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Construction of Hurwitz Stability Intervals for Matrix Families

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Abstract

In this work, matrix families which consist of linear sum and convex combination have been determined. In this process, Hurwitz stability, sensitivity and continuity theorems have been mentioned. The intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ of the matrix families have been determined so that the linear sum family $\mathcal{L}(A_1,A_2)$ and convex combination family $\mathcal{C}(A_1,A_2)$ are Hurwitz stable. A method which based on continuity theorem have been given to extend the intervals $\mathcal{I}_{\mathcal{L}}$. The extended interval $\mathcal{I}^e_{\mathcal{L}}$ which preserve Hurwitz stability was obtained. An algorithm which based on the method have been given. Finally, the results have been supported with the examples.

1 Introduction

Determining the stability of matrix families is one of the real problems of stability analysis. In this paper we will firstly present the matrix families $\mathcal{L}(A_1,A_2)$ and $\mathcal{C}(A_1,A_2)$ which consist of linear sum and convex combination, respectively. The intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ will be determined to make these matrix families Hurwitz stable. A method will be given to extend the intervals $\mathcal{I}_{\mathcal{L}}$. An algorithm which based on the method will be given to obtain the extended interval $\mathcal{I}^e_{\mathcal{L}}$. Although there are many studies on linear sum and convex combination in the literature [1, 2], the difference of this paper is the use of continuity theorems for Hurwitz stability.

In the literature, to be $A \in M_N(\mathbb{C})$, it is well known that the stability of the matrix A. According to Lyapunov's theorem, the matrix A is asymptotic stable if and only if eigenvalues of the matrix A lay in the left open half-plane, that is, $\operatorname{Re}\lambda_i(A) < 0$ for all $i = 1,2,\ldots,N$, where $\lambda_i (i = 1,2,\ldots,N)$ stands for the eigenvalues of the coefficient matrix A [3, 4]. Let's give the family of Hurwitz stable matrices as follows;

$$H_N = \{A \in M_N(\mathbb{C}) \mid \operatorname{Re}\lambda_i(A) < 0 \ (i = 1,2,...,N)\}. \tag{1}$$

$A \in H_N$ if $\sigma(A) \subset C_H = \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$ which is called spectral criterion, where $\sigma(A) = \{\lambda \mid \lambda = \lambda_i(A)\}$ is spectrum of the matrix A [5]. In practice it is not so easy to determine the eigenvalue therefore the eigenvalue problem isn't a well-posed problem. Let's take $A_w = (a_{ij}) \in M_N(\mathbb{R})$,

$$a_{ij} = \begin{cases} -N+i-1 & i=j \\ (i+1.5)\times 10 & j=i+1 \ (i \le N-1) \\ w & i=N,\ j=1 \\ 0 & others \end{cases}. \tag{2}$$

The matrix A_0 is the Hurwitz stable but the matrix $A_{1.5\times10^{1-N}}$ is non-Hurwitz stable since $0.5 \in \sigma(A_{1.5\times10^{1-N}})$ [3, 4, 6]. Therefore, for the determination of stability, it is more convenient to use the parameters calculated with the help of the solution of a linear algebraic equation which characterizing the stability. With this notion for linear systems, the stability problem is reduced to the problem of the existence of a positive definite solution.

$$A^*F + FA + I = 0\ ,\ F = \int_0^\infty \left(e^{tA}\right)^* e^{tA} dt\ ,\ F = F^* > 0 \tag{3}$$

According to Lyapunov's theorem, if the Lyapunov matrix equation, which determines the Hurwitz stability, has a solution then the matrix A is said to be Hurwitz stable [3, 4, 7]. However, existence of F does not enough to comment on quality. So we need a parameter which determines the quality of the Hurwitz stability.

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Table 1: The quality of Hurwitz stability of the matrix A_k

k	1	2	3	4
$\kappa(A_k)$	1	382.556	485348	4.98503e+008

Hurwitz stability parameter $\kappa(A)$ is defined as $\kappa(A) = 2\|A\|\|F\| \geq 1$, where $\|.\|$ is spectral norm and it is defined as $\|A\| = \max_{\|x\|=1} \|Ax\|$ [3, 8, 9]. The quality of the Hurwitz stability decreases as the values of the parameter $\kappa(A)$ increases.

It is obvious that the equality (1) is equal the follow statement,

$$H_N = \{A \in M_N(\mathbb{C}) \mid \kappa(A) < \infty\}.$$

Let's examine the following the matrices in order to see the notion of quality of Hurwitz stability more easily. Let's take $A_k \in H_N$ as follow

$$A_k = \begin{pmatrix} -1 & 10^{k-1}-1 \\ 0 & -1 \end{pmatrix}, \ k \in \mathbb{N}.$$

It is clear that, although $\sigma(A_k) = \{-1\}$ for $k \in \mathbb{N}$, it can be seen from the Table 1 that the values of $\kappa(A_k)$ also increase as the values of k increase. Also the quality of the Hurwitz stability increases as it approaches 1.

Moreover, κ^* to be the practical Hurwitz stability parameter, where $1 < \kappa^* \in \mathbb{R}$ and the users choose the value κ^* in view of their problem. If $\kappa(A) \leq \kappa^*$ then the matrix A is κ^*- Hurwitz stable matrix. Otherwise, the matrix A is κ^*- Hurwitz unstable matrix [8, 9, 10].

In this paper, Hurwitz stability of the matrix families which consist of linear sum and convex combination were discussed. In Section 2, the matrix families $\mathcal{L}(A_1,A_2)$ and $\mathcal{C}(A_1,A_2)$ were introduced, the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ were determined to make these matrix families Hurwitz stable. Examples related to the subject were given. In Section 3, a method and an algorithm which based on the continuity theorem to extend the intervals $\mathcal{I}_{\mathcal{L}}$ was given. At the end of, numerical examples were given.

2 Hurwitz Stability of New Matrix Families

Before giving matrix families, let's give the continuity theorem which determining the sensitivity of the stability. We use this theorem for Hurwitz stability. Let's remember the set of Hurwitz stable matrices as follows;

$$H_N = \{A \in M_N(\mathbb{C}) \mid \kappa(A) < \infty\}.$$

Theorem 2.1. *Let $A \in H_N$. If $\|B\| < \frac{\|A\|}{\kappa(A)}$ then the matrix $A+B \in H_N$ and*

$$\kappa(A+B) \leqslant \frac{\kappa(A)(\|A\|+\|B\|)}{\|A\|-\|B\|\kappa(A)}$$

holds [11].

The two theorems, which will be given below, can be obtained from Theorem 2.1. Let's take the matrix families $\mathcal{L}(A_1,A_2)$ and $\mathcal{C}(A_1,A_2)$ which consist of linear sum and convex combination, respectively, as follow,

$$\mathcal{L}(A_1,A_2) = \{A(r) = A_1 + rA_2 \mid A_1,\ A_2 \in M_N(\mathbb{C})\} \tag{4}$$

and

$$\mathcal{C}(A_1,A_2) = \{A(r) = (1-r)A_1 + rA_2 \mid A_1,\ A_2 \in M_N(\mathbb{C})\}. \tag{5}$$

Let give the conditions and the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ that will make the matrix families $\mathcal{L}(A_1,A_2)$ and $\mathcal{C}(A_1,A_2)$ Hurwitz stable as follow theorems.

Theorem 2.2. *Let's $A_1 \in H_N$, $A_2 \in M_N(\mathbb{C})$ and $r \in \mathcal{I}_{\mathcal{L}} = [\underline{r},\overline{r}]$ then the matrix family $\mathcal{L}(A_1,A_2)$ is Hurwitz stable, where $-l = u = \frac{\|A_1\|}{\|A_2\|\kappa(A_1)}$, $l < \underline{r} < \overline{r} < u$.*

Proof. If $A_2 = 0$ then $A(r) = A_1$ and we know that $A_1 \in H_N$ so $A(r) \in H_N$ too. Let us consider the given linear sum as follow

$$A(r) = A_1 + rA_2.$$

If we substitute it in the Lyapunov equation

$$(A_1 + rA_2)^* F + F(A_1 + rA_2) + I = 0$$

$$A_1^* F - FA_1 = -\left(I + r\left(A_2^* F + FA_2\right)\right).$$

At that rate,

$$C = I + r\left(A_2^* F + FA_2\right) > 0$$

$C = C^* > 0$ is available. The result obtained is written as follows

$$\|C\| \leq 1 + 2|r|\,\|A_2\|\,\|F\|$$

then, if the found inequality is substituted in the equation which is the solution of the Lyapunov equation

$$F = \int_0^\infty e^{tA^*} C e^{tA} dt$$

$$\|F\| \leq \|C\|\,\|F_1\|$$

$$\|F\| \leq (1 + 2|r|\,\|A_2\|\,\|F\|)\,\|F_1\|$$

$$\|F\| \leq \frac{\|F_1\|}{1 - 2|r|\,\|A_2\|\,\|F_1\|}$$

is obtained. We know that $\kappa(A_1) = 2\|A_1\|\,\|F_1\|$ and $\kappa(A) = 2\|A_1 + rA_2\|\,\|F\|$. If the inequality is arranged through these equalities, obtained as follows

$$\kappa(A) \leq \frac{(\|A_1\| + |r|\,\|A_2\|)\,\kappa(A_1)}{\|A_1\| - |r|\,\|A_2\|\,\kappa(A_1)}.$$

While $A_1 \in H_N$, the following condition must be verified for $A(r)$ to be Hurwitz stable

$$\frac{\|A_1\| - |r|\,\|A_2\|\,\kappa(A_1)}{(\|A_1\| + |r|\,\|A_2\|)\,\kappa(A_1)} > 0.$$

Then if the inequality found is arranged with according to r, Hurwitz stability intervals (l, u) of matrix $A(r)$ obtained, where

$$-l = u = \frac{\|A_1\|}{\|A_2\|\,\kappa(A_1)}.$$

□

Theorem 2.3. *Let's $A_1 \in H_N$, $A_2 \in M_N(\mathbb{C})$ and $r \in \mathcal{I}_{\mathcal{C}} = [\underline{r}, \overline{r}]$ then the matrix family $\mathcal{C}(A_1, A_2)$ is Hurwitz stable, where $-l = u = \frac{\|A_1\|}{\|A_2 - A_1\|\kappa(A_1)}$, $l < \underline{r} < \overline{r} < u$.*

Proof. If we write $A_2 - A_1$ instead of A_2 in Theorem 2.2, proof is clear from Theorem 2.2. □

Example 2.1. Let us consider

$$A_1 = \begin{pmatrix} -1 & \beta \\ \alpha & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For these matrices we know that $\|A_2\| = 1$. Also we will chose the matrix A_1 is Hurwitz stable. According to the Theorem 2.2, choosing $\alpha = \beta = 0$, we get the values $-l = u = 1$; choosing $\alpha = 10$, $\beta = 0$ or $\alpha = 0$, $\beta = 10$, we get the values $-l = u = 0.0560947$; choosing $\alpha = \beta = 1$, we get the values $-l = u = 0.381966$. As can be seen here, choosing the values α and β, we get different intervals which satisfy the Hurwitz stability of the matrix family $\mathcal{L}(A_1, A_2)$.

One of the important point of the article, as can be seen here, it is possible to write the convex combination as a special case of the linear sum. In other words, we can express the convex combination given as $A(r) = (1 - r)A_1 + rA_2$ as a linear sum as $A(r) = A_1 + r(A_2 - A_1)$. The other important point, in order to the matrix $A(r)$ to be Hurwitz stable, determine the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ using the Hurwitz stability of the matrix A_1. So, at the same time, there is no need the stability of the matrix A_2.

3 Extend of the Intervals

In the above section, the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ are found which preserve Hurwitz stability of matrix families. It is possible to extend these intervals with a certain rule. In this section, the extended interval for the matrix families are given which preserve the Hurwitz stability. On the other hand, in the literature, there are studies testing the Hurwitz stability of the interval matrices (see. [12], [13]). The extended intervals to be obtained in this section also allow us to introduce the interval matrices with Hurwitz stable, unlike testing the Hurwitz stability of the given interval matrices. To obtain the intervals, firstly, a method which based on continuity theorems is given and afterwards, an algorithm which based on the method is given in here. So it can be obtained bigger intervals which preserve the Hurwitz stability of matrix families. In this process, the stepsize is determined from the continuity theorems which are Theorem 2.2 and Theorem 2.3. At the end of processing, the extended intervals are obtained. These intervals are denoted by $\mathcal{I}^e_{\mathcal{L}}$ and $\mathcal{I}^e_{\mathcal{C}}$. Let's give a method as below.

3.1 A method to find the extended interval $\mathcal{I}^e_{\mathcal{L}}$

In this section, a method is given to extend the intervals obtained by Theorem 2.2 with the Hurwitz stable matrix A_1 and the matrix B, keeping the Hurwitz stability of the matrix family $\mathcal{L}(A_1, B)$.

- $\mathcal{I}_{\mathcal{L}} = [\underline{r}, \overline{r}]$ have been created as $l \lessapprox \underline{r}$ and $\overline{r} \lessapprox u$ in Theorem 2.2. For $r \in \mathcal{I}_{\mathcal{L}}$, the matrices $A(r) = A_r = A_1 + rB$ are Hurwitz stable. The parameter r which indicate the stepsize is used to extend the interval. The initial value of r is chosen as $r = \overline{r}$.
- To extend the upper bound $\overline{r}$ of the intervals $\mathcal{I}_{\mathcal{L}}$, the following steps are done,

$$A_k = A_{k-1} + r_{k-1}B \ , \ r_1 = \overline{r}, \ k \geq 2 \tag{6}$$

$$u_k = \frac{||A_k||}{||B|| \, \kappa(A_k)}, \tag{7}$$

$$r_k < u_k, \tag{8}$$

$$\overline{u}_k = \overline{u}_{k-1} + r_k \ , \ \overline{u}_1 = \overline{r} \tag{9}$$

  for $k \geq 2$.
- The new matrix A_k in the equality (6) is obtained as Hurwitz stable. u_k in equality (7) is calculated with Theorem 2.2. The stepsize r_k is chosen from the inequality (8). $\overline{u}_k$ is the upper bound of the extended interval obtained in step k. At the end of this process, the upper bound of the extended interval $\mathcal{I}^e_{\mathcal{L}}$ is obtained as $\overline{u}$.
- To extend the lower bound $\underline{r}$ of the intervals $\mathcal{I}_{\mathcal{L}}$, the following steps are done,

$$A_k = A_{k-1} - r_{k-1}B \ , \ r_1 = \overline{r}, \ k \geq 2 \tag{10}$$

$$u_k = \frac{||A_k||}{||B|| \, \kappa(A_k)}, \tag{11}$$

$$r_k < u_k, \tag{12}$$

$$\underline{l}_k = \underline{l}_{k-1} - r_k \ , \ \underline{l}_1 = \underline{r} \tag{13}$$

  for $k \geq 2$. Here, the matrix A_k given in (6) and (10) are the different matrices. At the end of this process, the lower bound of the extended interval $\mathcal{I}^e_{\mathcal{L}}$ is obtained as $\underline{l}$.

Remark 1. Let's take A_1 and B.

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From the Theorem 2.2, it is known that $u = 1$. This is also evident from the spectral criterion. Let's take $r = 0.9$ from the inequality (8). If the method is applied to get the upper bound, the stepsize is become smaller and the upper bound goes indefinitely as $\overline{u} = 0.\overline{9}$. Similarly, if the method is applied to get the lower bound, the stepsize is become bigger and as iteration continues indefinitely, the lower bound goes to the minus infinity. Because of these reasons, the working with unlimited intervals is non-practical. To avoid these situations, the stopping criteria are given in the next section. Also, there is a need for a certain rule regarding the selection of r_k since the r_k should be chosen close enough to the u_k. So, for $\gamma \lessapprox 1$, $r_k = \gamma . u_k$ is taken.

3.2 Algorithms

In this section, the stopping criteria are given and explains why these criteria are needed. These criteria prevent unnecessary processing. So these criteria helps the given method to run smoothly and to obtain bigger interval. After a certain step, the stepsize becomes too large or too small according to the criteria determined in each new values. Calculations with such values are not practical due to some reasons (i.e. floating point arithmetic). Let us give the stopping criteria to avoid endless loop as follows,

- r^* is called the practical parameter which chosen by user small enough [14]. If $r \geq r^*$ then the calculation continues. On the contrary, if $r < r^*$ then the calculation stops. If this parameter to be determined by the user is selected small enough, the interval can be obtained as a bigger interval.
- R is a positive number which chosen by the user big enough. If $r \leq R$ then the calculation continues. On the contrary, if $r > R$ then the calculation stops. If this parameter to be determined by the user is selected big enough, the interval can be obtained as a bigger interval.

- κ^* is practical Hurwitz stability parameter which chosen by user. If $\kappa(A_r) \leq \kappa^*$ then the calculation continues. On the contrary, if $\kappa(A_r) > \kappa^*$ then the calculation stops. Too large the value of $\kappa(A_r)$ degrades the quality of Hurwitz stability. If this parameter to be determined by the user is selected big enough, the interval can be obtained as a bigger interval.

  Let's give the following algorithms through the method.

Algorithm 1
Let's give the following algorithm to extend the upper bound of the intervals $\mathcal{I}_{\mathcal{L}}$.

1. Input;
   $A \in H_N$, B,
   r^* - small enough,
   κ^*, R - big enough,
   $\gamma \lessapprox 1$.
2. Calculate $\|B\|$.
3. Take $k=1$, $A_0 := A$, $r_0 := 0$, $u_0 := 0$.
4. Calculate;
   $A_k = A_{k-1} + r_{k-1}B$, $\|A_k\|$, $\kappa(A_k)$,
   $r_k = \gamma \frac{\|A_k\|}{\|B\|\kappa(A_k)}$,
   $u_k = u_{k-1} + r_k$.
5. If $r_k < r^*$ or $r_k > R$ or $\kappa(A_k) > \kappa^*$then finish the algorithm otherwise take $k := k+1$ and go 4. step.
6. Write the upper bound of interval $\overline{u} = u_k$.

Algorithm 2
Let's give the following algorithm to extend the lower bound of the intervals $\mathcal{I}_{\mathcal{L}}$.

1. Input;
   $A \in H_N$, B,
   r^* - small enough,
   κ^*, R - big enough,
   $\gamma \lessapprox 1$.
2. Calculate $\|B\|$.
3. Take $k=1$, $A_0 := A$, $r_0 := 0$, $l_0 := 0$.
4. Calculate;
   $A_k = A_{k-1} - r_{k-1}B$, $\|A_k\|$, $\kappa(A_k)$,
   $r_k = \gamma \frac{\|A_k\|}{\|B\|\kappa(A_k)}$,
   $l_k = l_{k-1} - r_k$.
5. If $r_k < r^*$ or $r_k > R$ or $\kappa(A_k) > \kappa^*$then finish the algorithm otherwise take $k := k+1$ and go 4. step.
6. Write the lower bound of interval $\underline{l} = l_k$.

Algorithm 3
Let's give the last algorithm to combine the found values and write them as an interval $\mathcal{I}^e_{\mathcal{L}}$.

1. Input;
   $A \in H_N$, B,
   r^* - small enough,
   κ^*, R - big enough,
   $\gamma \lessapprox 1$.
2. Calculate;
   $\overline{u}$ from Alg 1,
   $\underline{l}$ from Alg 2.
3. Write $\mathcal{I}^e_{\mathcal{L}} = \left[\underline{l}, \overline{u}\right]$.

Table 2: The upper bound of the interval $\mathcal{I}^{e}_{\mathcal{L}}$ obtained from the Algorithm 1 for $\gamma = 0.9$

A	B	κ^*	r^*	R	$\overline{r}$	$\overline{u}$	k	S.P.
A_1^1	$-2 \times E_{11} - E_{22}$	500	0.01	100	0.45	262.342	15	R
	$-2 \times E_{11} - E_{22}$	5000	0.001	200	0.45	552.676	17	R
	$E_{11} + E_{22}$	100	0.01	50	0.9	0.99	2	r^*
	$E_{11} + E_{22}$	200	0.001	500	0.9	0.999	3	r^*
A_1^2	$-2 \times E_{12}$	50	0.06	100	0.45	3.08373	21	κ^*
	$-2 \times E_{12}$	150	0.03	200	0.45	4.56651	57	r^*
	$-2 \times E_{11} - E_{22}$	10000	0.001	100	0.45	283.488	14	R
	$-2 \times E_{11} - E_{22}$	100000	0.0001	1000	0.45	2651.36	20	R
A_1^3	$-2 \times E_{11} - E_{21}$	100	0.01	100	0.248754	273.19	16	R
	$-2 \times E_{11} - E_{21}$	200	0.001	200	0.248754	575.484	18	R
	$E_{12} + E_{21}$	1000	0.01	100	0.497508	0.61345	3	κ^*
	$E_{12} + E_{21}$	10000	0.001	200	0.497508	0.61714	4	r^*

Table 3: The lower bound of the interval $\mathcal{I}^{e}_{\mathcal{L}}$ obtained from the Algorithm 2 for $\gamma = 0.9$

A	B	κ^*	r^*	R	$\underline{r}$	$\underline{l}$	k	S.P.
A_1^1	$-2 \times E_{11} - E_{22}$	500	0.01	100	-0.45	-0.495	2	r^*
	$-2 \times E_{11} - E_{22}$	5000	0.001	200	-0.45	-0.4995	3	κ^*
	$E_{11} + E_{22}$	100	0.01	50	-0.9	-88.3872	7	R
	$E_{11} + E_{22}$	200	0.001	500	-0.9	-612.107	10	R
A_1^2	$-2 \times E_{12}$	50	0.06	100	-0.45	-3.08373	21	κ^*
	$-2 \times E_{12}$	150	0.03	200	-0.45	-4.56651	57	r^*
	$-2 \times E_{11} - E_{22}$	10000	0.001	100	-0.45	-0.4995	3	r^*
	$-2 \times E_{11} - E_{22}$	100000	0.0001	1000	-0.45	-0.49995	4	κ^*
A_1^3	$-2 \times E_{11} - E_{21}$	100	0.01	100	-0.248754	-0.45949	6	r^*
	$-2 \times E_{11} - E_{21}$	200	0.001	200	-0.248754	-0.484445	10	κ^*
	$E_{12} + E_{21}$	1000	0.01	100	-0.497508	-1.60957	4	κ^*
	$E_{12} + E_{21}$	10000	0.001	200	-0.497508	-1.61771	6	r^*

Here, the upper bound $\overline{u}$ is extended with Algorithm 1. The lower bound $\underline{l}$ is extended with Algorithm 2. These values constitute of the Hurwitz stability interval $\mathcal{I}^{e}_{\mathcal{L}} = \left[\underline{l}, \overline{u}\right]$ of the $\mathcal{L}(A_1, B)$ matrix family. In the algorithm given above the larger bounds are obtained that preserve the Hurwitz stability of the given matrix family.

Remark 2. Let's give some reviews of algorithm as below;

1. If $k \to \infty$, $\kappa\left(A_{r_k}\right) \to \infty$ then the stepsize is $r_k = \gamma.u_k \to 0$, where k is the number of steps.
2. Let's $A_1 \in H_N, B$ such that A_1 and B are the diagonal matrices. $r \in (-\infty, u)$ or $r \in (l, \infty)$ where u is the upper bound and l is the lower bound obtained as a result of the algorithm. In this situation $\kappa(A_r) \to 1^+$. Similarly, this situation can be write for the lower triangular matrices or the upper triangular matrices. Specially if the matrices $A_1 = -\alpha \times I_N$ $(\alpha > 0)$, $B = \beta \times I_N$ are taken, where $I_N - N \times N$ identical matrix, then $r \in \left(-\infty, \frac{\alpha}{\beta}\right)$ $(\beta > 0)$ and $r \in \left(\frac{\alpha}{\beta}, \infty\right)(\beta < 0)$. In this situation $\kappa(A_r) = 1$.

Example 3.1. Let us consider the matrices A and B as follow,

$$A_1^1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, A_1^2 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, A_1^3 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

$$B = \sum_{i=1}^{2}\sum_{j=1}^{2} \delta_{ij} E_{ij}\ ,\ \sum_{i=1}^{2}\sum_{j=1}^{2} \delta_{ij}^2 \neq 0.$$

Here E_{ij} is a real matrix which the element in position (i, j) equals 1 and all other elements are 0.

The parameters κ^*, r^* and R selected by the users, and the matrices A and B are the input elements of the algorithms. In the examples, the lower and upper bounds of the intervals $\mathcal{I}_{\mathcal{L}} = [\underline{r}, \overline{r}]$ will be taken as $-\underline{r} = \overline{r} = \gamma.u$, where u is as defined in Theorem 2.2 and $\gamma \lessapprox 1$. The lower and upper bounds of the interval $[\underline{l}, \overline{u}]$ are also the values of the extended interval $\mathcal{I}^e_{\mathcal{L}}$ obtained by the Algorithm 1 and Algorithm 2. Stopping parameter (S.P.) indicates with which parameter the algorithms are stopped. k indicates how many steps the algorithms stopped.

For example, according to Table 2, starting with the upper bound $\overline{r} = 0.45$ ($\gamma = 0.9$, $u = 0.5$) for the matrices A^1_1, $B = -2 \times E_{11} - E_{22}$, the extended upper bound $\overline{u}$ is obtained as $\overline{u} = 262.342$ in 15 steps with stopping parameter R at the end of process, where the stopping criteria are chosen $\kappa^* = 250$, $r^* = 0.01$, $R = 100$. On the other hand, with same stopping criteria, according to Table 3, starting with the lower bound $\underline{r} = -0.45$ ($\gamma = 0.9$, $u = 0.5$) for the matrices A^1_1, $B = -2 \times E_{11} - E_{22}$, the extended lower bound $\underline{l}$ is obtained as $\underline{l} = -0.495$ in 2 steps with stopping parameter r^* at the end of process. The values obtained from the two tables constitute the extended interval matrix $\mathcal{I}^e_{\mathcal{L}} = [\underline{l}, \overline{u}] = [-0.495, 262.342]$ for the matrices A^1_1, $B = -2 \times E_{11} - E_{22}$.

Remark 3. Similar results are obtained if the operations for the intervals $\mathcal{I}_{\mathcal{L}}$ are also performed for the intervals $\mathcal{I}_{\mathcal{C}}$. So, in this paper, operations for the interval $\mathcal{I}^e_{\mathcal{C}}$ will not be repeated.

Remark 4. In this paper, calculations are made with the computer dialogue system MVC [15] .

4 Conclusion

In this study, new matrix families $\mathcal{L}(A_1, B)$ and $\mathcal{C}(A_1, B)$ based on linear sum and convex combination were created, respectively. The $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ intervals that make these families Hurwitz stable were determined in Theorem 2.2 and Theorem 2.3 and supported by examples. To obtain the extended the interval $\mathcal{I}^e_{\mathcal{L}}$, firstly, the method which based on continuity theorem were given and afterwards, the algorithms which based on the method were given. So bigger intervals which preserve the Hurwitz stability of matrix families were obtained. The method and algorithms given in Section 3 for finding the extended interval $\mathcal{I}^e_{\mathcal{L}}$ can also be easily used for finding the extended interval $\mathcal{I}^e_{\mathcal{C}}$ by making elementary arrangements. On the other hand, in many studies in the literature, the matrices A_1 and B were taken as Hurwitz stable but in this paper there is no need for the matrix B to be Hurwitz stable. The method and algorithms given in this article can also be used to construct the interval matrices with Hurwitz stable. So, this study, which is important for Hurwitz stable matrix families, has the feature of being a source due to the new results in it.

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