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Construction of Bivariate Modified Bernstein-Chlodowsky Operators and Approximation Theorems

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Abstract

In this paper, we modified Bernstein-Chlodowsky operators via weaker condition than the classical Bernstein-Chlodowsky operators' condition. We get more powerful results than classical ones. We obtain aproximation properties for these positive linear operators and their generalizations in this work. The rate of convergence of these operators is calculated by means of the modulus of continuity and Lipschitz class of the functions of f of two variables. Finally, we give some concluding remarks with q-calculus.

1 Introduction

Positive approximation processes appear in a very natural way in many problems, especially when one requires further qualitative properties, such as monotonicity, convexity, shape preservation and so on. The fundamental theorem of Korovkin [17] on approximation of continuous functions on a compact interval gives conditions in order to decide whether a sequence of positive linear operators converges to identity operator. Many authors have extended these type approximation theorems using different, new and strong convergence methods, mostly in the direction of statistical convergence ([13, 22]), power series method and also, in some interesting function spaces. For more details and examples, we refer the readers to [2, 4, 9, 10, 11, 12, 14, 18, 23, 24, 25].

The classical Bernstein-Chlodowsky polynomials have the following form

$$\tilde{B}_n(f;x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n}\alpha_n\right)\binom{n}{k}\left(\frac{x}{\alpha_n}\right)^k\left(1-\frac{x}{\alpha_n}\right)^{n-k}$$

where $0 \le x \le \alpha_n$ and $\{\alpha_n\}$ is a sequence of positive numbers such that $\lim\limits_{n\to\infty}\alpha_n=\infty$, $\lim\limits_{n\to\infty}\dfrac{\alpha_n}{n}=0$. These polynomials were introduced by Chlodowsky in 1932 as a generalization of Bernstein polynomials (1912) on an unbounded set. There are many investigations devoted to the problem of approximating continuous functions by classical Bernstein polynomials, as well as by two-dimensional Bernstein polynomials and their generalizations. Some generalizations of these polynomials may be found in [3], [5], [6], [7], [16]. By using regular summability methods Alemdar and Duman [1] studied the Bernstein-Chlodowsky operators and Korovkin-type approximation theory. Following that study we modified Bernstein-Chlodowsky operators via weaker condition than the classical Bernstein-Chlodowsky operators' condition.

2 Preliminaries

A double sequence $x=\left(x_{m,n}\right)$ is convergent to L in Pringsheim's sense (or simply P-convergent) if, for every $\varepsilon>0$, there exists $N=N(\varepsilon)\in\mathbb{N}$ such that $\left|x_{m,n}-L\right|<\varepsilon$ whenever $m,n>N$ and denoted by $P-\lim\limits_{m,n}x_{m,n}=L$ (see [20]). A double sequence is bounded if there exists a positive number M such that $\left|x_{m,n}\right|\le M$ for all $(m,n)\in\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$. As it is well known, a convergent single sequence is bounded whereas a convergent double sequence need not to be bounded.

Let $A=(a_{i,j,m,n})$ be a four-dimensional summability method. For a given double sequence $x=(x_{m,n})$, the A-transform of x, denoted by $Ax:=((Ax)_{i,j})$, is given by

$$(Ax)_{i,j}=\sum_{m,n=1}^{\infty}a_{i,j,m,n}x_{m,n}$$

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provided the double series converges in the Pringsheim's sense for $(i,j) \in \mathbb{N}^2$.

Recall that a four-dimensional matrix $A = (a_{i,j,m,n})$ is said to be *RH-regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison-Hamilton conditions (see also [15, 21]) state that a four-dimensional matrix $A = (a_{i,j,m,n})$ is RH-regular if and only if

(i) $P - \lim_{i,j} a_{i,j,m,n} = 0$ for each m and n,

(ii) $P - \lim_{i,j} \sum_{m,n=1}^{\infty} a_{i,j,m,n} = 1$,

(iii) $P - \lim_{i,j} \sum_{m=1}^{\infty} \left|a_{i,j,m,n}\right| = 0$ for each $n \in \mathbb{N}$,

(iv) $P - \lim_{i,j} \sum_{n=1}^{\infty} \left|a_{i,j,m,n}\right| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{m,n=1}^{\infty} \left|a_{i,j,m,n}\right|$ is P-convergent,

(vi) there exits finite positive integers M_1 and M_2 such that $\sum_{m,n>M_1} \left|a_{i,j,m,n}\right| < M_2$ holds for every $(i,j) \in \mathbb{N}^2$.

Another summability method is power series method. The power series method includes many known summability methods such as Abel and Borel (we give the details in Remark 1), both methods are not matrix methods.

Let $\left(p_{m,n}\right)$ be a double sequence of nonnegative numbers with $p_{0,0} > 0$ and such that the following power series

$$p(r_1, r_2) := \sum_{m,n=0}^{\infty} p_{m,n} r_1^m r_2^n$$

has radius of convergence R with $R \in (0, \infty]$ and $r_1, r_2 \in (0, R)$. If, for all $r_1, r_2 \in (0, R)$,

$$\lim_{r_1, r_2 \to R^-} \frac{1}{p(r_1, r_2)} \sum_{m,n=0}^{\infty} p_{m,n} r_1^m r_2^n x_{m,n} = L$$

is satisfied then we say that a double sequence $x = \left(x_{m,n}\right)$ is convergent to L in the sense of power series method [8]. The power series method for double sequences is regular if and only if

$$\lim_{r_1, r_2 \to R^-} \frac{\sum_{m=0}^{\infty} p_{m,\upsilon} r_1^m}{p(r_1, r_2)} = 0 \text{ and } \lim_{r_1, r_2 \to R^-} \frac{\sum_{n=0}^{\infty} p_{\mu,n} r_2^n}{p(r_1, r_2)} = 0, \text{ for any } \mu, \upsilon,$$

hold [8].

Throughout the paper we assume that power series method is regular.

Remark 1. Note that in the case of $R = 1$, $p_{m,n} = 1$ and $p_{m,n} = \frac{1}{(m+1)(n+1)}$, the power series method coincides with Abel summability method and logarithmic summability method, respectively. For $R = \infty$ and $p_{m,n} = \frac{1}{m!n!}$, the power series method coincides with Borel summability method.

It is worthwhile to point out that, Ünver and Orhan [26] have recently introduced P_p-density of $E \subset \mathbb{N}_0$ and the definition of P_p-statistical convergence for single sequences. Hence, they showed that statistical convergence and P_p-statistical convergence are incompatible. In view of their work, Yıldız et al. [27] have introduced the definitions of P_p^2-density of $F \subset \mathbb{N}_0^2 = \mathbb{N}_0 \times \mathbb{N}_0$ and P_p^2-statistical convergence for double sequences:

Definition 2.1. [27] Let $F \subset \mathbb{N}_0^2$. If the limit

$$\delta_{P_p}^2(F) := \lim_{r_1, r_2 \to R^-} \frac{1}{p(r_1, r_2)} \sum_{(m,n) \in F} p_{m,n} r_1^m r_2^n$$

exists, then $\delta_{P_p}^2(F)$ is called the P_p^2-density of F. Note that, from the definition of a power series method and P_p^2-density it is obvious that $0 \le \delta_{P_p}^2(F) \le 1$ whenever it exists.

Definition 2.2. [27] Let $x = \left(x_{m,n}\right)$ be a double sequence. Then x is said to be statistically convergent with respect to power series method (P_p^2-statistically convergent) to L if for any $\varepsilon > 0$

$$\lim_{r_1, r_2 \to R^-} \frac{1}{p(r_1, r_2)} \sum_{(m,n) \in F_\varepsilon} p_{m,n} r_1^m r_2^n = 0$$

where $F_\varepsilon = \left\{(m,n) \in \mathbb{N}_0^2 : \left|x_{m,n} - L\right| \ge \varepsilon\right\}$, that is $\delta_{P_p}^2(F_\varepsilon) = 0$ for any $\varepsilon > 0$. In this case we write $st_{P_p}^2 - \lim x_{m,n} = L$.

Example 2.1. Let $(p_{m,n})$ be defined as follows

$$p_{m,n} = \begin{cases} 1, & m = 2k \text{ and } n = 2l \\ 0, & \text{otherwise} \end{cases}, \; k, l = 1, 2, ...,$$

and take the sequence $(x_{m,n})$ defined by

$$x_{m,n} = \begin{cases} 1, & m = 2k \text{ and } n = 2l \\ \sqrt{mn}, & \text{otherwise} \end{cases}, \; k, l = 1, 2, ... \tag{1}$$

We can easily see that, since for any $\varepsilon > 0$,

$$\lim_{r_1, r_2 \to R^-} \frac{1}{p(r_1, r_2)} \sum_{\{(m,n): |x_{m,n} - 1| \geq \varepsilon\}} p_{m,n} r_1^m r_2^n = 0,$$

$(x_{m,n})$ is P_p^2-statistically convergent to 1. However, the sequence $(x_{m,n})$ is not Pringsheim convergent to 1.

3 Modified Operators and Approximation Theorems

In this section, we will modify the Bernstein-Cholodowsky operators defined for double sequences with the regular method and give their properties by a weaker condition than for the classical Bernstein-Chlodowsky operators:

$$C_{m,n}(f; y, z) = \sum_{k=1}^{m} \sum_{l=1}^{n} f\left(\frac{k\alpha_m}{m}, \frac{l\beta_n}{n}\right) \binom{m}{k} \left(\frac{y}{\alpha_m}\right)^k \left(1 - \frac{y}{\alpha_m}\right)^{m-k} \binom{n}{l} \left(\frac{z}{\beta_n}\right)^l \left(1 - \frac{z}{\beta_n}\right)^{n-l}$$

Now, we give the following new modified operator with the condition:

$$\widetilde{C}_{i,j}(f; y, z) = \sum_{m,n=1}^{\infty} a_{i,j,m,n} C_{m,n}(f; y, z), \tag{2}$$

$$P - \lim_{i,j} \sum_{m,n=1}^{\infty} a_{i,j,m,n} \frac{\alpha_m}{m} = 0, \; P - \lim_{i,j} \sum_{m,n=1}^{\infty} a_{i,j,m,n} \frac{\beta_n}{n} = 0. \tag{3}$$

Lemma 3.1. *Let $e_{kl}(y, z) = y^k z^l$, $k, l \in \mathbb{N}$, be the two-dimensional test functions. The bivariate modified Bernstein-Chlodowsky operators defined in (2) satisfy the equalities*

(i) $\widetilde{C}_{i,j}(e_{00}; y, z) = \sum_{m,n=1}^{\infty} a_{i,j,m,n}$,

(ii) $\widetilde{C}_{i,j}(e_{10}; y, z) = y \sum_{m,n=1}^{\infty} a_{i,j,m,n}$,

(iii) $\widetilde{C}_{i,j}(e_{01}; y, z) = z \sum_{m,n=1}^{\infty} a_{i,j,m,n}$,

(iv) $\widetilde{C}_{i,j}(e_{20}; y, z) = y^2 \sum_{m,n=1}^{\infty} a_{i,j,m,n} - y^2 \sum_{m,n=1}^{\infty} a_{i,j,m,n} \frac{1}{m} + y \sum_{m,n=1}^{\infty} a_{i,j,m,n} \frac{\alpha_m}{m}$,

(v) $\widetilde{C}_{i,j}(e_{02}; y, z) = z^2 \sum_{m,n=1}^{\infty} a_{i,j,m,n} - z^2 \sum_{m,n=1}^{\infty} a_{i,j,m,n} \frac{1}{n} + z \sum_{m,n=1}^{\infty} a_{i,j,m,n} \frac{\beta_n}{n}$.

Lemma 3.2. *The bivariate modified Bernstein-Chlodowsky operators (2) satisfy the relations*

$$\widetilde{C}_{i,j}((e_{10} - y)^2; y, z) = y \sum_{m,n=1}^{\infty} a_{i,j,m,n} \frac{\alpha_m}{m} - y^2 \sum_{m,n=1}^{\infty} a_{i,j,m,n} \frac{1}{m}, \tag{4}$$

$$\widetilde{C}_{i,j}((e_{01} - z)^2; y, z) = z \sum_{m,n=1}^{\infty} a_{i,j,m,n} \frac{\beta_n}{n} - z^2 \sum_{m,n=1}^{\infty} a_{i,j,m,n} \frac{1}{n}. \tag{5}$$

Proof. Since $\widetilde{C}_{i,j}$ is linear, we have

$$\widetilde{C}_{i,j}((e_{10} - y)^2; y, z) = \widetilde{C}_{i,j}(e_{20}; y, z) - 2y\widetilde{C}_{i,j}(e_{10}; y, z) + y^2 \widetilde{C}_{i,j}(e_{10}; y, z).$$

By applying Lemma 3.1, we get relation (4). Similary we have the equality (5). □

Let for any $\alpha_m > 0$ and $\beta_n > 0$, $I_{\alpha_m\beta_n} := [0,\alpha_m]\times[0,\beta_n]$ and the space $C(I_{\alpha_m\beta_n})$ be the set of all real-valued functions of bivariate continuous on $I_{\alpha_m\beta_n}$. $C(I_{\alpha_m\beta_n})$ is a linear normed space with the norm

$$\|f\|_{C(I_{\alpha_m\beta_n})} = \max_{(x,y)\in[0,\alpha_m]\times[0,\beta_n]} |f(x,y)|.$$

If we consider the condition (3) and Lemma 3.1, then we immediately get the following classical result for our new defined operators:

Corollary 3.3. *Let b,d are any suficiently large fixed positive real numbers such that $b \le \alpha_m$ and $d \le \beta_n$, $f \in C(I_{bd})$, then*

$$P - \lim_{i,j} \left\|\widetilde{C}_{i,j}(f) - f\right\|_{C(I_{bd})} = 0.$$

Yıldız et al. ([27]) have obtained an abstract version of the Korovkin type approximation theorems with respect to the concept of P_p-statistical convergence in modular spaces for double sequences of positive linear operators. As it is well-known results that all normed spaces are also modular spaces. Hence, this type Korovkin Theorem can be given as follows:

Theorem 3.4. *(see also[27]) Let $\left(L_{i,j}\right)$ be a double sequence of positive linear operators acting from $C([a,b]\times[c,d])$ into itself. Then, for all $f \in C([a,b]\times[c,d])$,*

$$st^2_{P_p} - \lim \left\|L_{i,j}(f) - f\right\|_{C([a,b]\times[c,d])} = 0$$

if and only if

$$st^2_{P_p} - \lim \left\|L_{i,j}(f_r) - f_r\right\|_{C([a,b]\times[c,d])} = 0, \quad r = 0,1,2,3,$$

where $f_0(y,z) = 1$, $f_1(y,z) = y$, $f_2(y,z) = z$ and $f_3(y,z) = y^2 + z^2$.

Let (α_m) and (β_n) are sequences of positive real numbers such that $\lim_m \alpha_m = \lim_n \beta_n = +\infty$ and also let $A = (a_{i,j,m,n})$ be a nonnegative RH-regular summability matrix such that

$$st^2_{P_p} - \lim \sum_{m,n=1}^{\infty} a_{i,j,m,n}\frac{\alpha_m}{m} = 0,\ st^2_{P_p} - \lim \sum_{m,n=1}^{\infty} a_{i,j,m,n}\frac{\beta_n}{n} = 0. \tag{6}$$

Based on this theorem, we get the following approximation result:

Theorem 3.5. *Let $f \in C(I_{bd})$ then*

$$st^2_{P_p} - \lim \left\|\widetilde{C}_{i,j}(f) - f\right\|_{C(I_{bd})} = 0.$$

Proof. We now claim that

$$st^2_{P_p} - \lim \left\|\widetilde{C}_{i,j}(f_r) - f_r\right\|_{C(I_{bd})} = 0, \quad r = 0,1,2,3. \tag{7}$$

Indeed, in view of (i) from Lemma 3.1 and RH-regularity of $A = (a_{i,j,m,n})$, it is clear that

$$st^2_{P_p} - \lim \left\|\widetilde{C}_{i,j}(f_0) - f_0\right\|_{C(I_{bd})} = 0$$

which guarantees that (7) holds true for $r = 0$. Similary we have

$$st^2_{P_p} - \lim \left\|\widetilde{C}_{i,j}(f_1) - f_1\right\|_{C(I_{bd})} = 0,$$
$$st^2_{P_p} - \lim \left\|\widetilde{C}_{i,j}(f_2) - f_2\right\|_{C(I_{bd})} = 0,$$

that is (7) holds true for $r = 1,2$. Finally, since

$$\begin{aligned}
&\left\|\widetilde{C}_{i,j}(f_3) - f_3\right\|_{C(I_{bd})} \\
&\le \left\|\widetilde{C}_{i,j}(e_{20}) - e_{20}\right\|_{C(I_{bd})} + \left\|\widetilde{C}_{i,j}(e_{02}) - e_{02}\right\|_{C(I_{bd})} \\
&\le \left(b^2 + d^2\right)\left|\sum_{m,n=1}^{\infty} a_{i,j,m,n} - 1\right| + b\sum_{m,n=1}^{\infty} a_{i,j,m,n}\frac{\alpha_m}{m} \\
&+ b^2\sum_{m,n=1}^{\infty} a_{i,j,m,n}\frac{1}{m} + d\sum_{m,n=1}^{\infty} a_{i,j,m,n}\frac{\beta_n}{n} + d^2\sum_{m,n=1}^{\infty} a_{i,j,m,n}\frac{1}{n}.
\end{aligned}$$

Then, from RH-regularity of $A = (a_{i,j,m,n})$ and condition (6), we get that

$$st^2_{P_p} - \lim \left\|\widetilde{C}_{i,j}(f_3) - f_3\right\|_{C(I_{bd})} = 0,$$

that is (7) holds true for $r = 3$. As a result $\left(\widetilde{C}_{i,j}\right)$ satisfies all hypothesis of Theorem 3.4 which finishes the proof. □

Example 3.1. Let us define the following type operators

$$L_{i,j}(f;y,z) = x_{i,j}\widetilde{C}_{i,j}(f;y,z)$$

where $(x_{i,j})$ given by (1) in Example 2.1. Then, by the $(x_{i,j})$ and Lemma 3.1 it can be easily seen that the operators $L_{i,j}$ satisfy all hypothesis of Theorem 3.5 and we obtain

$$st^2_{P_p} - \lim \left\| L_{i,j}(f) - f \right\|_{C(I_{bd})} = 0$$

but the operators $L_{i,j}$ do not satisfy Corollary 3.3. Hence, it is not Pringsheim convergent but statistically convergent with respect to power series method.

4 Rate of convergence

This section is devoted to computing the rate of P_p-statistical convergence with the aid of the modulus of continuity and Lipschitz class.

As it is well known, there are different types of modulus continuity. Now, we begin with followings:

The complete modulus of continuity is

$$\omega(f,\delta) = \sup_{\sqrt{(y_1-y_2)^2+(z_1-z_2)^2}\le\delta} |f(y_1,z_1) - f(y_2,z_2)| \quad (\delta > 0),\ f \in C(I_{bd})$$

and its partial continuity moduli with respect to y and z are

$$\omega_1(f,\delta) = \sup_{0\le z\le d}\ \sup_{|y_1-y_2|\le\delta} |f(y_1,z) - f(y_2,z)|,$$

$$\omega_2(f,\delta) = \sup_{0\le y\le b}\ \sup_{|z_1-z_2|\le\delta} |f(y,z_1) - f(y,z_2)|.$$

It is readily seen that, for any $\lambda > 0$, $\omega(f,\lambda\delta) \le (1+[\lambda])\,\omega(f,\delta)$, where $[\lambda]$ is defined to be the greatest integer less than or equal to λ. The same property is satisfied by the partial modulus of continuity.

Now, let us give the following result.

Theorem 4.1. *Let $f \in C(I_{bd})$. If $st^2_{P_p} - \lim \omega(f,\delta_{i,j}) = 0$, then*

$$st^2_{P_p} - \lim \left\| \widetilde{C}_{i,j}(f) - f \right\|_{C(I_{bd})} = 0,$$

where

$$\delta_{i,j} := \left\{ b \sum_{m,n=1}^{\infty} a_{i,j,m,n}\frac{\alpha_m}{m} + d \sum_{m,n=1}^{\infty} a_{i,j,m,n}\frac{\beta_n}{n} \right\}^{\frac{1}{2}}$$

for any positive integers $i, j \in \mathbb{N}$.

Proof. Utilizing the well-known properties of ω, and the positivity and linearity of $\widetilde{C}_{i,j}$, we get

$$\begin{aligned}
\left|\widetilde{C}_{i,j}(f;y,z) - f(y,z)\right| &\le \widetilde{C}_{i,j}\left(|f(s,t) - f(y,z)|;y,z\right) + |f(y,z)| \left| \sum_{m,n=1}^{\infty} a_{i,j,m,n} - 1 \right| \\
&\le \widetilde{C}_{i,j}\left(\left(1 + \frac{(s-y)^2 + (t-z)^2}{\delta^2}\right)\omega(f,\delta); y,z \right) \\
&+ |f(y,z)| \left| \sum_{m,n=1}^{\infty} a_{i,j,m,n} - 1 \right| \\
&= \omega(f,\delta)\sum_{m,n=1}^{\infty} a_{i,j,m,n} + \frac{\omega(f,\delta)}{\delta^2}\widetilde{C}_{i,j}\left((s-y)^2 + (t-z)^2; y,z\right) \\
&+ |f(y,z)| \left| \sum_{m,n=1}^{\infty} a_{i,j,m,n} - 1 \right|.
\end{aligned}$$

Then taking supremum over $(y,z)\in I_{bd}$, we have

$$\begin{aligned}
&\left\|\widetilde{C}_{i,j}(f)-f\right\|_{C(I_{bd})}\\
&\leq \omega(f,\delta)\sum_{m,n=1}^{\infty}a_{i,j,m,n}+\frac{\omega(f,\delta)}{\delta^2}\left\{\left\|\widetilde{C}_{i,j}\left((s-.)^2\right)\right\|_{C(I_{bd})}+\left\|\widetilde{C}_{i,j}\left((t-.)^2\right)\right\|_{C(I_{bd})}\right\}\\
&+\|f\|_{C(I_{bd})}\left|\sum_{m,n=1}^{\infty}a_{i,j,m,n}-1\right|\\
&\leq \omega(f,\delta)\sum_{m,n=1}^{\infty}a_{i,j,m,n}+\frac{\omega(f,\delta)}{\delta^2}\left\{b\sum_{m,n=1}^{\infty}a_{i,j,m,n}\frac{\alpha_m}{m}+\sum_{m,n=1}^{\infty}a_{i,j,m,n}d\frac{\beta_n}{n}\right\}\\
&+\|f\|_{C(I_{bd})}\left|\sum_{m,n=1}^{\infty}a_{i,j,m,n}-1\right|.
\end{aligned}$$

Taking

$$\delta=\delta_{i,j}:=\left\{b\sum_{m,n=1}^{\infty}a_{i,j,m,n}\frac{\alpha_m}{m}+d\sum_{m,n=1}^{\infty}a_{i,j,m,n}\frac{\beta_n}{n}\right\}^{\frac{1}{2}},$$

we get for any positive integers i,j that

$$\begin{aligned}
&\left\|\widetilde{C}_{i,j}(f)-f\right\|_{C(I_{bd})}\\
&\leq \omega(f,\delta_{i,j})\left(\sum_{m,n=1}^{\infty}a_{i,j,m,n}+1\right)+\|f\|_{C(I_{bd})}\left|\sum_{m,n=1}^{\infty}a_{i,j,m,n}-1\right|.
\end{aligned}$$

From the hypothesis and RH-regularity of $A=(a_{i,j,m,n})$ it follows that

$$st^2_{P_p}-\lim\left\|\widetilde{C}_{i,j}(f)-f\right\|_{C(I_{bd})}=0.$$

□

Theorem 4.2. *Let $f\in C(I_{bd})$. If*

$$st^2_{P_p}-\lim\omega_1(f,\delta_i)=0 \text{ and } st^2_{P_p}-\lim\omega_2(f,\delta_j)=0$$

then

$$st^2_{P_p}-\lim\left\|\widetilde{C}_{i,j}(f)-f\right\|_{C(I_{bd})}=0,$$

where

$$\delta_i=\left\{b\sum_{m,n=1}^{\infty}a_{i,j,m,n}\frac{\alpha_m}{m}\right\}^{\frac{1}{2}},\ \delta_j=\left\{d\sum_{m,n=1}^{\infty}a_{i,j,m,n}\frac{\beta_n}{n}\right\}^{\frac{1}{2}}$$

for any positive integers $i,j\in\mathbb{N}$.

Proof. In the same way, we obtain, for any positive integers i,j, that

$$\begin{aligned}
&\left\|\widetilde{C}_{i,j}(f)-f\right\|_{C(I_{bd})}\\
&\leq \left(\omega_1(f,\delta_i)+\omega_2(f,\delta_j)\right)\left(\sum_{m,n=1}^{\infty}a_{i,j,m,n}+1\right)+\|f\|_{C(I_{bd})}\left|\sum_{m,n=1}^{\infty}a_{i,j,m,n}-1\right|
\end{aligned}$$

where

$$\delta_i=\left\{b\sum_{m,n=1}^{\infty}a_{i,j,m,n}\frac{\alpha_m}{m}\right\}^{\frac{1}{2}},\ \delta_j=\left\{d\sum_{m,n=1}^{\infty}a_{i,j,m,n}\frac{\beta_n}{n}\right\}^{\frac{1}{2}}.$$

Hence, we get the desired result. □

Now, we give the rate of P^2_p-statistical convergence via Lipschitz class $Lip_M(\gamma)$ of the functions of f of two variables is given by

$$|f(y_1,z_1)-f(y_2,z_2)|\leq M\left\{(y_1-y_2)^2+(z_1-z_2)^2\right\}^{\frac{\gamma}{2}}$$

where $M>0$, $\gamma\in(0,1]$ and $f\in C(I_{bd})$.

Then, we have the following theorem.

Theorem 4.3. *Let $f\in Lip_M(\gamma)$. Then*

$$st^2_{P_p}-\lim\left\|\widetilde{C}_{i,j}(f)-f\right\|_{C(I_{bd})}=0.$$

Proof. Taking into account that $f \in Lip_M(\gamma)$. We have

$$\begin{aligned}\left|\widetilde{C}_{i,j}(f;y,z)-f(y,z)\right| &\leq \widetilde{C}_{i,j}(|f(s,t)-f(y,z)|;y,z)+|f(y,z)|\left|\sum_{m,n=1}^{\infty}a_{i,j,m,n}-1\right| \\ &\leq M\widetilde{C}_{i,j}\left(\left\{(y_1-y_2)^2+(z_1-z_2)^2\right\}^{\frac{\gamma}{2}};y,z\right) \\ &+|f(y,z)|\left|\sum_{m,n=1}^{\infty}a_{i,j,m,n}-1\right|.\end{aligned}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}\left|\widetilde{C}_{i,j}(f;y,z)-f(y,z)\right| &\leq M\left\{\widetilde{C}_{i,j}\left(\left\{(y_1-y_2)^2+(z_1-z_2)^2\right\};y,z\right)\right\}^{\frac{\gamma}{2}} \\ &+|f(y,z)|\left|\sum_{m,n=1}^{\infty}a_{i,j,m,n}-1\right|.\end{aligned}$$

Then taking supremum over $(y,z)\in I_{bd}$, we have

$$\begin{aligned}\left\|\widetilde{C}_{i,j}(f)-f\right\|_{C(I_{bd})} &\leq M\left\{b\sum_{m,n=1}^{\infty}a_{i,j,m,n}\frac{\alpha_m}{m}+\sum_{m,n=1}^{\infty}a_{i,j,m,n}d\frac{\beta_n}{n}\right\}^{\frac{\gamma}{2}} \\ &+\|f\|_{C(I_{bd})}\left|\sum_{m,n=1}^{\infty}a_{i,j,m,n}-1\right|.\end{aligned}$$

Hence, we get the desired result. □

Corollary 4.4. *If f satisfies the Lipschitz conditions*

$$\begin{aligned}|f(y_1,z)-f(y_2,z)| &\leq M_1|y_1-y_2|^{\gamma_1}, \\ |f(y,z_1)-f(y,z_2)| &\leq M_2|z_1-z_2|^{\gamma_2},\end{aligned}$$

where $M_1,M_2>0$, $\gamma_1,\gamma_2\in(0,1]$. Then

$$st^2_{P_p}-\lim\left\|\widetilde{C}_{i,j}(f)-f\right\|_{C(I_{bd})}=0.$$

5 An Application

Let $(p_{m,n})$ be defined as follows

$$p_{m,n}=\begin{cases}1, & m=2k \text{ and } n=2l \\ 0, & \text{otherwise}\end{cases}, \quad k,l=1,2,...,$$

and sequences of positive real numbers (α_m) and (β_n) given by

$$\alpha_m=\begin{cases}\sqrt{m}, & m \text{ is even,} \\ m, & m \text{ is odd,}\end{cases} \quad \text{and } \beta_n=\begin{cases}\sqrt{n}, & n \text{ is even,} \\ n, & n \text{ is odd.}\end{cases}$$

Also, *assume that $A=(a_{i,j,m,n})$ is a four dimensional infinite matrix defined by $a_{i,j,m,n}=\frac{1}{ij}$ if $1\leq m\leq i$, $1\leq n\leq j$ and $a_{i,j,m,n}=0$ otherwise. Then, we check that A* is a nonnegetive *RH*-regular method and $\lim\limits_m \alpha_m=\lim\limits_n \beta_n=+\infty$. We also obtain that

$$st^2_{P_p}-\lim\sum_{m,n=1}^{\infty}a_{i,j,m,n}\frac{\alpha_m}{m}=0,\ st^2_{P_p}-\lim\sum_{m,n=1}^{\infty}a_{i,j,m,n}\frac{\beta_n}{n}=0$$

which gives the condition (6). Our operators $\widetilde{C}_{i,j}$ map the function space $C(I_{bd})$ into itself. Then, all conditions of Theorem 3.5 are satisfied for these operators. Hence we get that

$$st^2_{P_p}-\lim\left\|\widetilde{C}_{i,j}(f)-f\right\|_{C(I_{bd})}=0.$$

6 Concluding Remarks

In recent years, many generalizations of well-known linear positive operators, based on q-calculus were introduced and studied by several authors. In 1996, Philips by using the q-binomial coefficients and the q-binomial theorem introduced a generalization of the Bernstein operators called q-Bernstein Operators [19]. q-Bernstein-Chlodowsky polynomials defined by Karsli Gupta in the one-dimensional case [16]. They investigated approximation properties for these new polynomials. Buyukyazici introduced the two-dimensional q analogue of Bernstein-Chlodowsky polynomial in [5].

With the thought in this study, the q-Bernstein-Chlodowsky operators are also modified by means of the RH-regular matrix, and similar results to those in this study are also obtained for this operator as well.

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