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Nonlinear multivariate sampling Kantorovich operators: quantitative estimates in functional spaces

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Abstract

In real world applications, signals can be suitably reconstructed by nonlinear procedures; this justifies the study of nonlinear approximation operators. In this paper, we prove some quantitative estimates for the nonlinear sampling Kantorovich operators in the multivariate setting using the modulus of smoothness of $L^p(\mathbb{R}^n)$. The above results have been then extended to the general case of Orlicz spaces $L^\varphi(\mathbb{R}^n)$, so obtaining quantitative estimates in several instances of well-known and useful spaces.

1 Introduction

In this paper, we focus our attention on the study of a general family of nonlinear operators, named *nonlinear multivariate sampling Kantorovich operators*, which are defined by

$$(K_w f)(\underline{x}) := \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u}\right), \quad \underline{x}\in\mathbb{R}^n,$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x}\in\mathbb{R}^n$, and $\chi:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ is a (nonlinear) kernel function. Here

$$R^w_{\underline{k}} := \left[\frac{t_{k_1}}{w}, \frac{t_{k_1+1}}{w}\right]\times\left[\frac{t_{k_2}}{w}, \frac{t_{k_2+1}}{w}\right]\times\cdots\times\left[\frac{t_{k_n}}{w}, \frac{t_{k_n+1}}{w}\right], \quad w>0,$$

where $(t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$ is a suitable sequence, with $A_{\underline{k}} := \Delta_{k_1}\cdot\Delta_{k_2}\cdots\Delta_{k_n}$, and $\Delta_{k_i} := t_{k_i+1} - t_{k_i} > 0$.
The interest in this topic is due to the fact that nonlinear operators play an important role in Signal Processing. In fact, the above operators are suitable, e.g., in order to describe nonlinear transformations generated by signals that, during their filtering process, produce new frequencies.
The pioneer works of the theory of nonlinear integral operators, in connection with approximation problems (see, e.g., [31]-[35]), can be reconducted to the Polish mathematician Julian Musielak. Further, the theory has been extensively developed in the monograph of Bardaro, Musielak and Vinti (see [8]), in relation to the abstract setting provided by the modular spaces. Other approximation results related to nonlinear operators can be found in [9, 42, 29, 4, 10, 5, 41].
The univariate version of nonlinear sampling Kantorovich operators has been introduced firstly in [43], where both uniform and modular convergence results have been established. The qualitative order of approximation has been studied in [20] considering functions in suitable Lipschitz classes both in the space of uniformly continuous and bounded functions and in Orlicz spaces. Results concerning the multidimensional case have been obtained in [19, 23]. As concerns inverse and saturation results and simultaneous approximation see [24, 25, 1, 12].
Concerning the problem of the order of approximation, quantitative estimates have been recently established in [13] in the one-dimensional case.
In the present paper, we prove some quantitative estimates in the multivariate setting using the modulus of smoothness of the Lebesgue space $L^p(\mathbb{R}^n)$. Here, a crucial role is played by basical properties of the modulus of smoothness. As a consequence of the previous estimates, the qualitative order of approximation is established for functions belonging to suitable Lipschitz classes. Further, the above results have been extended to the general frame of Orlicz spaces $L^\varphi(\mathbb{R}^n)$, which include the L^p-spaces as a particular case, besides the Zygmund spaces, the exponential spaces and others, therefore obtaining a unifying approach for the

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above approximation results.
Note that, in this setting, since the modulus of smoothness in $L^{\varphi}(\mathbb{R}^n)$ can not benefit from the same properties of the L^p-modulus of smoothness (used in Theorem 4.1), the quantitative estimates obtained are less sharp than those ones reached in $L^p(\mathbb{R}^n)$.
In the last part of the paper, we also give some concrete examples of nonlinear multivariate sampling Kantorovich operators constructed by using Fejér and B-spline kernels, establishing some particular results in these instances.

2 Preliminary notions

In this article we consider, on the multivariate space $\mathbb{R}^n$, the usual Euclidean norm $\|\cdot\|_2$, defined by $\|\underline{x}\|_2 := \left(x_1^2 + \cdots + x_n^2\right)^{1/2}$, where $\underline{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$.
Let now $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function. It is well-known that φ is said to be a *φ-function* if it satisfies the following conditions:

(Φ1) φ is a non decreasing and continuous function;

(Φ2) $\varphi(0) = 0$, $\varphi(u) > 0$ for every $u > 0$;

(Φ3) $\varphi(u) \to +\infty$ if $u \to +\infty$.

Now, in order to recall the definition of Orlicz spaces, we introduce the notion of the modular functional I^{φ} associated to the φ-function φ, defined by

$$I^{\varphi}[f] := \int_{\mathbb{R}^n} \varphi(|f(\underline{x})|)d\underline{x},$$

for every $f \in M(\mathbb{R}^n)$, where $M(\mathbb{R}^n)$ denotes the set of all Lebesgue measurable functions on $\mathbb{R}^n$. The Orlicz space generated by φ can now be defined by

$$L^{\varphi}(\mathbb{R}^n) := \{f \in M(\mathbb{R}^n) \ : I^{\varphi}[\lambda f] < +\infty, \text{ for some } \lambda > 0\}.$$

In Orlicz spaces, different notions of convergence can be introduced. In this paper, we recall the most natural notion of convergence in this setting, that is called *modular convergence*. A family (net) of functions $(f_w)_{w>0} \subset L^{\varphi}(\mathbb{R}^n)$ is modularly convergent to $f \in L^{\varphi}(\mathbb{R}^n)$, if there exists $\lambda > 0$ such that

$$I^{\varphi}\left[\lambda(f_w - f)\right] = \int_{\mathbb{R}^n} \varphi(\lambda\left|f_w(\underline{x}) - f(\underline{x})\right|)d\underline{x} \to 0,$$

as $w \to +\infty$. For further details concerning these spaces, see, e.g., [36, 30, 38, 28, 39, 8].
Now, in order to establish quantitative estimates for the order of approximation of a family of nonlinear multivariate operators, we recall the definition of the modulus of smoothness in Orlicz spaces $L^{\varphi}(\mathbb{R}^n)$, with respect to the modular I^{φ}. For any fixed $f \in L^{\varphi}(\mathbb{R}^n)$, we denote

$$\omega(f,\delta)_{\varphi} := \sup_{\|\underline{t}\|_2 \le \delta} I^{\varphi}\left[f(\cdot + \underline{t}) - f(\underline{t})\right],$$

with $\delta > 0$. It is well-known that, for $f \in L^{\varphi}(\mathbb{R}^n)$ there exists $\lambda > 0$ such that $\omega(\lambda f, \delta)_{\varphi} \to 0$, as $\delta \to 0^+$ (see [8], Theorem 2.4).

3 Nonlinear multivariate sampling Kantorovich type operators

Now, we recall the definition of the class of operators we work with.
Let $\Pi^n = (t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$ be a sequence of vectors defined by $t_{\underline{k}} = (t_{k_1}, \cdots, t_{k_n})$, where each $(t_{k_i})_{k_i\in\mathbb{Z}}$, $i = 1, \cdots, n$, is a sequence of real numbers with $-\infty < t_{k_i} < t_{k_i+1} < +\infty$, $\lim_{k_i\to\pm\infty} t_{k_i} = \pm\infty$, for every $i = 1, \cdots, n$ and such that there exist $\Delta, \delta > 0$ for which $\delta \le \Delta_{k_i} := t_{k_i+1} - t_{k_i} \le \Delta$, for every $i = 1, \cdots, n$. Moreover, we denote by

$$R_{\underline{k}}^w := \left[\frac{t_{k_1}}{w}, \frac{t_{k_1+1}}{w}\right] \times \left[\frac{t_{k_2}}{w}, \frac{t_{k_2+1}}{w}\right] \times \cdots \times \left[\frac{t_{k_n}}{w}, \frac{t_{k_n+1}}{w}\right] \quad (w > 0),$$

the n-dimensional parallelepipeds of $\mathbb{R}^n$ identified by the sequence Π^n. We note that the Lebesgue measure of $R_{\underline{k}}^w$ is given by $A_{\underline{k}}/w^n$, where $A_{\underline{k}} := \Delta_{k_1} \cdot \Delta_{k_2} \cdots \Delta_{k_n}$.

A function $\chi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ will be called *kernel* (for the nonlinear multivariate sampling Kantorovich operators) if it satisfies the following conditions:

(χ1) $\left(\chi(w\underline{x} - t_{\underline{k}}, u)\right)_{\underline{k}} \in \ell^1(\mathbb{Z}^n)$, for every $\underline{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $w > 0$;

(χ2) $\chi(\underline{x}, 0) = 0$, for every $\underline{x} \in \mathbb{R}^n$;

(χ3) χ is an (L,ψ)-Lipschitz kernel, i.e., there exist a measurable function $L : \mathbb{R}^n \to \mathbb{R}_0^+$ and a φ-function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\left|\chi(\underline{x}, u) - \chi(\underline{x}, v)\right| \le L(\underline{x})\psi(|u - v|),$$

for every $\underline{x} \in \mathbb{R}^n$ and $u, v \in \mathbb{R}$;

(χ4) there exists $\theta_0 > 0$ such that

$$\mathcal{T}_w(\underline{x}) := \sup_{u\neq 0} \left| \frac{1}{u} \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, u\right) - 1 \right| = O\left(w^{-\theta_0}\right),$$

as $w \to +\infty$, uniformly with respect to $\underline{x} \in \mathbb{R}^n$.

Moreover, we assume that the function L of condition (χ3) satisfies the following additional assumptions:

(L1) $L \in L^1(\mathbb{R}^n)$ and is bounded in a neighborhood of $\underline{0} \in \mathbb{R}^n$;

(L2) there exists a number $\beta_0 > 0$ such that

$$m_{\beta_0,\Pi^n}(L) := \sup_{\underline{x}\in\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(\underline{x} - t_{\underline{k}}) \left\|\underline{x} - t_{\underline{k}}\right\|_2^{\beta_0} < +\infty,$$

i.e., the absolute moment of order β_0 is finite.

Now, we recall the definition of the family of operators considered in this paper. The nonlinear multivariate sampling Kantorovich operators for a given kernel χ are defined by

$$(K_w f)(\underline{x}) := \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u}\right), \quad \underline{x} \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the series is convergent for every $\underline{x} \in \mathbb{R}^n$. We now recall the following lemma that will be useful in the next sections. For a proof, see [18].

Lemma 3.1. *Let L be a function satisfying conditions (L1) and (L2). We have*

$$m_{0,\Pi^n}(L) := \sup_{\underline{x}\in\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(\underline{x} - t_{\underline{k}}) < +\infty.$$

4 Main results

In this section, we establish a quantitative estimate for the nonlinear multivariate sampling Kantorovich operators in $L^p(\mathbb{R}^n)$ by using the modulus of smoothness of $L^p(\mathbb{R}^n)$-spaces. In order to obtain the above mentioned result, we recall, for $f \in L^p(\mathbb{R}^n)$, the definition of the first order L^p-modulus of smoothness of f, given by

$$\omega(f,\delta)_p := \sup_{\|\underline{h}\|_2\leq\delta} \left\|f(\cdot + \underline{h}) - f(\cdot)\right\|_p = \sup_{\|\underline{h}\|_2\leq\delta} \left(\int_{\mathbb{R}^n} |f(\underline{t} + \underline{h}) - f(\underline{t})|^p d\underline{t} \right)^{1/p},$$

with $\delta > 0$, $f \in L^p(\mathbb{R}^n)$, $1 \leq p < +\infty$. From the theory developed in [43], we know that if the function ψ of condition (χ3) is of the form $\psi(u) = u$, $u \in \mathbb{R}$, (i.e., the case of a strongly Lipschitz condition), the operators K_w map the whole space $L^p(\mathbb{R}^n)$ into itself, i.e., K_w are well-defined in $L^p(\mathbb{R}^n)$. Therefore we can prove the following estimate.

Theorem 4.1. *Suppose that (χ3) is satisfied with $\psi(u) = u$, $u \in \mathbb{R}$, and*

$$M^p(L) := \int_{\mathbb{R}^n} L(\underline{u}) \left\|\underline{u}\right\|_2^p d\underline{u} < +\infty, \tag{1}$$

for some $1 \leq p < +\infty$. Then, for every $f \in L^p(\mathbb{R}^n)$, the following quantitative estimate holds

$$\|K_w f - f\|_p \leq T\,\omega\left(f, \frac{1}{w}\right)_p + M_2 \|f\|_p\, w^{-\theta_0},$$

where $T := \delta^{-\frac{n}{p}} \left(m_{0,\Pi^n}(L)\right)^{\frac{p-1}{p}} \left\{2^{\frac{p-1}{p}} m_{0,\Pi^n}(\tau)^{1/p} \left[\|L\|_1 + M^p(L)\right]^{\frac{1}{p}} + \left(m_{0,\Pi^n}(L)\right)^{\frac{1}{p}} \Delta^{\frac{n}{p}} (1 + \sqrt{n}\Delta)\right\}$, for sufficiently large $w > 0$, where τ denotes the characteristic function of the set $[0,1]^n$, $m_{0,\Pi^n}(\tau) < +\infty$ since τ is bounded and with compact support, $m_{0,\Pi^n}(L) < +\infty$ in view of Lemma 3.1 and M_2, $\theta_0 > 0$ are the constants of condition (χ4).

Proof. Using the Minkowsky inequality, the concavity (hence the subadditivity) of the function $|\cdot|^{\frac{1}{p}}$, and applying condition

$(\chi 3)$, we have

$$\begin{aligned}
\|K_w f - f\|_p &= \left(\int_{\mathbb{R}^n} \left| K_w f(\underline{x}) - f(\underline{x}) \right|^p d\underline{x} \right)^{1/p} \\
&\le \left(\int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} \left| \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u}) d\underline{u} \right) - \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) d\underline{u} \right) \right| \right]^p d\underline{x} \right)^{1/p} \\
&+ \left(\int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} \left| \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) d\underline{u} \right) - \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x}) \right) \right| \right]^p d\underline{x} \right)^{1/p} \\
&+ \left(\int_{\mathbb{R}^n} \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})) - f(\underline{x}) \right|^p d\underline{x} \right)^{1/p} \\
&\le \left(\int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) \right| d\underline{u} \right]^p d\underline{x} \right)^{1/p} \\
&+ \left(\int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left| f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) - f(\underline{x}) \right| d\underline{u} \right]^p d\underline{x} \right)^{1/p} \\
&+ \left(\int_{\mathbb{R}^n} \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})) - f(\underline{x}) \right|^p d\underline{x} \right)^{1/p} =: I_1 + I_2 + I_3,
\end{aligned}$$

where $\frac{t_{\underline{k}}}{w} = \left(\frac{t_{k_1}}{w}, \frac{t_{k_2}}{w}, \cdots, \frac{t_{k_n}}{w} \right)$.

Now, we estimate I_1. Using Jensen inequality twice (see, e.g., [17]), and Fubini-Tonelli theorem, we obtain

$$\begin{aligned}
I_1^p &= \int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) \right| d\underline{u} \right]^p d\underline{x} \\
&\le \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} m_{0,\Pi^n}(L) \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) \right| d\underline{u} \right]^p d\underline{x} \\
&\le m_{0,\Pi^n}(L)^{p-1} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) \right|^p d\underline{u} \right] d\underline{x}.
\end{aligned}$$

Now, denoting by $\tau(\underline{u})$ the characteristic function of the set $[0,1]^n$, i.e., $\tau(\underline{u}) = 1$, if $\underline{u} \in [0,1]^n$, and $\tau(\underline{u}) = 0$ otherwise, the change of variable $\underline{y} = \underline{x} - t_{\underline{k}}/w$ and Fubini-Tonelli theorem, we get:

$$\begin{aligned}
I_1^p &\le m_{0,\Pi^n}(L)^{p-1} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{\mathbb{R}^n} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) \right|^p \tau(w\underline{u} - t_{\underline{k}}) d\underline{u} \right] d\underline{x} \\
&= m_{0,\Pi^n}(L)^{p-1} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{y}) \left[\frac{w^n}{A_{\underline{k}}} \int_{\mathbb{R}^n} \left| f(\underline{u}) - f\left(\underline{u} + \underline{y} \right) \right|^p \tau(w\underline{u} - t_{\underline{k}}) d\underline{u} \right] d\underline{y} \\
&\le \delta^{-n} m_{0,\Pi^n}(L)^{p-1} \int_{\mathbb{R}^n} w^n L(w\underline{y}) \left[\int_{\mathbb{R}^n} \left| f(\underline{u}) - f\left(\underline{u} + \underline{y} \right) \right|^p \sum_{\underline{k}\in\mathbb{Z}^n} \tau(w\underline{u} - t_{\underline{k}}) d\underline{u} \right] d\underline{y} \\
&\le \delta^{-n} m_{0,\Pi^n}(L)^{p-1} m_{0,\Pi^n}(\tau) \int_{\mathbb{R}^n} w^n L(w\underline{y}) \left[\int_{\mathbb{R}^n} \left| f(\underline{u}) - f\left(\underline{u} + \underline{y} \right) \right|^p d\underline{u} \right] d\underline{y} \\
&\le \delta^{-n} m_{0,\Pi^n}(L)^{p-1} m_{0,\Pi^n}(\tau) \int_{\mathbb{R}^n} w^n L(w\underline{y})\, \omega\left(f, \left\| \underline{y} \right\|_2 \right)_p^p d\underline{y},
\end{aligned}$$

where the constant $m_{0,\Pi^n}(\tau) < +\infty$ since τ is bounded and with compact support (see, e.g., [23]). Exploiting the well-known inequality $\omega(f,\lambda\delta)_p \leq (1+\lambda)\omega(f,\delta)_p$, with $\delta, \lambda > 0$, we finally get

$$\begin{aligned}
I_1^p &\leq \delta^{-n} m_{0,\Pi^n}(L)^{p-1} m_{0,\Pi^n}(\tau) \int_{\mathbb{R}^n} w^n L(w\underline{y}) \left(1 + w\left\|\underline{y}\right\|_2\right)^p \omega\left(f, \frac{1}{w}\right)_p^p d\underline{y} \\
&\leq \delta^{-n} m_{0,\Pi^n}(L)^{p-1} m_{0,\Pi^n}(\tau) 2^{p-1} \omega\left(f, \frac{1}{w}\right)_p^p \int_{\mathbb{R}^n} w^n L(w\underline{y}) \left(1 + w^p \left\|\underline{y}\right\|_2^p\right) d\underline{y} \\
&= \delta^{-n} m_{0,\Pi^n}(L)^{p-1} m_{0,\Pi^n}(\tau) 2^{p-1} \omega\left(f, \frac{1}{w}\right)_p^p \left\{\int_{\mathbb{R}^n} w^n L(w\underline{y}) d\underline{y} + \int_{\mathbb{R}^n} w^n L(w\underline{y}) \left(w\left\|\underline{y}\right\|_2\right)^p d\underline{y}\right\} \\
&= \delta^{-n} m_{0,\Pi^n}(L)^{p-1} m_{0,\Pi^n}(\tau) 2^{p-1} \omega\left(f, \frac{1}{w}\right)_p^p \{\|L\|_1 + M^p(L)\},
\end{aligned}$$

for every $w > 0$, where $\|L\|_1$ and $M^p(L)$ are both finite, in view of $(L1)$ and (1).
Now we estimate I_2. Using Jensen inequality twice, the change of variable $\underline{y} = \underline{u} - t_{\underline{k}}/w$ and Fubini-Tonelli theorem, we have

$$\begin{aligned}
I_2^p &= \int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} \left| f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) - f(\underline{x}) \right| d\underline{u}\right]^p d\underline{x} \\
&\leq \int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{\widetilde{R}_{\underline{k}}^w} \left| f\left(\underline{x} + \underline{y}\right) - f(\underline{x}) \right| d\underline{y}\right]^p d\underline{x} \\
&\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{\widetilde{R}_{\underline{k}}^w} m_{0,\Pi^n}(L) \left| f\left(\underline{x} + \underline{y}\right) - f(\underline{x}) \right| d\underline{y}\right]^p d\underline{x} \\
&\leq \delta^{-n} m_{0,\Pi^n}(L)^{p-1} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \left[w^n \int_{\Delta_w} \left| f\left(\underline{x} + \underline{y}\right) - f(\underline{x}) \right|^p d\underline{y}\right] d\underline{x} \\
&\leq \delta^{-n} m_{0,\Pi^n}(L)^p \int_{\mathbb{R}^n} w^n \int_{\Delta_w} \left| f\left(\underline{x} + \underline{y}\right) - f(\underline{x}) \right|^p d\underline{y}\, d\underline{x} \\
&= \delta^{-n} m_{0,\Pi^n}(L)^p \int_{\Delta_w} w^n \left[\int_{\mathbb{R}^n} \left| f\left(\underline{x} + \underline{y}\right) - f(\underline{x}) \right|^p d\underline{x}\right] d\underline{y} \\
&\leq \delta^{-n} m_{0,\Pi^n}(L)^p \int_{\Delta_w} w^n \left[\omega\left(f, \sqrt{n}\frac{\Delta}{w}\right)_p\right]^p d\underline{y} \\
&\leq \delta^{-n} m_{0,\Pi^n}(L)^p \Delta^n \omega\left(f, \sqrt{n}\frac{\Delta}{w}\right)_p^p \\
&\leq \delta^{-n} m_{0,\Pi^n}(L)^p \Delta^n (1 + \sqrt{n}\Delta)^p \omega\left(f, \frac{1}{w}\right)_p^p,
\end{aligned}$$

where $\widetilde{R}_{\underline{k}}^w := \left[0, \frac{\Delta_{k_1}}{w}\right] \times \ldots \times \left[0, \frac{\Delta_{k_n}}{w}\right]$ and $\Delta_w := \left[0, \frac{\Delta}{w}\right]^n$.
Finally, denoted by $A_0 \subseteq \mathbb{R}^n$, the set of all points of $\mathbb{R}^n$ for which $f \neq 0$ a.e., we immediately obtain, for sufficiently large $w > 0$,

$$\begin{aligned}
I_3^p &= \int_{A_0} \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})) - f(\underline{x}) \right|^p d\underline{x} \\
&= \int_{A_0} |f(\underline{x})|^p \left| \frac{1}{f(\underline{x})} \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})) - 1) \right|^p d\underline{x} \\
&\leq \int_{A_0} |f(\underline{x})|^p \left[\mathcal{T}_w(\underline{x})\right]^p d\underline{x} \\
&\leq \int_{A_0} |f(\underline{x})|^p M_2^p w^{-p\theta_0} d\underline{x} \\
&\leq M_2^p w^{-p\theta_0} \int_{\mathbb{R}^n} |f(\underline{x})|^p d\underline{x} \\
&= M_2^p w^{-p\theta_0} \|f\|_p^p,
\end{aligned}$$

for the positive constants M_2 and θ_0 from condition $(\chi 4)$. This proves the theorem. □

Remark 1. Note that, in the papers [3, 2] quantitative estimates in the L^1 and L^p settings respectively, have been established for the linear and nonlinear univariate versions of the sampling Kantorovich operators. We highlight that the estimates achieved in the above quoted papers (and then also the related proofs) should be updated with the introduction of the finite constant $m_{0,\Pi^n}(\tau)$, as made in Theorem 4.1.

Remark 2. Note that in condition $(\chi 3)$, the (L,ψ)-Lipschitz kernel is suitable when one deals with the frame of Orlicz spaces (as done later), while in the present setting it is natural to assume a strongly Lipschitz condition, i.e., with $\psi(u)=u$, $u\in\mathbb{R}$.

From the above quantitative estimate we can directly deduce the qualitative order of approximation, assuming f in suitable Lipschitz spaces.

Firstly, we recall that the Lipschitz class of Zygmund-type in L^p-spaces, with $0<\alpha\leq 1$, are defined as follows

$$Lip(\alpha,p):=\{f\in L^p(\mathbb{R}^n)\ :\ \left\|f(\cdot+\underline{t})-f(\cdot)\right\|_p=O(\left\|\underline{t}\right\|_2^{\alpha}),\text{ as }\left\|\underline{t}\right\|_2\to 0\}.$$

Now, we can state the following result.

Corollary 4.2. *Under the assumptions of Theorem 4.1, for every $f\in Lip(\alpha,p)$, with $0<\alpha\leq 1$, $1\leq p<+\infty$, the following qualitative estimate holds*

$$\|K_w f-f\|_p\leq T C_1\frac{1}{w^{\alpha}}+M_2\|f\|_p\, w^{-\theta_0},$$

for sufficiently large $w>0$, where T is the constant of Theorem 4.1, M_2, $\theta_0>0$ are the constants of condition $(\chi 4)$ and $C_1>0$ is the constant arising from the class $Lip(\alpha,p)$.

Remark 3. We remark that, if $\chi(\underline{x},u)=L(\underline{x})u$, where L satisfies $(L1)$ and $(L2)$, condition $(\chi 4)$ becomes

$$\begin{aligned}\mathcal{T}_w(\underline{x})&=\sup_{u\neq 0}\left|\frac{1}{u}\sum_{\underline{k}\in\mathbb{Z}^n}L(w\underline{x}-t_{\underline{k}})u-1\right|\\&=\left|\sum_{\underline{k}\in\mathbb{Z}^n}L(w\underline{x}-t_{\underline{k}})-1\right|=O(w^{-\theta_0}),\end{aligned}\tag{2}$$

as $w\to+\infty$, uniformly with respect to $\underline{x}\in\mathbb{R}^n$, for some $\theta_0>0$, that is we deal with the linear case. Quantitative estimates for the multivariate sampling Kantorovich operators in the linear case have been considered in details in [2]. For more references, see, e.g., [6, 7, 27, 40, 37]. In the general theory of sampling type operators, a slightly stronger condition (instead of (2)) is required, that is

$$\sum_{\underline{k}\in\mathbb{Z}^n}L(\underline{u}-t_{\underline{k}})=1,\tag{3}$$

for every $\underline{u}\in\mathbb{R}^n$. If (3) holds, condition $(\chi 4)$ turns out to be satisfied for every $\theta_0>0$. When the uniform spaced sequence $t_{\underline{k}}=\underline{k}$ is considered and L is continuous, it is well known that (3) is equivalent to

$$\widehat{L}(2\pi\underline{k}):=\begin{cases}0, & \underline{k}\in\mathbb{Z}^n\setminus\{\underline{0}\},\\ 1, & \underline{k}=\underline{0},\end{cases}$$

where $\widehat{L}(\underline{v}):=\int_{\mathbb{R}^n}L(\underline{u})e^{-i\underline{v}\cdot\underline{u}}d\underline{u}$, $\underline{v}\in\mathbb{R}^n$, denotes the Fourier transform of L (see, e.g., [11]). Such condition is known in literature with the name of Strang-Fix type condition.

Now, in order to obtain quantitative estimates for K_w in a broader context, we extend the above results to the general setting of Orlicz spaces.

Now, for our operators to be well-defined in $L^{\varphi}(\mathbb{R}^n)$, we need to introduce a growth condition on the composition of the function φ, which generates the Orlicz space, and the function ψ of the (L,ψ)-Lipschitz condition. We assume what follows.

Let φ be a fixed φ-function; we suppose that there is a φ-function η such that, for every $\lambda\in(0,1)$, there exists a constant $C_\lambda\in(0,1)$ satisying

$$\varphi(C_\lambda\psi(u))\leq\eta(\lambda u),\tag{H}$$

for every $u\in\mathbb{R}_0^+$, where ψ is the φ-function of the condition $(\chi 3)$. For more details concerning condition (H), see, e.g., [8].

Now we can prove the main result of this section.

Theorem 4.3. *Let φ be a convex φ-function. Suppose that φ satisfies condition (H) with η convex, $f\in L^{\varphi+\eta}(\mathbb{R}^n)$ and that for any fixed $0<\alpha<1$, we have*

$$w^n\int_{\left\|\underline{y}\right\|_2>\frac{1}{w^{\alpha}}}L(w\underline{y})d\underline{y}\leq M_1 w^{-\alpha_0},\tag{4}$$

as $w\to+\infty$, for suitable positive constants M_1, α_0 depending on α and L. Then, there exist $\mu>0$, $\lambda>0$ and $\lambda_0>0$ such that

$$\begin{aligned}I^{\varphi}[\mu(K_w f-f)]\leq{}&\frac{\|L\|_1\, m_{0,\Pi^n}(\tau)}{3\delta^n m_{0,\Pi^n}(L)}\omega\left(\lambda f,\frac{1}{w^{\alpha}}\right)_{\eta}+\frac{M_1 m_{0,\Pi^n}(\tau)I^{\eta}[\lambda_0 f]}{3\delta^n m_{0,\Pi^n}(L)}w^{-\alpha_0}\\&+\frac{\Delta^n}{3\delta^n}\omega\left(\lambda f,\sqrt{n}\frac{\Delta}{w}\right)_{\eta}+\frac{I^{\varphi}[\lambda_0 f]}{3}w^{-\theta_0},\end{aligned}$$

for sufficiently large $w > 0$, where $m_{0,\Pi^n}(L) < +\infty$ in view of Lemma 3.1, $m_{0,\Pi^n}(\tau) < +\infty$, τ being the characteristic function of $[0,1]^n$, and $\theta_0 > 0$ is the constant of condition $(\chi 4)$. In particular, for $\mu > 0$ and $\lambda > 0$ sufficiently small, the above inequality implies the modular convergence of nonlinear multivariate sampling Kantorovich operators $K_w f$ to f.

Proof. Let $\lambda_0 > 0$ such that $I^\varphi[\lambda_0 f] < +\infty$. Further, we also fix $\lambda > 0$ such that

$$\lambda < \min\left\{1, \frac{\lambda_0}{2}\right\}.$$

In correspondence to λ, by condition (H), we know that there exists $C_\lambda \in (0,1)$ such that $\varphi(C_\lambda \psi(u)) \leq \eta(\lambda u)$, $u \in \mathbb{R}_0^+$, while by $(\chi 4)$, there exist constants θ_0, $M_2 > 0$ such that

$$\mathcal{T}_w(\underline{x}) \leq M_2 w^{-\theta_0},$$

uniformly with respect to $\underline{x} \in \mathbb{R}^n$, for sufficiently large $w > 0$. Now, we choose $\mu > 0$ such that

$$\mu \leq \min\left\{\frac{C_\lambda}{3m_{0,\Pi^n}(L)}, \frac{\lambda_0}{3M_2}\right\}.$$

Taking into account that φ is convex and non-decreasing, for $\mu > 0$, we can write

$$\begin{aligned}
I^\varphi[\mu(K_w f - f)] &= \int_{\mathbb{R}^n} \varphi\left(\mu\left|(K_w f)(\underline{x}) - f(\underline{x})\right|\right) d\underline{x} \\
&\leq \frac{1}{3}\left\{\int_{\mathbb{R}^n} \varphi\left(3\mu\left|(K_w f)(\underline{x}) - \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right)\right|\right) d\underline{x}\right. \\
&+ \int_{\mathbb{R}^n} \varphi\left(3\mu\left|\sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right) - \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right)\right|\right) d\underline{x} \\
&+ \left.\int_{\mathbb{R}^n} \varphi\left(3\mu\left|\sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right) - f(\underline{x})\right|\right) d\underline{x}\right\} =: I_1 + I_2 + I_3.
\end{aligned}$$

Now, we estimate I_1. Applying condition $(\chi 3)$, we have

$$\begin{aligned}
3I_1 &= \int_{\mathbb{R}^n} \varphi\left(3\mu\left|(K_w f)(\underline{x}) - \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right)\right|\right) d\underline{x} \\
&\leq \int_{\mathbb{R}^n} \varphi\left(3\mu\sum_{\underline{k}\in\mathbb{Z}^n}\left|\chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} f(\underline{u})\, d\underline{u}\right) - \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right)\right|\right) d\underline{x} \\
&\leq \int_{\mathbb{R}^n} \varphi\left(3\mu\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\psi\left(\left|\frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right|\right)\right) d\underline{x}.
\end{aligned}$$

Using Jensen inequality twice, Fubini-Tonelli theorem, condition (H), and the change of variable $\underline{y} = \underline{x} - \frac{t_{\underline{k}}}{w}$, we obtain

$$\begin{aligned}
3I_1 &\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\, \varphi\left(3\mu m_{0,\Pi^n}(L)\psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} \left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u}\right)\right) d\underline{x} \\
&= \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}})\, \varphi\left(3\mu m_{0,\Pi^n}(L)\psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} \left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u}\right)\right) d\underline{x} \\
&\leq \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}})\, \varphi\left(C_\lambda \psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} \left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u}\right)\right) d\underline{x} \\
&\leq \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}})\, \eta\left(\lambda \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} \left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u}\right) d\underline{x} \\
&\leq \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} \eta\left(\lambda \left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right|\right) d\underline{u}\, d\underline{x} \\
&= \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}}\int_{\mathbb{R}^n} \eta\left(\lambda \left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right|\right) \tau(w\underline{u} - t_{\underline{k}}) d\underline{u}\, d\underline{x} \\
&= \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{y}) \frac{w^n}{A_{\underline{k}}}\int_{\mathbb{R}^n} \eta\left(\lambda \left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{y}\right)\right|\right) \tau(w\underline{u} - t_{\underline{k}}) d\underline{u}\, d\underline{y} \\
&\leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} w^n L(w\underline{y}) \int_{\mathbb{R}^n} \eta\left(\lambda \left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{y}\right)\right|\right) \sum_{\underline{k}\in\mathbb{Z}^n} \tau(w\underline{u} - t_{\underline{k}}) d\underline{u} d\underline{y} \\
&\leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)} m_{0,\Pi^n}(\tau) \int_{\mathbb{R}^n} w^n L(w\underline{y}) \int_{R^n} \eta\left(\lambda \left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{y}\right)\right|\right) d\underline{u} d\underline{y} \\
&= \frac{\delta^{-n}}{m_{0,\Pi^n}(L)} m_{0,\Pi^n}(\tau) \int_{\mathbb{R}^n} w^n L(w\underline{y})\, I^\eta\left[\lambda\left(f(\cdot) - f(\cdot + \underline{y})\right)\right] d\underline{y},
\end{aligned}$$

where τ denotes again the characteristic function of the set $[0,1]^n$. Now, let $0 < \alpha < 1$ be fixed. We now split the above integral as follows

$$\frac{w^n \delta^{-n}}{m_{0,\Pi^n}(L)} m_{0,\Pi^n}(\tau) \left\{ \int_{\left\|\underline{y}\right\|_2 \leq \frac{1}{w^\alpha}} + \int_{\left\|\underline{y}\right\|_2 > \frac{1}{w^\alpha}} \right\} L(w\underline{y})\, I^\eta\left[\lambda\left(f(\cdot) - f(\cdot + \underline{y})\right)\right] d\underline{y} =: I_{1,1} + I_{1,2}.$$

For $I_{1,1}$, one has

$$\begin{aligned}
I_{1,1} &\leq \frac{w^n \delta^{-n}}{m_{0,\Pi^n}(L)} m_{0,\Pi^n}(\tau) \int_{\left\|\underline{y}\right\|_2 \leq \frac{1}{w^\alpha}} L(w\underline{y}) \omega\left(\lambda f, \left\|\underline{y}\right\|_2\right)_\eta d\underline{y} \\
&\leq \frac{w^n \delta^{-n}}{m_{0,\Pi^n}(L)}\, m_{0,\Pi^n}(\tau) \omega\left(\lambda f, \frac{1}{w^\alpha}\right)_\eta \int_{\left\|\underline{y}\right\|_2 \leq \frac{1}{w^\alpha}} L(w\underline{y}) d\underline{y} \\
&\leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)}\, m_{0,\Pi^n}(\tau) \omega\left(\lambda f, \frac{1}{w^\alpha}\right)_\eta \|L\|_1 .
\end{aligned}$$

On the other hand, taking into account that η is convex, for $I_{1,2}$ we can write

$$I_{1,2} \leq \frac{w^n \delta^{-n}}{m_{0,\Pi^n}(L)} m_{0,\Pi^n}(\tau) \int_{\left\|\underline{y}\right\|_2 > \frac{1}{w^\alpha}} L(w\underline{y}) \frac{1}{2}\left(I^\eta[2\lambda f(\cdot)] + I^\eta[2\lambda f(\cdot + \underline{y})]\right) d\underline{y}.$$

Now, observing that

$$I^\eta[2\lambda f(\cdot)] = I^\eta[2\lambda f(\cdot + \underline{y})],$$

for every $\underline{y}$, using (4), we finally get

$$\begin{aligned}
I_{1,2} &\leq \frac{w^n \delta^{-n}}{m_{0,\Pi^n}(L)} m_{0,\Pi^n}(\tau) \int_{\left\|\underline{y}\right\|_2 > \frac{1}{w^\alpha}} L(w\underline{y})\, I^\eta[2\lambda f] d\underline{y} \\
&\leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)}\, m_{0,\Pi^n}(\tau) I^\eta[\lambda_0 f]\, M_1 w^{-\alpha_0},
\end{aligned}$$

for $w > 0$ sufficiently large and for $M_1 > 0$. Now we can proceed estimating I_2. Using the assumption ($\chi 3$) we immediately have

$$3I_2 = \int_{\mathbb{R}^n} \varphi\left(3\mu \left|\sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right) - \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right)\right|\right) d\underline{x}$$
$$\leq \int_{\mathbb{R}^n} \varphi\left(3\mu \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\, \psi\left(\left|\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u} - f(\underline{x})\right|\right)\right) d\underline{x}.$$

Now, by the change of variable $\underline{y} = \underline{u} - \frac{t_{\underline{k}}}{w}$, we have

$$3I_2 \leq \int_{\mathbb{R}^n} \varphi\left(3\mu \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\, \psi\left(\frac{w^n}{A_{\underline{k}}}\int_{\widetilde{R}^w_{\underline{k}}} \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right| d\underline{y}\right)\right) d\underline{x},$$

where the symbol $\widetilde{R}^w_{\underline{k}} := \left[0, \frac{\Delta_{k_1}}{w}\right] \times \cdots \left[0, \frac{\Delta_{k_n}}{w}\right]$ for every $\underline{k} \in \mathbb{Z}^n$ and $w > 0$. Hence, applying Jensen inequality twice as above, recalling that $3\mu m_{0,\Pi^n}(L) \leq C_\lambda$ and condition (H), we get

$$3I_2 \leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\, \varphi\left(3\mu m_{0,\Pi^n}(L)\, \psi\left(\frac{w^n}{A_{\underline{k}}}\int_{\widetilde{R}^w_{\underline{k}}} \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right| d\underline{y}\right)\right) d\underline{x}$$
$$\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\, \eta\left(\lambda \frac{w^n}{A_{\underline{k}}}\int_{\widetilde{R}^w_{\underline{k}}} \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right| d\underline{y}\right) d\underline{x}$$
$$\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}}\int_{\widetilde{R}^w_{\underline{k}}} \eta\left(\lambda \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right|\right) d\underline{y}\, d\underline{x}$$
$$\leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} w^n \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \int_{\Delta_w} \eta\left(\lambda \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right|\right) d\underline{y}\, d\underline{x},$$

where $\Delta_w := \left[0, \frac{\Delta}{w}\right]^n$. Then, by the Fubini-Tonelli theorem, we get

$$3I_2 \leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} w^n m_{0,\Pi^n}(L) \int_{\Delta_w} \eta\left(\lambda \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right|\right) d\underline{y}\, d\underline{x}$$
$$= \delta^{-n} w^n \int_{\Delta_w}\int_{\mathbb{R}^n} \eta\left(\lambda \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right|\right) d\underline{x}\, d\underline{y}$$
$$= \delta^{-n} w^n \int_{\Delta_w} I^\eta\left[\lambda\left(f\left(\cdot + \underline{y}\right) - f(\cdot)\right)\right] d\underline{y}$$
$$\leq \delta^{-n} w^n \omega\left(\lambda f, \sqrt{n}\frac{\Delta}{w}\right)_\eta \int_{\Delta_w} d\underline{y}$$
$$= \delta^{-n} \Delta^n \omega\left(\lambda f, \sqrt{n}\frac{\Delta}{w}\right)_\eta.$$

For I_3, denoted by $A_0 \subseteq \mathbb{R}^n$ the set of all points of $\mathbb{R}^n$ for which $f \neq 0$ almost everywhere, we obtain

$$3I_3 = \int_{A_0} \varphi\left(3\mu \left|\sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right) - f(\underline{x})\right|\right) d\underline{x}$$
$$= \int_{A_0} \varphi\left(3\mu |f(\underline{x})| \left|\frac{1}{f(\underline{x})} \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right) - 1\right|\right) d\underline{x}$$
$$\leq \int_{A_0} \varphi\left(3\mu |f(\underline{x})| \mathcal{T}_w(x)\right) d\underline{x}.$$

By the convexity of φ and condition $(\chi 4)$, we have

$$\begin{aligned}3I_3 &\leq \int_{A_0} \varphi\left(3\mu M_2 w^{-\theta_0}|f(\underline{x})|\right)d\underline{x}\\ &\leq w^{-\theta_0}\int_{\mathbb{R}^n} \varphi\left(3\mu M_2|f(\underline{x})|\right)d\underline{x}\\ &\leq w^{-\theta_0} I^{\varphi}[3\mu M_2 f]\\ &\leq w^{-\theta_0} I^{\varphi}[\lambda_0 f],\end{aligned}$$

for positive constants M_2 and θ_0. This completes the proof. □

Remark 4. Note that, considerations similar to those given in Remark 1, must be given also for the quantitative estimates in the setting of Orlicz spaces achieved in the following paper [20, 21, 22, 23, 26].

Remark 5. Note that, condition (4) is obviously fullfilled when the kernel χ satisfies condition $(\chi 3)$ with L having compact support. Indeed, if $supp\, L \subset B(\underline{0}, R) \subset \mathbb{R}^n$, $R > 0$, we have

$$w^n \int_{\|\underline{y}\|_2 > \frac{1}{w^\alpha}} L(w\underline{y})d\underline{y} = \int_{\|\underline{u}\|_2 > w^{1-\alpha}} L(\underline{u})d\underline{u} = 0,$$

for every $w > R^{1/(1-\alpha)}$. The above consideration implies that the term $I_{1,2}$ in the proof of Theorem 4.3 is null, for sufficiently large $w > 0$. Moreover, in this case, we also have that condition $(L2)$ is satisfied for every $\beta_0 > 0$.

Corollary 4.4. *Let χ be a kernel satisfying condition $(\chi 3)$ with L having compact support. Let φ be a convex φ-function satisfying condition (H) with η convex and $f \in L^{\varphi+\eta}(\mathbb{R}^n)$. Then, for every $0 < \alpha < 1$, there exist constants $\mu > 0$, $\lambda > 0$ and $\lambda_0 > 0$ such that*

$$I^{\varphi}[\mu(K_w f - f)] \leq \frac{\|L\|_1\, m_{0,\Pi^n}(\tau)}{3\delta^n m_{0,\Pi^n}(L)}\omega\left(\lambda f, \frac{1}{w^\alpha}\right)_\eta + \frac{\Delta^n}{3\delta^n}\omega\left(\lambda f, \sqrt{n}\frac{\Delta}{w}\right)_\eta + \frac{I^{\varphi}[\lambda_0 f]}{3} w^{-\theta_0},$$

for sufficiently large $w > 0$, where $m_{0,\Pi^n}(L) < +\infty$, $m_{0,\Pi^n}(\tau) < +\infty$, and $\theta_0 > 0$ is the constant of condition $(\chi 4)$.

Remark 6. Note that, if L has not compact support, we may require the following condition:

$$M^{\nu}(L) := \int_{\mathbb{R}^n} L(\underline{u}) \left\|\underline{u}\right\|_2^{\nu} d\underline{u} < +\infty, \tag{5}$$

for $\nu > 0$, which results a sufficient condition for (4). Indeed, for every $0 < \alpha < 1$, we can write what follows

$$\begin{aligned}w^n \int_{\|\underline{y}\|_2 > \frac{1}{w^\alpha}} L(w\underline{y})d\underline{y} &= \int_{\|\underline{u}\|_2 > w^{1-\alpha}} L(\underline{u})d\underline{u} \leq \frac{1}{w^{\nu(1-\alpha)}} \int_{\|\underline{u}\|_2 > w^{1-\alpha}} \left\|\underline{u}\right\|_2^{\nu} L(\underline{u})d\underline{u}\\ &\leq \frac{M^{\nu}(L)}{w^{\nu(1-\alpha)}} = O(w^{\nu(\alpha-1)}),\end{aligned}$$

as $w \to +\infty$. Hence, (4) is satisfied with $\alpha_0 = (1-\alpha)\nu$ and $M_1 = M^{\nu}(L)$.

Now, as made in the particular context of $L^p(\mathbb{R}^n)$ spaces, we recall the definition of Lipschitz classes in Orlicz spaces $L^{\varphi}(\mathbb{R}^n)$. We define by $Lip_{\varphi}(\nu)$, $0 < \nu \leq 1$, the set of all functions $f \in L^{\varphi}(\mathbb{R}^n)$ such that there exists $\lambda > 0$ with

$$I^{\varphi}\left[\lambda\left(f(\cdot) - f(\cdot + \underline{t})\right)\right] = \int_{\mathbb{R}^n} \varphi\left(\lambda\left|f(\underline{x}) - f(\underline{x} + \underline{t})\right|\right)d\underline{x} = O(\left\|\underline{t}\right\|_2^{\nu}),$$

as $\left\|\underline{t}\right\|_2 \to 0$. From Theorem 4.3, we immediately obtain the following corollary.

Corollary 4.5. *Under the assumptions of Theorem 4.3 with $0 < \alpha < 1$ and for any $f \in Lip_{\eta}(\nu)$, $0 < \nu \leq 1$, there exist $S > 0$ and $\mu > 0$ such that*

$$I^{\varphi}[\mu(K_w f - f)] \leq S w^{-l},$$

for sufficiently large $w > 0$, where $l := \min\{\alpha\nu, \alpha_0, \theta_0\}$.

Note that the quantitative estimate established in Orlicz spaces (Theorem 4.3) and consequently also the qualitative one achieved in Corollary 4.5 turns out to be less sharp than that one achieved in Lebesgue spaces, even if we choose $L^{\varphi+\eta}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ (with $\psi(u) = u, u \in \mathbb{R}$). This is the price to pay to get the above generalization, that is due to the fact that, in general, the φ-modulus of smoothness does not satisfy the property $\omega(f, \lambda\delta) \leq (1+\lambda)\omega(f, \delta)$, which instead holds for $\omega(f, \cdot)_p$.

Now, we conclude this section recalling other useful examples of spaces $L^{\varphi}(\mathbb{R}^n)$. If we consider the convex φ-function $\varphi_\alpha(u) = e^{u^\alpha} - 1$, for $\alpha > 0$, the corresponding Orlicz spaces are the exponential spaces. Other well-known example of Orlicz spaces are, e.g., the Zygmund (or interpolation) spaces $L^\alpha \log^\beta L(\mathbb{R}^n)$ which are generated by the φ-functions $\varphi_{\alpha,\beta}(u) = u^\alpha \log^\beta(u+e)$, with $1 \leq \alpha < +\infty$ and $\beta \in \mathbb{R}^+$.

In the particular case of Theorem 4.3 applied to L^p-spaces, if the function ψ of condition $(\chi 3)$ is $\psi(u) = u^{q/p}$, $1 \leq q \leq p$, condition (H) turns out to be satisfied with $\eta(u) = u^q$ and $C_\lambda = \lambda^{q/p}$. So we have $L^{\varphi+\eta}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, which is a proper subspace of $L^p(\mathbb{R}^n)$.

5 Examples of kernels

In this section, we discuss about a suitable procedure in order to construct examples of kernels for the nonlinear multivariate sampling Kantorovich operators. In general, we consider kernel functions of the form

$$\chi(w\underline{x}-t_{\underline{k}},u)=L(w\underline{x}-t_{\underline{k}})\,g_w(u),$$

where $(g_w)_{w>0}$, $g_w:\mathbb{R}\to\mathbb{R}$ is a family of functions satisfying $g_w(u)\to u$ uniformly as $w\to+\infty$ and such that there exists a φ-function ψ with

$$|g_w(u)-g_w(v)|\le\psi(|u-v|), \tag{6}$$

for every $u,v\in\mathbb{R}$ and $w>0$.
For a sake of clarity, all the assumptions made in Section 2 on χ and L can be summarized as follows:

(i) $\left(L(w\underline{x}-t_{\underline{k}})\right)_{\underline{k}}\in\ell^1(\mathbb{Z}^n)$, for every $\underline{x}\in\mathbb{R}^n$ and $w>0$, $L\in L^1(\mathbb{R}^n)$ is bounded in a neighborhood of $\underline{0}\in\mathbb{R}^n$ and there exists a number $\beta_0>0$ such that

$$m_{\beta_0,\Pi^n}(L):=\sup_{\underline{x}\in\mathbb{R}^n}\sum_{\underline{k}\in\mathbb{Z}^n}L(\underline{x}-t_{\underline{k}})\left\|\underline{x}-t_{\underline{k}}\right\|_2^{\beta_0}<+\infty;$$

(ii) $g_w(0)=0$, for every $w>0$;

(iii) for every $j\in\mathbb{N}^+$, there exists $\theta_0>0$ such that

$$\mathcal{T}_j^w(\underline{x})=\sup_{u\neq 0}\left|\frac{g_w(u)}{u}\sum_{\underline{k}\in\mathbb{Z}^n}L(w\underline{x}-t_{\underline{k}})-1\right|=O(w^{-\theta_0}),$$

as $w\to+\infty$, uniformly with respect to $\underline{x}\in\mathbb{R}^n$.

An example of a family $(g_w)_{w>0}$ satisfying all the above assumptions is defined, for $w>0$, by $g_w(u)=u^{1-1/w}$ for every $a<u<1$, with $0<a<1/e$, and $g_w(u)=u$ otherwise (see, e.g., [43, 13]).
If instead $g_w(u)\equiv u$ for every $w>0$, we reduce again to the linear case already studied in [2]. For a sake of simplicity, in what follows, we will consider only the case of the uniform sequence $t_{\underline{k}}=\underline{k}$, $\underline{k}\in\mathbb{Z}^n$.
A first typical example of nonlinear multivariate sampling Kantorovich operator is based on the multivariate Fejér kernel $\mathcal{F}_n(\underline{x}):=\prod_{i=1}^n F(x_i)$, where F is the well-known Fejér kernel of one variable

$$F(x):=\frac{1}{2}\mathrm{sinc}^2\left(\frac{x}{2}\right),\quad x\in\mathbb{R},$$

where the sinc-function is defined by

$$\mathrm{sinc}(x):=\begin{cases}\frac{\sin(\pi x)}{\pi x} & x\in\mathbb{R}\setminus\{0\},\\ 1, & x=0.\end{cases}$$

We have that $\mathcal{F}_n$ is continuous, non-negative and bounded, belongs to $L^1(\mathbb{R}^n)$ and satisfies all the other required conditions. In particular, it is possible to see that (3) holds in view of the Strang-Fix condition recalled in Remark 3. In this case, we can assume

$$\chi(w\underline{x}-\underline{k},u):=\mathcal{F}_n(w\underline{x}-\underline{k})\,g_w(u),$$

and therefore, condition (iii) reduces to $\sup_{u\neq 0}\left|\frac{g_w(u)}{u}-1\right|=O(w^{-\theta_0})$, as $w\to+\infty$, for some $\theta_0>0$, which is obviously satisfied.
The corresponding nonlinear multivariate sampling Kantorovich operators take now the following form

$$(K_w^{\mathcal{F}_n}f)(\underline{x})=\sum_{\underline{k}\in\mathbb{Z}^n}\mathcal{F}_n(w\underline{x}-\underline{k})\,g_w\left(\frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w}f(\underline{u})d\underline{u}\right),$$

for every $w>0$, where $f:\mathbb{R}^n\to\mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x}\in\mathbb{R}^n$. As shown in Remark 6, it is easy to see that $\mathcal{F}_n$ satisfies (5) for every $0<\nu<1$. Hence, for $K_w^{\mathcal{F}_n}$, from Theorem 4.3 we can state the following corollary.

Corollary 5.1. *Let φ be a convex φ-function. Suppose that φ satisfies condition (H) with η convex, $f\in L^{\varphi+\eta}(\mathbb{R}^n)$. Then, for every $0<\nu<1$, $0<\alpha<1$, there exist constants $\mu>0$, $\lambda>0$ and $\lambda_0>0$, such that*

$$\begin{aligned}&I^\varphi[\mu(K_w^{\mathcal{F}_n}f-f)]\\&\le\frac{1}{3}\left\{m_{0,\Pi^n}(\tau)\omega\left(\lambda f,\frac{1}{w^\alpha}\right)_\eta+M_1m_{0,\Pi^n}(\tau)I^\eta[\lambda_0 f]w^{-\alpha_0}+\omega\left(\lambda f,\frac{\sqrt{n}}{w}\right)_\eta+I^\varphi[\lambda_0 f]w^{-\theta_0}\right\},\end{aligned}$$

for sufficiently large $w>0$, where $\alpha_0=(1-\alpha)\nu$, $m_{0,\Pi^n}(\tau)<+\infty$, $M_1>0$ and $\theta_0>0$ is the constant of condition (iii).

Another useful class of kernels is given by the so-called Jackson type kernels of order $s \in \mathbb{N}$, defined in the univariate case by

$$J_s(x) := c_s \,\mathrm{sinc}^{2s}\left(\frac{x}{2s\pi\alpha}\right), \quad x \in \mathbb{R},$$

with $\alpha \geq 1$ and c_s is a non-zero normalization coefficient, given by

$$c_s := \left[\int_{\mathbb{R}} \mathrm{sinc}^{2s}\left(\frac{u}{2s\pi\alpha}\right) du\right]^{-1}.$$

The multivariate Jackson type kernel is given by the n-fold product of the corresponding univariate function, as follows

$$\mathcal{J}_s^n(\underline{x}) = \prod_{i=1}^{n} J_s(x_i), \quad \underline{x} \in \mathbb{R}^n.$$

It is easy to prove that all the required assumptions are satisfied and the corresponding multivariate nonlinear sampling Kantorovich operators are given by

$$(K_w^{\mathcal{J}_n} f)(\underline{x}) = \sum_{\underline{k}\in\mathbb{Z}^n} \mathcal{J}_n(w\underline{x}-\underline{k})\, g_w\left(\frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} f(\underline{u})d\underline{u}\right),$$

for every $w > 0$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$. For $K_w^{\mathcal{J}_n}$, we can obtain an analogous result to that one achieved for $K_w^{\mathcal{F}_n}$.
For what concerns examples of function L with compact support, we can consider the well-known central B-spline (univariate) of order $s \in \mathbb{N}$, defined by

$$M_s(x) := \frac{1}{(s-1)!}\sum_{j=0}^{s}(-1)^j\binom{s}{j}\left(\frac{s}{2}+x-j\right)_+^{s-1},$$

where $x_+ := \max\{x, 0\}$ is the positive part of x. The Fourier transform of M_s is given by

$$\widehat{M_s}(v) = \mathrm{sinc}^s\left(\frac{v}{2\pi}\right), \quad v \in \mathbb{R},$$

and then, we have $\sum_{k\in\mathbb{Z}} M_s(u-k) = 1$, for every $u \in \mathbb{R}$, by Remark 3, and therefore, condition (iii) is again satisfied. Obviously, each M_s is bounded on $\mathbb{R}$, with compact support on $[-s/2, s/2]$, and hence $M_s \in L^1(\mathbb{R})$, for all $s \in \mathbb{N}$, with $\|M_s\|_1 = 1$. Further, condition (i) is fulfilled for every $\beta_0 > 0$. Thus we can define the multivariate central B-spline of order s, as follows

$$\mathcal{M}_s^n(\underline{x}) := \prod_{i=1}^{n} M_s(x_i), \quad \underline{x} \in \mathbb{R}^n,$$

and the corresponding multivariate nonlinear sampling Kantorovich operators are given by

$$(K_w^{\mathcal{M}_s^n} f)(\underline{x}) = \sum_{\underline{k}\in\mathbb{Z}^n} \mathcal{M}_s^n(w\underline{x}-\underline{k})\, g_w\left(\frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} f(\underline{u})d\underline{u}\right),$$

for every $w > 0$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$. For $K_w^{\mathcal{M}_s^n}$, from Corollary 4.4 we obtain the following.

Corollary 5.2. *Let φ be a convex φ-function. Suppose that φ satisfies condition (H) with η convex, $f \in L^{\varphi+\eta}(\mathbb{R}^n)$. Then, for every $0 < \alpha < 1$, there exist constants $\mu > 0$, $\lambda > 0$ and $\lambda_0 > 0$, such that*

$$I^{\varphi}[\mu(K_w^{\mathcal{M}_s^n} f - f)] \leq \frac{1}{3}\left\{m_{0,\Pi^n}(\tau)\,\omega\left(\lambda f, \frac{1}{w^\alpha}\right)_\eta + \omega\left(\lambda f, \frac{\sqrt{n}}{w}\right)_\eta + I^{\varphi}[\lambda_0 f]w^{-\theta_0}\right\},$$

for sufficiently large $w > 0$, where $m_{0,\Pi^n}(\tau) < +\infty$ and $\theta_0 > 0$ is the constant of condition (iii).

Finally, an example of non-product kernels can be of radial type, e.g., represented by the so-called Bochner-Riesz kernel of order $s > 0$, defined as follows

$$b_s^n(\underline{x}) := \frac{2^s}{\sqrt{(2\pi)^n}}\Gamma(s+1)\left\|\underline{x}\right\|_2^{-s-n/2} J_{s+n/2}(\left\|\underline{x}\right\|_2), \quad \underline{x} \in \mathbb{R}^n,$$

where J_λ is the Bessel function of order λ, with $\lambda > \frac{n-1}{2}$, and Γ is the usual Euler gamma function.
Since it is well-known that $J_\lambda(\left\|\underline{x}\right\|_2) = O(\left\|\underline{x}\right\|_2^{-n/2})$, as $\left\|\underline{x}\right\|_2 \to +\infty$, hence $b_s^n(\underline{x}) = O(\left\|\underline{x}\right\|_2^{-s-n})$, as $\left\|\underline{x}\right\|_2 \to +\infty$, then $b_s^n \in L^1(\mathbb{R}^n)$. Its Fourier transform is given by:

$$\widehat{b_s^n}(\underline{v}) = \begin{cases}(1-\left\|\underline{v}\right\|_2^2)^s, & \left\|\underline{v}\right\|_2 \leq 1,\\ 0, & \left\|\underline{v}\right\|_2 > 1,\end{cases} \quad \underline{v} \in \mathbb{R}^n,$$

namely, b_s^n is bandlimited (i.e., it belongs to the Bernstein class $B_1^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$). The corresponding nonlinear multivariate sampling Kantorovich operators take the following form

$$(K_w^{b_s^n} f)(\underline{x}) = \sum_{\underline{k}\in\mathbb{Z}^n} b_s^n(w\underline{x} - \underline{k})\, g_w\left(\frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} f(\underline{u})d\underline{u}\right),$$

for every $w > 0$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$. For other examples of kernels, the readers can see, e.g., [14, 15, 16].

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