4m) = 0, \quad x_i > x_j\right\} \subset \mathbb{P}_m.$$

In the sequel we will refer to (7) as the *Ordinary Product Rule* (OPR in short) and, we will refer to $\{\widetilde{M}_i(y)\}_{i\in\mathbb{N}}$ as *Ordinary Modified Moments* (in short OMMs) [6] (see, e.g., [30]). Setting

$$\overline{u}(x) = e^{-x/2} x^\gamma (1+x)^\delta, \quad \gamma, \delta \geq 0, \tag{9}$$

and assuming $f \in C_{\overline{u}}$, where $C_{\overline{u}}$ is defined as in (2) by replacing u with $\overline{u}$, the following error estimate was proved in a more general context in [18, Th. 3.2].

Theorem 1. With ρ defined in (1) and $\overline{u}$ in (9), under the assumptions

$$\sup_{y\in S} \frac{\rho}{\sqrt{w\varphi}} k_y \in L\log^+ L, \quad \sup_{x\geq 0} \frac{\sqrt{w(x)\varphi(x)}}{\overline{u}(x)} < +\infty, \tag{10}$$

the rule (7) is stable in $C_{\overline{u}}$, i.e.

$$\sup_{y\in S} \sum_{i=1}^{j} \frac{|C_i(y)|}{\overline{u}(x_i)} < +\infty, \tag{11}$$

and for $f \in C_{\overline{u}}$ the error in (7) is estimated as follows:

$$\sup_{y\in S} |e_m(f,y)| \leq \mathcal{C}\left\{E_M(f)_{\overline{u}} + e^{-\mathcal{A}m}\|f\overline{u}\|_\infty\right\}, \tag{12}$$

where $M = \left\lfloor \frac{\theta}{1+\theta} m \right\rfloor$, and $0 < \mathcal{A} \neq \mathcal{A}(m,f)$, $\mathcal{C} \neq \mathcal{C}(m,f)$.

Remark 2. For any $f \in W_r(\overline{u})$, by (12) and taking into account (3), the following error estimate holds:

$$\sup_{y\in S} |e_m(f,y)| \leq \mathcal{C} \frac{\|f\|_{W_r(\overline{u})}}{(\sqrt{m})^r}, \quad \mathcal{C} \neq \mathcal{C}(m,f).$$

About the implementation of the rule (7), the main effort in evaluating the coefficients $C_k(y)$ is due to the exact computation of $\{\widetilde{M}_n(y)\}_{n=0,1,\ldots}$, being them usually deduced by recurrence relations dependent on the kernel $k(x,y)$ (see e. g.[6, 30]). In the next section we state such recurrence relations for some kernels.

3.1 Modified Moments recurrence relations for some kernels

First we recall the well-known three-term recurrence relation to generate the sequence of orthonormal Laguerre polynomials $\{p_n\}_n$ w.r.t. the weight $w(x) = e^{-x}x^\alpha$ (see e.g. [31]):

$$\begin{cases} p_{-1}(x) = 0, \\ p_0(x) = \dfrac{1}{\sqrt{\Gamma(\alpha+1)}}, \\ a_{n+1}p_{n+1}(x) = (x-b_n)p_n(x) - a_n p_{n-1}(x), \qquad n = 0,1,\ldots, \end{cases} \tag{13}$$

where $\Gamma(z)$ is the Euler's Gamma function and

$$a_n = \sqrt{n(n+\alpha)}, \quad b_n = 2n+\alpha+1, \quad n = 0,1,\ldots.$$

Using (13), it's not difficult to prove that $\{\widetilde{M}_n(y)\}_{n=0,1,\ldots}$ are related to the moments $\{M_n(y)\}_{n=0,1,\ldots}$

$$M_n(y) := \int_0^{+\infty} p_n(x)k(x,y)\rho(x)\,dx, \tag{14}$$

through the following relation:

$$\begin{cases} \widetilde{M}_0(y) = (4m-b_0)M_0(y) - a_1 M_1(y), \\ \widetilde{M}_n(y) = (4m-b_n)M_n(y) - a_{n+1}M_{n+1}(y) - a_n M_{n-1}(y). \end{cases}$$

Hence, we focus now on the computation of $\{M_i(y)\}_{i\in\mathbb{N}}$. They are generated via recurrence relations essentially based on (13), the repeated application of the integration by part rule and the relation

$$x\frac{d}{dx}p_n(x) = np_n(x) + a_n p_{n-1}(x).$$

We have selected kernels which occur more frequently in the applications, determining recurrence relations that the related moments $\{M_i(y)\}_{i\in\mathbb{N}}$ satisfy. Such relations, already found in [20], are here revisited, according to the initial assumptions on the orthonormal sequence.

3.2 Recurrence relations for $k(x,y) = \sin(yx)$, $k(x,y) = \cos(yx)$, $\gamma = 0$, $S = \mathbb{R}$

Denoted by $M_n^{Sin}(y) = \int_0^{+\infty} p_n(x)\sin(yx)e^{-x}\,dx$ and $M_n^{Cos}(y) = \int_0^{+\infty} p_n(x)\cos(yx)e^{-x}\,dx$, they satisfy the following relations:

$$\begin{cases} M_{-1}^{Sin}(y) = 0, \qquad M_0^{Sin}(y) = c_0\dfrac{y}{1+y^2}, \\ M_{-1}^{Cos}(y) = 0, \qquad M_0^{Cos}(y) = c_0\dfrac{1}{1+y^2}, \\ a_{n+1}M_{n+1}^{Sin}(y) = -\left[\tau_n M_n^{Sin}(y) - d_n M_n^{Cos}(y) + e_n M_{n-1}^{Sin}(y) - f_n M_{n-1}^{Cos}(y)\right], \\ a_{n+1}M_{n+1}^{Cos}(y) = -\left[\tau_n M_n^{Cos}(y) + d_n M_n^{Sin}(y) + e_n M_{n-1}^{Cos}(y) + f_n M_{n-1}^{Sin}(y)\right], \end{cases}$$

where $c_0 = \dfrac{1}{\sqrt{\Gamma(\alpha+1)}}$, $\quad \tau_n = b_n - \dfrac{n+1}{1+y^2}$, $\quad d_n = (n+1)\dfrac{y}{1+y^2}$, $\quad e_n = a_n\dfrac{y^2}{1+y^2}$, $\quad f_n = a_n\dfrac{y}{1+y^2}$, $\quad n = 0,1,\ldots$.

3.3 Recurrence relation for $k(x,y) = (x+y)^\mu$, $S = \mathbb{R}^+$

For the function $k(x,y) = (x+y)^\mu$, we set $M_n(y) = \int_0^{+\infty} p_n(x)(x+y)^\mu \rho(x)\,dx$ and $\overline{M}_n(y) = \int_0^{+\infty} p_n(x)(x+y)^{\mu+1}\rho(x)\,dx$, so that we obtain the following recurrence formula:

$$\begin{cases} M_{-1}(y) = 0, \\ M_0(y) = c_0\Gamma(\gamma+1)y^{\gamma+\mu+1}U(\gamma+1,\gamma+\mu+2,y), \\ \overline{M}_0(y) = c_0\Gamma(\gamma+1)y^{\gamma+\mu+2}U(\gamma+1,\gamma+\mu+3,y), \\ a_{n+1}M_{n+1}(y) = \overline{M}_n(y) - (y+b_n)M_n(y) - a_n M_{n-1}(y), \\ a_{n+1}\overline{M}_{n+1}(y) = g_n\overline{M}_n(y) - y(\mu+1)M_n(y), \end{cases}$$

where $U(a,b,y)$ is the Tricomi's confluent hypergeometric function and $c_0 = \dfrac{1}{\sqrt{\Gamma(\alpha+1)}}$, $\quad g_n = \gamma+\mu-n-\alpha+1, \quad n = 0,1,\ldots.$

3.4 Recurrence relation for $k(x,y) = \log(x+y)$, $\gamma = 0$, $S = \mathbb{R}^+ \setminus \{0\}$

To construct the modified moments $M_n(y)$ first we need to know the modified moments $\{C_n(y)\}_{n=0,1,\ldots}$ w.r.t. the kernel $\xi(x,y) := \dfrac{1}{x+y}$, i.e. $C_n(y) = \int_0^{+\infty} p_n(x)\xi(x,y)e^{-x}\,dx$. They satisfy the recurrence formula:

$$\begin{cases} C_{-1}(y) = 0, \\ C_0(y) = -c_0\left(e^y \mathrm{Ei}(-y)\right), \\ a_{n+1}C_{n+1}(y) = c_n - (y+b_n)C_n(y) - a_n C_{n-1}(y), \end{cases}$$

where $\mathrm{Ei}(y)$ is the exponential integral function and the coefficients $\{c_n\}_{n=0,1,\ldots}$ are so defined:

$$c_0 = \frac{1}{\sqrt{\Gamma(\alpha+1)}}, \quad c_{n+1} = -\frac{\alpha+n}{a_{n+1}}c_n = (-1)^{n+1}\frac{\alpha(\alpha+1)\ldots(\alpha+n)}{a_1a_2\ldots a_{n+1}}c_0, \quad n = 0,1,\ldots.$$

Hence, the modified moments $M_n(y)$ satisfy the recurrence relation:

$$\begin{cases} M_{-1}(y) = 0, \\ M_0(y) = c_0\left(\log(y) - e^y\mathrm{Ei}(-y)\right), \\ a_{n+1}M_{n+1}(y) = c_n - yC_n(y) - (n+\alpha)M_n(y). \end{cases}$$

3.5 Recurrence relation for $k(x,y) = \log|x-y|$, $\gamma = 0$, $S = \mathbb{R}^+ \setminus \{0\}$

To construct the modified moments $M_n(y)$ first we need to know the modified moments $\{A_n(y)\}_{n=0,1,\ldots}$ w.r.t. the kernel $\xi(x,y) := \dfrac{1}{x-y}$, i.e. $A_n(y) = \int_0^{+\infty} p_n(x)\xi(x,y)e^{-x}\,dx$. They satisfy the recurrence formula:

$$\begin{cases} A_{-1}(y) = 0, \\ A_0(y) = -c_0e^{-y}\mathrm{Ei}(y), \\ a_{n+1}A_{n+1}(y) = c_n + (y-b_n)A_n(y) - a_nA_{n-1}(y), \end{cases}$$

where $\mathrm{Ei}(y)$ is the exponential integral function and and the coefficients $\{c_n\}_{n=0,1,\ldots}$ are so defined:

$$c_0 = \frac{1}{\sqrt{\Gamma(\alpha+1)}}, \quad c_{n+1} = -\frac{\alpha+n}{a_{n+1}}c_n = (-1)^{n+1}\frac{\alpha(\alpha+1)\ldots(\alpha+n)}{a_1a_2\ldots a_{n+1}}c_0, \quad n = 0,1,\ldots.$$

Hence, the modified moments $M_n(y)$ satisfy the recurrence relation:

$$\begin{cases} M_{-1}(y) = 0, \\ M_0(y) = c_0\left(\log(y) - e^{-y}\mathrm{Ei}(y)\right), \\ a_{n+1}M_{n+1}(y) = c_n + yA_n(y) - (n+\alpha)M_n(y). \end{cases}$$

3.6 Recurrence relations for $k(x,y)=|x-y|^{\lambda}$, $\lambda>-1$, $S=\mathbb{R}^+$

To compute the modified moments for the function $k(x,y)=|x-y|^{\lambda}$, we write

$$\begin{aligned} M_n(y) &= \int_0^y p_n(x)(y-x)^{\lambda}\rho(x)\,dx + \int_y^{+\infty} p_n(x)(x-y)^{\lambda}\rho(x)\,dx \\ &=: M_n^-(y)+M_n^+(y). \end{aligned}$$

Denoted by $\widehat{M}_n^-(y)=\int_0^y p_n(x)(y-x)^{\lambda+1}\rho(x)\,dx$ and $\widehat{M}_n^+(y)=\int_y^{+\infty} p_n(x)(x-y)^{\lambda+1}\rho(x)\,dx$, we obtain the following recurrence formulas:

$$\begin{cases} M_{-1}^-(y)=0, \\ M_0^-(y)=c_0B(\lambda+1,\gamma+1)y^{\gamma+\lambda+1}\,{}_1F_1(\gamma+1,\gamma+\lambda+2,-y), \\ \widehat{M}_0^-(y)=c_0B(\lambda+2,\gamma+1)y^{\gamma+\lambda+2}\,{}_1F_1(\gamma+1,\gamma+\lambda+3,-y), \\ a_{n+1}M_{n+1}^-(y)=-\widehat{M}_n^-(y)+(y-b_n)M_n^-(y)-a_nM_{n-1}^-(y), \\ a_{n+1}\widehat{M}_{n+1}^-(y)=g_n\widehat{M}_n^-(y)-y(\lambda+1)M_n^-(y), \end{cases}$$

$$\begin{cases} M_{-1}^+(y)=0, \\ M_0^+(y)=c_0\Gamma(\lambda+1)e^{-y}U(-\gamma,-(\gamma+\lambda),y), \\ \widehat{M}_0^+(y)=c_0\Gamma(\lambda+2)e^{-y}U(-\gamma,-(\gamma+\lambda+1),y), \\ a_{n+1}M_{n+1}^+(y)=\widehat{M}_n^+(y)+(y-b_n)M_n^+(y)-a_nM_{n-1}^+(y), \\ a_{n+1}\widehat{M}_{n+1}^+(y)=g_n\widehat{M}_n^+(y)+y(\lambda+1)M_n^+(y), \end{cases}$$

where ${}_1F_1(a,b,y)$ is the Kummer's confluent hypergeometric function , $B(x,y)$ the Euler's Beta function, $U(a,b,y)$ is the Tricomi's confluent hypergeometric function and $c_0=\dfrac{1}{\sqrt{\Gamma(\alpha+1)}}$, $\quad g_n=\gamma+\lambda-n-\alpha+1, \quad n=0,1,\ldots.$

4 A new extended product rule

Now we introduce the following truncated *Extended Product Rule* (briefly EPR) obtained by approximating f in (1) with the extended Lagrange polynomial in (5) interpolating f at the zeros of $Q_{2m+1}(x)(4m-x)$

$$\mathcal{I}(f,y)=\int_0^{+\infty} f(x)k(x,y)\rho(x)\,dx=\Sigma_{2m+1}(f,y)+\mathcal{E}_{2m+1}(f,y),$$

$$\Sigma_{2m+1}(f,y):=\sum_{k=1}^{j}(\mathcal{A}_k(y)f(x_k)+\mathcal{B}_k(y)f(y_k)), \tag{15}$$

where

$$\mathcal{A}_k(y)=\int_0^{+\infty}\ell_{2m+2,k}(x)k(x,y)\rho(x)\,dx,$$

$$\mathcal{B}_k(y)=\int_0^{+\infty}\hat{\ell}_{2m+2,k}(x)k(x,y)\rho(x)\,dx,$$

and $\mathcal{E}_{2m+1}(f,y)$ is the quadrature error. In view of the Remark 1, the rule is exact in $\tilde{\mathcal{P}}^*_{2m+1}$, i.e.

$$\mathcal{E}_{2m+1}(f,y)=0, \quad \forall f\in\tilde{\mathcal{P}}^*_{2m+1}.$$

Note that by (5), the quadrature weights $\{\mathcal{A}_k(y),\mathcal{B}_k(y)\}_{k=1}^{j}$ can be rewritten as

$$\begin{aligned} \mathcal{A}_k(y) &= \frac{\lambda_{m,k}}{p_{m+1}(x_k)(4m-x_k)}\sum_{j=0}^{m-1}p_j(x_k)M_j^{(m+1)}(y), \\ \mathcal{B}_k(y) &= \frac{\lambda_{m+1,k}}{p_m(y_k)(4m-y_k)}\sum_{j=0}^{m}p_j(y_k)M_j^{(m)}(y), \end{aligned} \tag{16}$$

where

$$M_j^{(h)}(y) = \int_0^\infty p_h(x)p_j(x)k(x,y)(4m-x)\rho(x)\,dx \tag{17}$$

will be called *Generalized Modified Moments* (shortly GMMs). By (15-16), we finally have

$$\Sigma_{2m+1}(f,y) = \sum_{k=1}^{j}\left(f(x_k)\frac{\lambda_{m,k}}{p_{m+1}(x_k)(4m-x_k)}\sum_{j=0}^{m-1} p_j(x_k)M_j^{(m+1)}(y) + f(y_k)\frac{\lambda_{m+1,k}}{p_m(y_k)(4m-y_k)}\sum_{j=0}^{m} p_j(y_k)M_j^{(m)}(y)\right). \tag{18}$$

Given the OMMs $\{\widetilde{M}_n(y)\}_{n=0,1,\dots}$ and using (13), the computation of the GMMs $\{M_n^{(h)}(y)\}_{n,h=0,1,\dots}$ can be performed through the following recurrence scheme:

$$\begin{cases} M_n^{(0)}(y) = c_0\widetilde{M}_n(y), \\ a_1M_n^{(1)}(y) = a_nM_{n-1}^{(0)}(y) + (b_n-b_0)M_n^{(0)}(y) + a_{n+1}M_{n+1}^{(0)}(y), \\ a_hM_n^{(h)}(y) = a_nM_{n-1}^{(h-1)}(y) + (b_n-b_{h-1})M_n^{(h-1)}(y) + a_{n+1}M_{n+1}^{(h-1)}(y) - a_{h-1}M_n^{(h-2)}(y), \end{cases}$$

with $n = 0,1,\dots$ and $h = 2,3,\dots$.

The above recurrence relation works regardless of the given kernel k and can be applied once the OMMs are computed, whether they are of the type (14) or (8). However, despite its simple form, we have tested the algorithm for different kernels observing a progressive loss of accuracy. To overcome this instability in implementing the algorithm we decided to carry out the computation of all the moments, including the starting OMMs, by using 32-digits precision (i.e. in quadruple precision), and this choice has been enough to compensate the aforesaid inaccuracy.
Finally, next theorem provides conditions assuring stability and convergence of the rule in the space C_u:

Theorem 2. Let $f \in C_u$ with $u(x) = x^\gamma(1+x)^\delta e^{-x}$, $\rho(x) = x^\gamma e^{-x}$, with $\gamma \geq 0, \delta > 1$, $\alpha \geq -\frac{1}{2}$. Then, under the assumptions

$$\sup_{y\in S}\frac{\rho}{w\varphi}k_y \in L\log^+ L, \qquad \frac{w\varphi}{u} \in L^\infty(\mathbb{R}^+)\cap L^1(\mathbb{R}^+), \tag{19}$$

we have

$$\sup_{y\in S}\sum_{i=1}^{j}\left(\frac{|A_i(y)|}{u(x_i)} + \frac{|B_i(y)|}{u(y_i)}\right) < +\infty. \tag{20}$$

and

$$\sup_{y\in S}|\mathcal{E}_{2m+1}(f,y)| \leq \mathcal{C}\left[E_{\hat{M}}(f)_u + e^{-\mathcal{A}m}\|fu\|_\infty\right], \tag{21}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$, $\mathcal{A} \neq \mathcal{A}(m,f)$ and $\hat{M} = \left\lfloor \frac{\theta}{1+\theta}(2m+1)\right\rfloor$.

Remark 3. The mixed scheme studied in [26] for integrals similar to (1) is different from that we propose here. Indeed, the extended rule employed there is based on Laguerre polynomials, orthogonal w.r.t. two different weight functions. Hence, besides a little more elaborated computation of the rule coefficients, also the conditions assuring the stability and convergence proved there (see [26, Th. 3.2]) are different from those established here in Theorem 2. This can turn useful in all the cases where w_α and u are fixed and the quadrature scheme in [26] doesn't converge.

4.1 A compounded sequence of Ordinary and Extended product rules

We start by proving that under suitable assumptions, both sequences $\{\mathcal{I}_m(f,y)\}_m$ and $\{\Sigma_{2m+1}(f,y)\}_m$ converge to $\mathcal{I}(f,y)$, with the same rate of convergence. Indeed, observing that $C_{\overline{u}} \subset C_u$, the following theorem holds:

Theorem 3. Let $f \in C_u$ with $u(x) = x^\gamma(1+x)^\delta e^{-x}$, $\rho(x) = x^\gamma e^{-x}$, with $0 \leq \gamma, \delta > 1$, $\alpha \geq -\frac{1}{2}$. Then, (11-12) and (20-21) hold true under the assumptions

$$\sup_{y\in S}\frac{\rho}{w\varphi}k_y \in L\log^+ L, \quad \frac{w\varphi}{u} \in L^1(\mathbb{R}^+), \quad \max\left(0, \alpha+\frac{1}{2}-\delta\right) \leq \gamma \leq \frac{\alpha}{2}+\frac{1}{4}. \tag{22}$$

The theorem follows by taking into account that the assumptions (22) imply either those in (10) and (19)).
The previous result represents the cornerstone on which it is based the idea of a compounded sequence of ordinary and extended quadrature rules in order to achieve a saving in the number of function computations, a reduction of the elapsed time and the overcoming of instability problems arising in the computation of zeros and Christoffel numbers for high degree orthogonal polynomials.
For a fixed $m \geq 2$, and suitably organizing the algorithm, once obtained $\mathcal{I}_m(f,y)$, by (18) the computation of $\Sigma_{2m+1}(f,y)$ requires j additional function evaluations. In the practice, instead of computing $\mathcal{I}_{2m+1}(f,y)$ requiring $2j$ function evaluations, we

double the quadrature formula order with only j additional function evaluations. Thus continuing, we consider the sequence $\mathcal{I}_m(f,y)$, $\Sigma_{2m+1}(f,y)$, $\mathcal{I}_{4m}(f,y)$, $\Sigma_{8m+1}(f,y)$, … instead of the sequence $\mathcal{I}_m(f,y), \mathcal{I}_{2m+1}(f,y), \mathcal{I}_{4m}(f,y), \mathcal{I}_{8m+1}(f,y), \dots$.

In details, for a given $q \in \mathbb{N}$, consider the sequences $\{\mathcal{I}_{2^{2k}m}(f,y), \mathcal{I}_{2^{2k+1}m+1}(f,y)\}_{k=0}^{q-1}$, and $\{\mathcal{I}_{2^{2k}m}(f,y), \Sigma_{2^{2k+1}m+1}(f,y)\}_{k=0}^{q-1}$. To perform an analysis of the computational cost, let us assume to implement the formula without truncation, i.e. $j = m+1$ in both rules. Then, the construction of the first sequence requires $m(2^{2q}-1)$ evaluations of f, while the compounded scheme only $q + \frac{2}{3}m(2^{2q}-1)$. Hence, almost one third of the function evaluations is spared by using the mixed sequence.

In what follows we will consider the sequence

$$\mathcal{T}_{2^n m}(f,y) = \begin{cases} \mathcal{I}_{2^n m}(f,y), & n = 0,2,4,6,\dots, \\ \Sigma_{2^n m+1}(f,y), & n \text{ odd}. \end{cases} \tag{23}$$

About the stability and the convergence of this mixed scheme we are able to prove the following:

Theorem 4. Let $f \in C_u$ and fix $n \in \mathbb{N}$. Under the assumptions of Theorem 3we have

$$\sup_{y\in S} |\mathcal{T}_{2^n m}(f,y)| \le \mathcal{C}\|fu\|_\infty$$

and

$$\sup_{y\in S} |\mathcal{I}(f,y) - \mathcal{T}_{2^n m}(f,y)| \le \mathcal{C}E_N(f)_u\,, \tag{24}$$

where $N = \left\lfloor \frac{\theta}{1+\theta} 2^n m \right\rfloor$, and in both formulas $\mathcal{C} \ne \mathcal{C}(m,f)$.

Remark 4. By (24), and taking into account (3), the following error estimate holds

$$\sup_{y\in S} |\mathcal{I}(f,y) - \mathcal{T}_{2^n m}(f,y)| \le \mathcal{C}\frac{\|f\|_{W_r(\overline{u})}}{(\sqrt{2^n m})^r}, \quad \mathcal{C} \ne \mathcal{C}(n,f).$$

5 Numerical tests

In this section we show some numerical tests to highlight the advantages of the compounded sequence by Ordinary and Extended product quadrature rules. In each example we approximate the given integral for two different values of the parameter y. In the tables we report the approximated values of the integrals achieved by the *Ordinary Sequence* (7) and the *Compounded Sequence* (23). To be more precise, for increasing values of m we display the exact digits of the corresponding approximation and the number of the effective function evaluations (*# f eval.*) by adopting truncation techniques. Finally, for comparison we considered as exact the values of the integrals attained by the built-in Wolfram Mathematica function `NIntegrate`.

About the computation, we precise that all the routines have been written in Wolfram Mathematica 13 and run on an M1 MacBook Pro under the macOS operating system. Moreover, once all the Modified Moments (OMMs and GMMs) have been computed in quadruple precision, all the remaining computation were carried out in double precision. Finally, the truncation index j has been empirically detected by means of the following condition

$$j = \begin{cases} \min\limits_{1\le k\le m} |f(x_k)C_k(y)| > 10^{-20}, & \textit{OP Rule} \\ \max\left(\min\limits_{1\le k\le m} |f(x_k)A_k(y)| > 10^{-20}, \ \min\limits_{1\le k\le m+1} |f(y_k)B_k(y)| > 10^{-20}\right), & \textit{EP Rule} \end{cases}$$

being $C_k(y)$ defined in (8) and $A_k(y)$, $B_k(y)$ in (16).

Example 1.

$$\mathcal{I}(f,y) = \int_0^{+\infty} \sin(x)\frac{|x-y|^{-\frac{1}{10}}}{x^2+25} x^{\frac{1}{4}} e^{-x}\,dx, \quad \alpha = \frac{1}{2}, \gamma = \frac{1}{4}, \delta = 2.$$

Example 2.

$$\mathcal{I}(f,y) = \int_0^{+\infty} \arctan(1+x)\frac{\sin(yx)}{(x+y)^2} e^{-x}\,dx, \quad \alpha = \frac{1}{2}, \gamma = 0, \delta = \frac{5}{4}.$$

Example 3.

$$\mathcal{I}(f,y) = \int_0^{+\infty} \log(3x+5)\frac{\cos(yx)}{(1+x)^3} e^{-x}\,dx, \quad \alpha = -\frac{1}{2}, \gamma = 0, \delta = \frac{3}{2}.$$

Example 4.

$$\mathcal{I}(f,y) = \int_0^{+\infty} \cos(x)x^{\frac{1}{3}}(x+y)^{-\frac{7}{4}} e^{-x}\,dx, \quad \alpha = 0, \gamma = \frac{1}{3}, \delta = \frac{11}{10}.$$

Example 5.

$$\mathcal{I}(f,y) = \int_0^{+\infty} (x^2+1)^{\frac{7}{2}}\frac{\log(x+y)}{x^2+y} e^{-x}\,dx, \quad \alpha = -\frac{1}{2}, \gamma = 0, \delta = \frac{13}{10}.$$

Example 6.

$$\mathcal{I}(f,y)=\int_0^{+\infty}\arctan(x)^{\frac{21}{4}}\frac{\log|x-y|}{(x^2+y^2)^2}e^{-x}\,dx,\quad \alpha=0,\gamma=0,\delta=\frac{3}{2}+\frac{1}{100}.$$

5.1 Comments to the numerical tests

All the reported examples consider different types of kernels, functions and choices of y. The tables show that the Compounded Sequence is always to be preferred to the Ordinary one since it allows to increase the number of exact digits, reduce the amount of function evaluations and also decreasing the elapsed time. In some cases, for high values of m the Compounded Sequence kept fixing even more correct digits, while its Ordinary counterpart experienced a loss of accuracy. This behavior probably depends on a certain instability arising in the computation of high degree zeros of Laguerre polynomials by means of the Golub-Welsh algorithm. About the order of convergence, all the obtained results are coherent with our theoretical estimates and in some cases they are even better than what we expected. Furthermore, it is evident the importance of implementing truncation techniques that ensure a significant reduction of the function evaluation numbers. For instance, this could be a really determinant factor in the economy of the construction of the linear system corresponding to a Nyström-type method for the treatment of Fredholm Integral Equations on the real semi-axis.

y = 1					
Rule	# f eval.	**Ordinary Sequence**	Rule	# f eval.	**Compounded Sequence**
$\mathcal{I}_4$	4	0.02	$\mathcal{I}_4$	4	0.02
$\mathcal{I}_9$	9	0.02108	Σ_9	4	0.021
$\mathcal{I}_{16}$	14	0.021089	$\mathcal{I}_{16}$	14	0.021089
$\mathcal{I}_{33}$	22	0.02109315	Σ_{33}	14	0.02109315
$\mathcal{I}_{64}$	31	0.0210931521	$\mathcal{I}_{64}$	31	0.0210931521
$\mathcal{I}_{129}$	45	0.021093152190035	Σ_{129}	31	0.021093152190035
y = 6					
Rule	# f eval.	**Ordinary Sequence**	Rule	# f eval.	**Compounded Sequence**
$\mathcal{I}_4$	4	0.01	$\mathcal{I}_4$	4	0.01
$\mathcal{I}_9$	9	0.015	Σ_9	4	0.015
$\mathcal{I}_{16}$	14	0.0158	$\mathcal{I}_{33}$	14	0.0158
$\mathcal{I}_{33}$	22	0.01589102	Σ_{33}	14	0.015891023
$\mathcal{I}_{64}$	31	0.015891023255	$\mathcal{I}_{64}$	31	0.015891023255
$\mathcal{I}_{129}$	45	0.0158910232558858	Σ_{129}	31	0.0158910232558858

Table 1: Example 1: Evaluation of $\mathcal{I}(f,y)$ with $y=1$ and $y=6$.

y = 15					
Rule	# f eval.	**Ordinary Sequence**	Rule	# f eval.	**Compounded Sequence**
$\mathcal{I}_4$	4	0.0002	$\mathcal{I}_4$	4	0.0002
$\mathcal{I}_9$	9	0.00023	Σ_9	4	0.00023
$\mathcal{I}_{16}$	14	0.000233	$\mathcal{I}_{16}$	14	0.000233
$\mathcal{I}_{33}$	22	0.0002334783	Σ_{33}	14	0.0002334783
$\mathcal{I}_{64}$	32	0.00023347838	$\mathcal{I}_{64}$	32	0.00023347838
$\mathcal{I}_{129}$	45	0.000233478386382	Σ_{129}	32	0.00023347838638288
$\mathcal{I}_{256}$	64	0.000233478386382	$\mathcal{I}_{256}$	64	0.000233478386382
$\mathcal{I}_{513}$	91	0.00023347838638	Σ_{513}	64	0.000233478386382885
y = 27					
Rule	# f eval.	**Ordinary Sequence**	Rule	# f eval.	**Compounded Sequence**
$\mathcal{I}_4$	4	0.0000	$\mathcal{I}_4$	4	0.0000
$\mathcal{I}_9$	9	0.00003	Σ_9	4	0.00003
$\mathcal{I}_{16}$	14	0.0000399	$\mathcal{I}_{16}$	14	0.0000399
$\mathcal{I}_{33}$	22	0.00003994	Σ_{33}	14	0.0000399480
$\mathcal{I}_{64}$	31	0.0000399480	$\mathcal{I}_{64}$	31	0.0000399480
$\mathcal{I}_{129}$	45	0.00003994809009	Σ_{129}	31	0.000039948090099180
$\mathcal{I}_{256}$	64	0.00003994809009	$\mathcal{I}_{256}$	64	0.00003994809009
$\mathcal{I}_{513}$	91	0.00003994809009	Σ_{513}	64	0.0000399480900991801

Table 2: Example 2: Evaluation of $\mathcal{I}(f,y)$ with $y=15$ and $y=27$.

y = 40					
Rule	# *f eval.*	**Ordinary Sequence**	Rule	# *f eval.*	**Compounded Sequence**
$\mathcal{I}_{16}$	14	0.003	$\mathcal{I}_{16}$	14	0.003
$\mathcal{I}_{33}$	21	0.0035	Σ_{33}	14	0.003598
$\mathcal{I}_{64}$	30	0.003598	$\mathcal{I}_{64}$	30	0.003598
$\mathcal{I}_{129}$	43	0.0035984	Σ_{129}	30	0.003598479953
$\mathcal{I}_{256}$	60	0.003598479953	$\mathcal{I}_{256}$	60	0.003598479953
$\mathcal{I}_{513}$	85	0.00359847995384	Σ_{513}	60	0.003598479953844
y = 90					
Rule	# *f eval.*	**Ordinary Sequence**	Rule	# *f eval.*	**Compounded Sequence**
$\mathcal{I}_{16}$	14	0.000	$\mathcal{I}_{16}$	14	0.000
$\mathcal{I}_{33}$	21	0.00071	Σ_{33}	14	0.000718
$\mathcal{I}_{64}$	30	0.000718	$\mathcal{I}_{64}$	30	0.000718
$\mathcal{I}_{129}$	43	0.000718713	Σ_{129}	30	0.000718713998
$\mathcal{I}_{256}$	60	0.000718713998	$\mathcal{I}_{256}$	60	0.000718713998
$\mathcal{I}_{513}$	85	0.0007187139985814	Σ_{513}	60	0.0007187139985814

Table 3: Example 3: Evaluation of $\mathcal{I}(f,y)$ with $y=40$ and $y=90$.

y = 1/5					
Rule	# *f eval.*	**Ordinary Sequence**	Rule	# *f eval.*	**Compounded Sequence**
$\mathcal{I}_{4}$	4	1.	$\mathcal{I}_{4}$	4	1.
$\mathcal{I}_{9}$	9	1.26	Σ_{9}	4	1.26
$\mathcal{I}_{16}$	15	1.268	$\mathcal{I}_{16}$	15	1.268
$\mathcal{I}_{33}$	24	1.268838	Σ_{33}	15	1.268838518
$\mathcal{I}_{64}$	34	1.26883851820	$\mathcal{I}_{64}$	34	1.26883851820
$\mathcal{I}_{129}$	48	1.268838518202	Σ_{129}	34	1.268838518202609
$\mathcal{I}_{256}$	68	1.26883851820260	$\mathcal{I}_{256}$	68	1.26883851820260
$\mathcal{I}_{513}$	96	1.2688385182026	Σ_{513}	68	1.2688385182026094
y = 1					
Rule	# *f eval.*	**Ordinary Sequence**	Rule	# *f eval.*	**Compounded Sequence**
$\mathcal{I}_{4}$	4	0.20	$\mathcal{I}_{4}$	4	0.20
$\mathcal{I}_{9}$	9	0.20	Σ_{9}	4	0.20
$\mathcal{I}_{16}$	15	0.2069	$\mathcal{I}_{16}$	15	0.2069
$\mathcal{I}_{33}$	24	0.2069223	Σ_{33}	15	0.20692235321
$\mathcal{I}_{64}$	34	0.206922353217	$\mathcal{I}_{64}$	34	0.206922353217
$\mathcal{I}_{129}$	48	0.2069223532172	Σ_{129}	34	0.20692235321729
$\mathcal{I}_{256}$	68	0.20692235321729	$\mathcal{I}_{256}$	68	0.20692235321729
$\mathcal{I}_{513}$	96	0.206922353217292	Σ_{513}	68	0.206922353217292

Table 4: Example 4: Evaluation of $\mathcal{I}(f,y)$ with $y=\frac{1}{5}$ and $y=1$.

6 The proofs

With Q_{2m+1} and $\{z_i\}_{i=1}^{2m+1}$ defined in (6), in the proof we use the Lagrange expression

$$\mathcal{L}^*_{2m+2}(w,f,x)=\sum_{k=1}^{j}\frac{Q_{2m+1}(x)(4m-x)}{Q'_{2m+1}(x)(4m-z_k)(x-z_k)}f(z_k),$$

equivalent to (5).

In the next the following two estimates will be needed:

$$|Q_{2m+1}(x)|e^{-x}\left(x+\frac{\mathcal{C}}{m}\right)^{\alpha+\frac{1}{2}}\sqrt{|4m-x|+\mathcal{C}m^{\frac{1}{3}}}\le\mathcal{C},\quad 0\le x\le\mathcal{C}(4m+m^{\frac{1}{3}}),\tag{25}$$

$$\frac{1}{|Q'_{2m+1}(z_k)|u(z_k)}\sim\sqrt{m}\,\frac{w(z_k)\varphi(z_k)}{u(z_k)}\Delta z_k,\quad k\le j,\tag{26}$$

with j defined in (4). Their proof is omitted, since deducible by [31] (see also [16, (3.3)]) and [25, (40)].

y = 3/4					
Rule	# *f eval.*	**Ordinary Sequence**	Rule	# *f eval.*	**Compounded Sequence**
$\mathcal{I}_{16}$	16	247.719	$\mathcal{I}_{16}$	16	247.719
$\mathcal{I}_{33}$	27	247.7193111	Σ_{33}	16	247.7193111
$\mathcal{I}_{64}$	40	247.719311110	$\mathcal{I}_{64}$	40	247.719311110
$\mathcal{I}_{129}$	57	247.719311109	Σ_{129}	40	247.7193111094
$\mathcal{I}_{256}$	82	247.7193111094	$\mathcal{I}_{256}$	82	247.7193111094
$\mathcal{I}_{513}$	116	247.71931110943	Σ_{513}	82	247.719311109438
y = 100					
Rule	# *f eval.*	**Ordinary Sequence**	Rule	# *f eval.*	**Compounded Sequence**
$\mathcal{I}_{16}$	16	162.687	$\mathcal{I}_{16}$	16	162.687
$\mathcal{I}_{33}$	27	162.687271	Σ_{33}	16	162.6872713
$\mathcal{I}_{64}$	40	162.68727132	$\mathcal{I}_{64}$	40	162.68727132
$\mathcal{I}_{129}$	57	162.68727132	Σ_{129}	40	162.687271325570
$\mathcal{I}_{256}$	81	162.687271325	$\mathcal{I}_{256}$	81	162.687271325
$\mathcal{I}_{513}$	116	162.6872713255	Σ_{513}	81	162.6872713255708

Table 5: Example 5: Evaluation of $\mathcal{I}(f,y)$ with $y=\frac{3}{4}$ and $y=100$.

y = 2/3					
Rule	# *f eval.*	**Ordinary Sequence**	Rule	# *f eval.*	**Compounded Sequence**
$\mathcal{I}_{16}$	14	−0.05	$\mathcal{I}_{16}$	14	−0.05
$\mathcal{I}_{33}$	21	−0.059	Σ_{33}	14	−0.059
$\mathcal{I}_{64}$	29	−0.059	$\mathcal{I}_{64}$	29	−0.059
$\mathcal{I}_{129}$	42	−0.0597	Σ_{129}	29	−0.0597
$\mathcal{I}_{256}$	59	−0.05971006	$\mathcal{I}_{256}$	59	−0.05971006
$\mathcal{I}_{513}$	84	−0.059710068504	Σ_{513}	59	−0.05971006850436
y = 5					
Rule	# *f eval.*	**Ordinary Sequence**	Rule	# *f eval.*	**Compounded Sequence**
$\mathcal{I}_{4}$	4	0.000	$\mathcal{I}_{4}$	4	0.000
$\mathcal{I}_{9}$	9	0.0005	Σ_{9}	4	0.0005
$\mathcal{I}_{16}$	14	0.000574	$\mathcal{I}_{16}$	14	0.000574
$\mathcal{I}_{33}$	21	0.0005742	Σ_{33}	14	0.000574
$\mathcal{I}_{64}$	29	0.00057420	$\mathcal{I}_{64}$	29	0.00057420
$\mathcal{I}_{129}$	42	0.00057420677	Σ_{129}	29	0.0005742067
$\mathcal{I}_{256}$	59	0.0005742067786936	$\mathcal{I}_{256}$	59	0.0005742067786936
$\mathcal{I}_{513}$	84	0.000574206778693	Σ_{513}	59	0.0005742067786936

Table 6: Example 6: Evaluation of $\mathcal{I}(f,y)$ with $y=\frac{2}{3}$ and $y=5$.

Moreover, we recall a formula for the inversion of the integration order related to the Hilbert transform $H_B(g,t)$ of a function g on the compact set B (see, e.g., [33])

$$H_B(g,t)=\int_B \frac{g(x)}{x-t}dx.$$

Indeed, if $G\in L^\infty(B), F\log^+ F\in L^1(B)$, then the following useful estimate holds true

$$\|GH_B(F)\|_1 \le \mathcal{C}+\left\|F\log^+ F\right\|_1, \quad \mathcal{C}\neq\mathcal{C}(F). \tag{27}$$

Proof of Theorem 2. First of all we prove that for any $f\in C_u$, under the assumptions (19),

$$\sup_{y\in S}\left\|\mathcal{L}^*_{2m+2}(w,f)k_y\rho\right\|_1 \le \mathcal{C}\|fu\|_\infty, \tag{28}$$

with $0<\mathcal{C}\neq\mathcal{C}(m,f)$.
To prove (28), fixed $y\in S$ and set $g_m=\operatorname{sgn}\left(\mathcal{L}^*_{2m+2}(w,f)k_y\right)$, we have

$$\begin{aligned}\left\|\mathcal{L}^*_{2m+2}(w,f)k_y\rho\right\|_1 &= \left(\int_0^{4m}+\int_{4m}^{8m}+\int_{8m}^{+\infty}\right)\mathcal{L}^*_{2m+2}(w,f,x)k_y(x)g_m(x)\rho(x)\,dx\\ &=: I_1(y)+I_2(y)+I_3(y)\end{aligned} \tag{29}$$

Defining

$$\Pi^*(t) = \int_0^{4m} \frac{Q_{2m+1}(x)(4m-x)q(x) - Q_{2m+1}(t)(4m-t)q(t)}{(x-t)} \frac{g_m(x)k_y(x)\rho(x)}{q(x)}\,dx,$$

being q an arbitrary polynomial of degree ml, l a fixed integer, by (26) we have

$$I_1(y) = \left|\sum_{k=1}^{j} \frac{f(z_k)}{Q'_{2m+1}(z_k)(4m-z_k)} \Pi^*(z_k)\right| \le \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{m}} \sum_{k=1}^{j} \Delta z_k \frac{z_k^{\alpha+\frac{1}{2}-\gamma}}{(1+z_k)^\delta} |\Pi^*(z_k)|.$$

Taking into account $\Pi^* \in \mathbb{P}_{2m+1+ml}$, we use [7, Lemma 4.3] (see also [26, Lemma 6.4]) with $p=1$, and $\theta_1 > \theta$ s.t. $4m\theta < z_j < 4m\theta_1$. Hence we get

$$\begin{aligned}
I_1(y) &\le \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{m}} \int_{z_1}^{4m\theta_1} \frac{t^{\alpha+\frac{1}{2}-\gamma}}{(1+t)^\delta} |\Pi^*(t)|\,dt \\
&\le \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{m}} \left\{ \int_{z_1}^{4m\theta_1} \frac{w(t)\varphi(t)}{u(t)} \left|H_{[0,4m]}(F_{m,y},t)\right| dt \right. \\
&+ \left. \int_{z_1}^{4m\theta_1} \frac{w(t)\varphi(t)}{u(t)} \left|Q_{2m+1}(t)(4m-t)q(t)H_{[0,4m]}(G_{m,y},t)\right| dt \right\} \\
&=: \mathcal{C}\|fu\|_\infty \left\{I_{1,1}(y) + I_{1,2}(y)\right\},
\end{aligned} \tag{30}$$

being

$$\begin{aligned}
F_{m,y}(t) &= Q_{2m+1}(t)e^{-t}\left(t+\frac{\mathcal{C}}{m}\right)^{\alpha+\frac{1}{2}} (4m-t)\frac{g_m(t)k_y(t)\rho(t)}{e^{-t}\left(t+\frac{\mathcal{C}}{m}\right)^{\alpha+\frac{1}{2}}}, \\
G_{m,y}(t) &= \frac{g_m(t)k_y(t)\rho(t)}{q(t)}.
\end{aligned}$$

By (25) we deduce for $0 \le t \le 4m$

$$\frac{|F_{m,y}(t)|}{\sqrt{m}} \le \mathcal{C}\frac{\rho(t)}{w(t)\varphi(t)}|k_y(t)| \in L\log^+ L$$

and under the assumptions in (19), by using (27)

$$I_{1,1}(y) \le \mathcal{C} + \mathcal{C}\left\|\frac{k_y\rho}{w\varphi}\log^+\left(\frac{k_y\rho}{w\varphi}\right)\right\|_1 \le \mathcal{C}.$$

Hence, we can conclude

$$\sup_{y\in S} I_{1,1}(y) \le \mathcal{C}. \tag{31}$$

In order to estimate $I_{1,2}$ by a result in [19], we choose the polynomial $q \in \mathbb{P}_{ml}$, such that $q(x) \sim e^{-x}x^{\alpha+\frac{1}{2}}$, being $\alpha + \frac{1}{2} \ge 0$. By (25) for $x \in [z_1, 4m\theta_1]$, we have

$$|Q_{2m+1}(t)(4m-t)q(t)| \le \mathcal{C}\sqrt{4m}.$$

Therefore

$$I_{1,2}(y) \le \mathcal{C}\int_{z_1}^{4m\theta_1} \frac{w(t)\varphi(t)}{u(t)} \left|H_{[0,4m]}(G_{m,y},t)\right| dt$$

and taking into account

$$\left|G_{m,y}(t)\right| \le \mathcal{C}\frac{k_y(t)\rho(t)}{w(t)\varphi(t)} \in L\log^+ L,$$

by (27) once again, under the assumptions (19), we get

$$\sup_{y\in S} I_{1,2}(y) \le \mathcal{C} + \mathcal{C}\sup_{y\in S}\left\|\frac{k_y\rho}{w\varphi}\log^+\left(\frac{k_y\rho}{w\varphi}\right)\right\|_1 \le \mathcal{C}. \tag{32}$$

Combining (31),(32) with (30) it follows

$$\sup_{y\in S} I_1(y) \le \mathcal{C}\|fu\|_\infty. \tag{33}$$

Now we estimate $I_2(y)$. By (26),

$$\begin{aligned} I_2(y) &\le \int_{4m}^{8m} |\mathcal{L}^*_{2m+2}(w,f,x)k_y(x)|\rho(x)\,dx \\ &\le \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{m}}\sum_{k=1}^{j}\Delta z_k \frac{w(z_k)\varphi(z_k)}{u(z_k)}\int_{4m}^{8m}\frac{\left|Q_{2m+1}(x)k_y(x)\right|(x-4m)\rho(x)}{x-z_k}\,dx \end{aligned}$$

and being $x-z_k>(1-\theta)4m$, $\left\|\frac{w\varphi}{u}\right\|_\infty\le\mathcal{C}$ and $\sum_{k=1}^{j}\Delta z_k\le 4m$, we get

$$I_2(y)\le\mathcal{C}\frac{\|fu\|_\infty}{\sqrt{m}}\int_{4m}^{8m}\left|Q_{2m+1}(x)k_y(x)\right|(x-4m)\rho(x)dx.$$

Now, by (25)

$$|Q_{2m+1}(x)|(x-4m)\rho(x)\le\mathcal{C}\frac{(x-4m)}{\sqrt{x-4m+m^{\frac{1}{3}}}}\frac{\rho(x)}{w(x)\varphi(x)}$$

and using

$$\frac{(x-4m)}{\sqrt{x-4m+m^{\frac{1}{3}}}}\le\mathcal{C}\sqrt{m},$$

it follows

$$I_2(y)\le\mathcal{C}\|fu\|_\infty\int_0^{+\infty}\left|k_y(x)\right|\frac{\rho(x)}{w(x)\varphi(x)}dx\le\mathcal{C}\|fu\|_\infty,$$

under the first assumption in (19) and hence

$$\sup_{y\in S} I_2(y)\le\mathcal{C}\|fu\|_\infty\int_0^{+\infty}\left|k_y(x)\right|\frac{\rho(x)}{w(x)\varphi(x)}dx\le\mathcal{C}\|fu\|_\infty, \tag{34}$$

In the end we estimate $I_3(y)$. By (26) we have

$$\begin{aligned} I_3(y) &\le \int_{8m}^{+\infty}|\mathcal{L}^*_{2m+2}(w,f,x)k_y(x)|\rho(x)\,dx \\ &\le \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{m}}\sum_{k=1}^{j}\Delta z_k\frac{w(z_k)\varphi(z_k)}{u(z_k)}\left|\int_{8m}^{+\infty}\frac{Q_{2m+1}(x)(x-4m)k_y(x)\rho(x)}{x-z_k}\,dx\right| \\ &\le \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{m}}\max_{x\ge 8m}|Q_{2m+1}(x)|w(x)\varphi(x)\sqrt{x-4m} \\ &\times \sum_{k=1}^{j}\Delta z_k\frac{w(z_k)\varphi(z_k)}{u(z_k)}\int_{8m}^{+\infty}|k_y(x)|\frac{\rho(x)}{w(x)\varphi(x)}\frac{\sqrt{x-4m}}{x-z_k}\,dx. \end{aligned}$$

Since $x-z_k\ge\mathcal{C}4m$, and by using [17, Lemma 2.2]

$$\max_{x\ge 8m}\left|Q_{2m+1}(x)\sqrt{x-4m}\,w(x)\,\varphi(x)\right|\le\mathcal{C}e^{-Am}\max_{x\le 4m\theta}\left|Q_{2m+1}(x)\sqrt{4m-x}\,w(x)\,\varphi(x)\right|\le\mathcal{C},$$

we get

$$I_3(y) \le \mathcal{C}\frac{\|fu\|_\infty}{m}\sum_{k=1}^{j}\Delta z_k\frac{w(z_k)\varphi(z_k)}{u(z_k)}\int_{8m}^{+\infty}\frac{\left|k_y(x)\right|\rho(x)}{w(x)\varphi(x)}\,dx,$$

and taking into account the assumptions (19), we can conclude

$$\sup_{y\in S} I_3(y)\le\mathcal{C}\|fu\|_\infty\int_0^{+\infty}\frac{w(t)\varphi(t)}{u(t)}\,dt\le\mathcal{C}\|fu\|_\infty. \tag{35}$$

Estimate (28) follows by combining (33), (34), (35) with (29).
We omit the proof of (20) since it can be easily deduced by standard arguments by (28). In order to prove (21), let $\widetilde{P}\in\widetilde{\mathcal{P}}^*_{2m+1}$, s.t.

$$\|(f-\widetilde{P})u\|_\infty=\inf_{\widetilde{\mathcal{P}}\in\widetilde{\mathcal{P}}^*_{2m+1}}\|(f-\widetilde{\mathcal{P}})u\|_\infty=:\widetilde{E}_{2m+1}(f)_u.$$

Then

$$\begin{aligned}
|\mathcal{E}_{2m+1}(f)| &\leq \int_0^{+\infty} \left|(f(x)-\widetilde{P}(x))k_y(x)\right| \rho(x)\,dx \\
&+ \int_0^{+\infty} \left|\mathcal{L}^*_{2m+2}(w, f-\widetilde{P}, x)k_y(x)\right| \rho(x)\,dx \\
&\leq \|(f-\widetilde{P})u\|_\infty \int_0^{+\infty} \frac{|k_y(x)|\,\rho(x)}{u(x)}\,dx + \mathcal{C}\|(f-\widetilde{P})u\|_\infty \\
&\leq \mathcal{C}\tilde{E}_{2m+1}(f)_u
\end{aligned}$$

Recalling that [18]

$$\tilde{E}_{2m+1}(f)_u \leq \mathcal{C}\left\{E_M(f)_u + e^{-Am}\|fu\|_\infty\right\},$$

where $M = \left\lfloor 2m\left(\frac{\theta}{1+\theta}\right)^\beta \right\rfloor$ and the constants $0 < A \neq A(m,f)$, $0 < \mathcal{C} \neq \mathcal{C}(m,f)$, estimate (21) follows. □

Acknowledgments. The authors are grateful to the anonymous referee for his helpful suggestions. This work has been supported by University of Basilicata (local funds) and by GNCS-INDAM Project 2020 "Approssimazione multivariata ed equazioni funzionali per la modellistica numerica".

References

[1] A.S. Cvetkovic, G.V. Milovanović. The Mathematica Package "OrthogonalPolynomials". *Facta Univ. Ser. Math. Inform.* 19, 17–36, 2004.

[2] A.S. Cvetkovic, G.V. Milovanović. Special classes of orthogonal polynomials and corresponding quadratures of Gaussian type. *Math. Balkanica* 26, 169–184, 2012.

[3] M.C. De Bonis, G. Mastroianni. Nyström method for systems of integral equations on the real semiaxis. *IMA J. Numer. Anal.*, 29: 632–650, 2009

[4] M.C. De Bonis, G. Mastroianni, M. Viggiano. K-functionals, Moduli of Smoothness and Weighted Best Approximation on the semiaxis. *Functions, Series, Operators, Alexits Memorial Conference*, Janos Bolyai Mathematical Society, 181–211, 2002.

[5] L. Fermo, C. Van Der Mee, S. Seatzu. Scattering data computation for the Zakharov-Shabat system. *Calcolo*: 533, 487–520, 2016.

[6] W. Gautschi. On the Construction of Gaussian Quadrature Rules from Modified Moments. *Math. Comput.*,24: 245–260,1970.

[7] C. Laurita, G. Mastroianni. L^p−convergence of Lagrange interpolation on the semiaxis. *Acta Math. Hungar.*,120, 3: 249–273, 2008.

[8] C. Laurita, G. Mastroianni. Projection Methods in L^p-norm for Fredholm Integral Equations on the real semiaxis. *Jour. of Integ. Equat. and Appl.* 21 (4):559–596, 2009.

[9] G. Mastroianni, G.V. Milovanović. Interpolation Processes – Basic Theory and Applications. *Springer Monographs in Mathematics*, Springer: Berlin, Germany, 2008.

[10] G. Mastroianni, G.V. Milovanović. Some numerical methods for second-kind Fredholm integral equations on the real semiaxis. *IMA Journal of Numerical Analysis*, 29: 1046–1066, 2009.

[11] G. Mastroianni, G.V. Milovanović, I. Notarangelo. A Nyström method for a class of Fredholm integral equations on the real semiaxis. *Calcolo*, 54,2:567–585, 2017.

[12] G. Mastroianni, G. Monegato. Convergence of product integration rules over $(0,\infty)$ for functions with weak singularities at the origin. *Mathematics of Computation*, 64,209:237-249, 1995.

[13] G. Mastroianni, G. Monegato. Some new applications of truncated Gauss-Laguerre quadrature formulas. *Numer. Algor.* 49: 283-297, 2008.

[14] G. Mastroianni, G. Monegato. Nyström interpolants based on zeros of Laguerre polynomials for some Wiener-Hopf equations. *IMA J. Numer. Anal.*, 17: 621–642 , 1997

[15] G. Mastroianni, G. Monegato. Truncated quadrature rules over $(0,\infty)$ and Nyström-type methods. SIAM J. NUMER. ANAL., 41 No. 5: 1870–1892, 2003.

[16] G. Mastroianni, D. Occorsio. Lagrange interpolation at Laguerre zeros in some weighted uniform spaces. *Acta Math. Hungar.*, 90, Issue 1-2: 27–52, 2001.

[17] G. Mastroianni, D. Occorsio. Numerical approximation of weakly singular integrals on the half line, *Jour. of Comput. and Appl. Math.*, 140: 587–598, 2002.

[18] G. Mastroianni, D. Occorsio. Some quadrature formulae with nonstandard weights. *Jour. of Comput. and Appl. Math.* , 235: 602–614, 2020.

[19] G. Mastroianni, J. Szabados. Polynomial approximation on the real semiaxis with generalized Laguerre weights, *Studia Univ. "Babes-Bolyai" Mathematica*, LII: 105–128, 2007

[20] N. Mastronardi, D. Occorsio. Product integration rules on the semiaxis. *Rend. Circ. Mat. di Palermo, Serie II*, Suppl. 52:605–618, 1998.

[21] D. Mezzanotte, M.G. Russo, D. Occorsio. Combining Nyström methods for a fast solution of Fredholm integral equations of the second kind. *Mathematics* 9, Article number 2652, 2021.

[22] G.V. Milovanović. Some orthogonal polynomials on the finite interval and Gaussian quadrature rules for fractional Riemann-Liouville integrals. *Math. Methods Appl. Sci.* 44, no. 1, 493–516, 2021.

[23] G.V. Milovanović. Quadrature processes and new applications. *Bull. Cl. Sci. Math. Nat. Sci. Math.* 38, 83–120, 2013.

[24] G.V. Milovanović, N. Vasovic. Orthogonal polynomials and generalized Gauss-Rys quadrature formulae. *Kuwait J. Sci.* 49 (1), 1–17, 2022.

[25] D. Occorsio. Extended Lagrange interpolation in weighted uniform norm. *Appl. Math. and Comput.* 211: 10–22, 2009.

[26] D. Occorsio, M.G. Russo. A new quadrature scheme based on an Extended Lagrange Interpolation process. *Appl. Numer. Math.*, 124: 57–75, 2018.

[27] D. Occorsio, M.G. Russo. Nyström methods for bivariate Fredholm integral equations on unbounded domains. *Appl. Math. and Comput.*, 318: 19–34, 2018.

[28] D. Occorsio, M.G. Russo. A mixed scheme of product integration rules in $(-1, 1)$. *Appl. Numer. Math.*, 149: 113–123, 2020.

[29] D. Occorsio, M.G. Russo. A mixed collocation scheme for solving second kind Fredholm integral equations in $(-1, 1)$. *Electronic Transactions on Numerical Analysis* 54: 443–459, 2021.

[30] R. Piessens, M. Branders. Numerical Solution of Integral equations of Mathematical Physics using Chebyshev Polynomials. *J. Comput. Phys.* ,21:178–196, 1976.

[31] G. Szegö. Orthogonal Polynomials. *Amer. Math. Soc. Colloq. Publ.* 23: 4th ed., Amer. Math. Soc., Providence, 1975.

[32] C. Van Der Mee, S. Seatzu, D. Theis. Structured matrix algorithms for inverse scattering on the line. *Calcolo*, 44: 59–88, 2007.

[33] P. Nevai, P. Hilbert transform and Lagrange interpolation. *J. Approx. Theory, 60: 360–363, 1990.*