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Strongly convex squared norms

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Abstract

Normed spaces for which the squared norm is strongly conves are intensively studied in literature. This note is motivated by the fact that a strongly convex squared norm plays a role in quantitative Korovkin approximation. We are concerned especially with the strong convexity of $\|\cdot\|_p^2$ on $\mathbb{R}^2$, $1 < p < 2$.

1 Introduction

Let $(E, \|\cdot\|)$ be a real normed space. For $0 < \varepsilon \leq 2$ let $\delta_E(\varepsilon) := \inf\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\}$.

$(E, \|\cdot\|)$ is called *uniformly convex* if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. It is called *strictly convex* if the unit sphere does not contain segments; equivalently, if $\delta_E(2) = 1$.

Let $c > 0$. $\|\cdot\|^2$ is called *c-strongly convex* if

$$\begin{aligned}(x, a, y; \|\cdot\|^2) &:= (1-a)\|x\|^2 + a\|y\|^2 - \|(1-a)x + ay\|^2 \\ &\geq ca(1-a)\|x-y\|^2,\ \forall x, y \in E,\ a \in [0,1].\end{aligned}$$

Normed spaces and norms with such properties are intensively studied in literature (see, e.g., [1, 2, 4, 6]) and the references therein. This note is motivated by a result from [5], according to which if $\|\cdot\|^2$ is c-strongly convex then it is useful in quantitative Korovkin approximation.

The elementary proof of the main result (Th. 3.2) was given by Andrzej Komisarski in [3]. At the end of the paper, a conjecture with geometric flavor is presented.

2 Preliminaries

Remark 1. (i) $\|\cdot\|^2$ is a convex function.

(ii) $(E, \|\cdot\|)$ is strictly convex iff $\|\cdot\|^2$ is strictly convex.

(iii) If $\|\cdot\|^2$ is c-strongly convex, then $c \leq 1$ and $\delta_E(\varepsilon) \geq 1 - \sqrt{1 - c\frac{\varepsilon^2}{4}} > 0$, and consequently $(E, \|\cdot\|)$ is uniformly convex.

(iv) If $(E, \|\cdot\|)$ is an inner-product space, then $\|\cdot\|^2$ is 1-strongly convex. Conversely, if $\|\cdot\|^2$ is 1-strongly convex, then according to (iii) and the Day-Nordlander Theorem,

$$1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \leq \delta_E(\varepsilon) \leq 1 - \sqrt{1 - \frac{\varepsilon^2}{4}},$$

and thus $(E, \|\cdot\|)$ is an inner-product space (see [2, p. 60]).

Remark 2. $(\mathbb{R}^2, \|\cdot\|_p)$ with $p > 2$ is strictly convex and finite dimensional, hence uniformly convex. However, $\|\cdot\|_p^2$ is not strongly convex.

Indeed, $\lim_{x \searrow 0} x^{-2}(1 - (1 - x^p)^{2/p}) = 0$, hence

$$\forall n \in \mathbb{N}\ \exists x_n \in (0,1):\ 1 - (1 - x_n^p)^{2/p} < \frac{1}{n} x_n^2.$$

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Let $y_n = \left(1 - x_n^p\right)^{1/p}$. Set $u_n = (-x_n, y_n)$, $v_n = (x_n, y_n)$. Then $\left(u_n, \frac{1}{2}, v_n; \|\cdot\|_p^2\right) < \frac{1}{n}\frac{1}{4}\|u_n - v_n\|^2$, $n \in \mathbb{N}$, which shows that $\|\cdot\|_p^2$ is not strongly convex.

Remark 3. In relation with the above definitions, let us recall that $(E, \|\cdot\|)$ is said to be q-convex for some $q \geq 2$ if $\exists d > 0$:

$$\left\|\frac{x+y}{2}\right\|^q \leq \frac{1}{2}(\|x\|^q + \|y\|^q) - \frac{d}{2}\|x - y\|^q, \; x, y \in E.$$

See [7], [6, p. 86].

3 Main result

So, for $p = 1$ or $p > 2$, $\|\cdot\|_p^2$ is not strongly convex as a norm on $\mathbb{R}^2$, while $\|\cdot\|_2^2$ is 1-strongly convex. It remains to study the case when $1 < p < 2$.

Theorem 3.1. *Let* $1 < p < 2$ *and* $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) := \left(\frac{|x|^p + |y|^p}{2}\right)^{2/p} - (p-1)\frac{x^2 + y^2}{2}, \; (x, y) \in \mathbb{R}^2,$$

Then f *is convex and* $f(x, y) > 0$ *for* $(x, y) \neq (0, 0)$.

Proof. Let us remark that

$$2^{1-\frac{2}{p}} > e^{1-\frac{2}{p}} > 1 + \left(1 - \frac{2}{p}\right) > p - 1. \tag{1}$$

On the other hand, the function $x \longmapsto x^{\frac{p}{2}}$ is subadditive on $[0, \infty)$, so that

$$\begin{aligned} f(x, y) &= \left(\frac{\left(x^2\right)^{p/2} + \left(y^2\right)^{p/2}}{2}\right)^{2/p} - (p-1)\frac{x^2 + y^2}{2} \geq \\ &\geq \left(\frac{\left(x^2 + y^2\right)^{p/2}}{2}\right)^{2/p} - (p-1)\frac{x^2 + y^2}{2} = \\ &= \frac{x^2 + y^2}{2}\left(2^{1-\frac{2}{p}} - (p-1)\right) > 0, \end{aligned}$$

for all $(x, y) \neq (0, 0)$. Now let's prove that f is convex on $(\mathbb{R}^+)^2$. Let $x > 0$, $y > 0, u := \frac{2x^p}{x^p + y^p}$, $v := \frac{2y^p}{x^p + y^p}$. Then $u > 0$, $v > 0$, $u + v = 2$. By a straightforward computation we find that the Hesse matrix of f is H with

$$\begin{aligned} H_{11} &= \frac{2-p}{2}u^{2-\frac{2}{p}} + (p-1)\left(u^{1-\frac{2}{p}} - 1\right), \\ H_{22} &= \frac{2-p}{2}v^{2-\frac{2}{p}} + (p-1)\left(v^{1-\frac{2}{p}} - 1\right), \\ H_{12} = H_{21} &= \frac{2-p}{2}u^{1-\frac{1}{p}}v^{1-\frac{1}{p}}. \end{aligned}$$

To prove that $H_{11} > 0$, let us remark that the function $\varphi(t) = u^t$ is convex, and so

$$\varphi\left[\left(\frac{2}{p} - 1\right)\left(2 - \frac{2}{p}\right) + \left(2 - \frac{2}{p}\right)\left(1 - \frac{2}{p}\right)\right] \leq \left(\frac{2}{p} - 1\right)\varphi\left(2 - \frac{2}{p}\right) + \left(2 - \frac{2}{p}\right)\varphi\left(1 - \frac{2}{p}\right).$$

This yields

$$1 \leq \left(\frac{2}{p} - 1\right)u^{2-\frac{2}{p}} + \left(2 - \frac{2}{p}\right)u^{1-\frac{2}{p}};$$

now

$$\begin{aligned} H_{11} &\geq \frac{2-p}{2}u^{2-\frac{2}{p}} + (p-1)u^{1-\frac{2}{p}} - (p-1)\left[\left(\frac{2}{p} - 1\right)u^{2-\frac{2}{p}} + \left(2 - \frac{2}{p}\right)u^{1-\frac{2}{p}}\right] \\ &= \frac{(2-p)^2}{2p}u^{2-\frac{2}{p}} + \frac{(p-1)(2-p)}{p}u^{1-\frac{2}{p}} > 0. \end{aligned}$$

Similarly $H_{22} > 0$. It remains to show that $\det H \geq 0$. A direct calculation reveals that

$$\begin{aligned} \det H = (p-1)^2\left[1 - u^{1-\frac{2}{p}} - v^{1-\frac{2}{p}}\right] - \frac{(p-1)(2-p)}{2}\left(u^{2-\frac{2}{p}} + v^{2-\frac{2}{p}}\right) \\ + (p-1)u^{1-\frac{2}{p}}v^{1-\frac{2}{p}}. \end{aligned}$$

Denote $\det H = g(u)$, where $u \in (0,2)$, $v = 2-u$. Then

$$g'(u) = \frac{(p-1)^2(2-p)}{p}(v-u)u^{-\frac{2}{p}}v^{-\frac{2}{p}}\left[\frac{1}{2}\left(u^{\frac{2}{p}}+v^{\frac{2}{p}}\right)-\frac{1}{p-1}\right].$$

The function $\psi(t) := t^{\frac{2}{p}}$ is convex on $[0,\infty)$, which implies $\psi(u)+\psi(v) \le \psi(0)+\psi(u+v)$, i.e., $u^{\frac{2}{p}}+v^{\frac{2}{p}} \le 2^{\frac{2}{p}}$. Now according to (1),

$$\frac{1}{2}(u^{\frac{2}{p}}+v^{\frac{2}{p}})-\frac{1}{p-1} \le 2^{\frac{2}{p}-1}-\frac{1}{p-1} < 0.$$

It follows that $g'(u) < 0$ iff $v-u > 0$, i.e., iff $u \in (0,1)$. Since $g(1) = 0$, we have $\det H = g(u) > 0$ for $u \neq 1$, which shows that f is a (strictly) convex function on $(\mathbb{R}^+)^2$. Then it is clearly convex on $(\mathbb{R}^-)^2$, $\mathbb{R}^+ \times \mathbb{R}^-$, $\mathbb{R}^- \times \mathbb{R}^+$. Consider now the semiaxis $\{(0,b)|\ b \ge 0\}$. It is easy to verify that the surface $z = f(x,y)$ has a tangent plane at the point $(0,b,f(0,b))$, namely the plane of equation

$$z = b(2^{1-\frac{2}{p}}+1-p)(y-\frac{b}{2}).$$

A similar reasoning involving the other semiaxes leads to the conclusion that f is a convex function. □

Theorem 3.2. *Consider the space $(\mathbb{R}^2, \|\cdot\|_p)$ with $1 < p < 2$. Then $\|\cdot\|_p^2$ is $(p-1)$-strongly convex, and $p-1$ is the largest constant with this property.*

Proof. From Theorem 3.1 we deduce that $\sqrt{f}$ is a norm on $\mathbb{R}^2$, and so $|\cdot| := \sqrt{\frac{2}{p-1}f}$ is also a norm. This implies

$$\|\cdot\|_p^2 = (p-1)2^{\frac{2}{p}-1}(|\cdot|^2+\|\cdot\|_2^2).$$

Now Lemma 1 shows that

$$(x,a,y;\|\cdot\|_p^2) \ge (p-1)2^{\frac{2}{p}-1}a(1-a)\,\|x-y\|_2^2,\ x,y \in \mathbb{R}^2.$$

It is easy to verify that $\|\cdot\|_2^2 \ge 2^{1-\frac{2}{p}}\,\|\cdot\|_p^2$, and so $(x,a,y;\|\cdot\|_p^2) \ge (p-1)a(1-a)\,\|x-y\|_p^2$, for all $x,y \in \mathbb{R}^2$, $a \in [0,1]$. Therefore, $\|\cdot\|_p^2$ is $(p-1)$-strongly convex. Suppose that $\|\cdot\|_p^2$ is c-strongly convex. Then

$$2(\|x\|_p^2+\|y\|_p^2)-\|x+y\|_p^2 \ge c\,\|x-y\|_p^2,\ x,y \in \mathbb{R}^2.$$

For $x = (1+\varepsilon, 1-\varepsilon)$, $y = (1-\varepsilon, 1+\varepsilon)$, this leads to

$$c \cdot 2^{\frac{2}{p}} \le \lim_{\varepsilon \searrow 0}\frac{[(1+\varepsilon)^p+(1-\varepsilon)^p]^{2/p}-2^{2/p}}{\varepsilon^2} = (p-1)2^{2/p},$$

and so $c \le p-1$. □

Conjecture 1. Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$, and $C := \{x \in \mathbb{R}^2|\ \|x\| = 1\}$. Then $\|\cdot\|^2$ is strongly convex iff $\exists M > 0$ such that $\forall x_1,x_2,x_3 \in C$ with Ox_1, Ox_2, Ox_3 pairwise distinct, the conic with center O and passing through x_1,x_2,x_3 is an ellipse with axes $\le M$.

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