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Spectral norms in spaces of polynomials

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Abstract

We consider a very general case of vector spaces of multivariate polynomials equipped with some norms. Between them we single out a class of spectral norms, that satisfy the condition $\|P^k\| = \|P\|^k$ for all positive integer k. In spaces of polynomials one can consider some linear operators that are usually unbounded, for example derivations, inclusions and multiplying by a fixed polynomial. Bounds for norms of derivatives of polynomials are related to Markov type inequality and Markov's exponent. We introduce a new concept of an asymptotic Markov's exponent and show that it is equal to Markov's exponent for a wide class of norms. However it is not true for all norms in the space of polynomials. We give some examples to show this. We prove an important and very useful inequality, which says that Markov's exponent for a norm with Nikolskii's property related to a compact set E is not less than Markov's exponent for the supremum norm on the set E. As a consequence we obtain a lower bound for the optimal exponent in Markov's inequality considered with L^p norms and other norms possessing a Nikolskii type property. Our result was used in the paper of Tomasz Beberok published in the Dolomites Research Notes on Approximation and it seems to be useful for future research. One of the main theorems shows a nice application of the Dedania theorem.

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1 Introduction

Let $(\mathbb{A}, \|\cdot\|)$ be a commutative complex normed algebra with unity e. The well-known and important quantity is the spectral seminorm $\|x\|_\sigma$ of an element $x \in \mathbb{A}$, where

$$\|x\|_\sigma := \lim_{n\to\infty} \|x^n\|^{1/n}.$$

Let us note an obvious property $\|x^k\|_\sigma = \|x\|_\sigma^k$. By the Bohenblust-Karlin theorem in [14] (cf. [44], [15])

$$\|x\|_\sigma = \inf\{\|x\|_a :\ \|\cdot\|_a \in D\},$$

where D is the set of all accessible norms $\|x\|_a$ (i.e. those that are equivalent to the original norm $\|x\|$ in $\mathbb{A}$, are submultiplicative $\|xy\|_a \le \|x\|_a\|y\|_a$ and $\|e\|_a = 1$). If a spectral seminorm is a norm then the completion of $\mathbb{A}$ is a semi-simple Banach algebra. There is well-known that in such algebras, two norms for which $\mathbb{A}$ is completed, are equivalent. However, very often there are considered non-completed algebras, in particular normed rings with a gradation. One of the most important case form polynomial classes. In normed algebras polynomials help to define various notions and quantities. One of them is the capacity of an element x defined by P. Halmos [26] (cf. [15] for the explanation) in the following way. Let $\widehat{P}_n(\mathbb{C})$ be the set of complex monic polynomials of degree $n \ge 1$ and let

$$\mathrm{cap}_n(x) = \inf\{\|P(x)\| :\ P \in \widehat{P}_n(\mathbb{C})\}^{1/n},\ \mathrm{cap}(x) = \inf_{n\ge1} \mathrm{cap}_n(x) = \lim_{n\to\infty} \mathrm{cap}_n(x).$$

$\mathrm{cap}(x)$ is invariant with respect to equivalent norms. One can also consider

$$\mathrm{Spcap}_n(x) = \inf\{\|P(x)\|_\sigma :\ P \in \widehat{P}_n(\mathbb{C})\}^{1/n},\ \mathrm{Spcap}(x) = \inf_{n\ge1} \mathrm{Spcap}_n(x) = \lim_{n\to\infty} \mathrm{Spcap}_n(x).$$

It is proved in [15] that those two quantities coincide. And both are equal to the logarithmic capacity of the spectrum $\sigma(x)$ in $\mathbb{C}$ (which is considered in the completion of $\mathbb{A}$):

$$\mathrm{cap}(x) = \mathrm{Spcap}(x) = C(\sigma(x)).$$

Let us observe that, with $P^{(n)}$ the n-th derivative of P, we have $\|P^{(n)}(x)\| = n!a_n$ where a_n is the leading coefficient of P and hence

$$\mathrm{cap}_n(x) = \inf\{\|n!P(x)\|/\|P^{(n)}(x)\| :\ \deg P = n\}^{1/n}.$$

Therefore, $\mathrm{cap}(x) > 0$ if and only if there exists a positive constant A such that

$$\|P^{(n)}(x)\| \le A^n n! \|P(x)\|,\ n = \deg P \ge 1.$$

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It is a time to come into the world of Markov's inequality. Let us recall that a compact subset of $\mathbb{C}$ is a Markov's (planar) set (cf. [18, 25] for the most classical cases) if there exist positive constants $\alpha, A = A(\alpha)$ such that

$$\|P'\|_E \le A(\deg P)^{\alpha}\|P\|_E.$$

Markov's exponent $m(E)$ is the infimum of α for which the above inequality (usually called *Markov's inequality* or more precisely an *Andriej Markov type inequality*) is satisfied. Let us recall the remarkable L. Białas-Cież theorem [11]:

if E is a planar Markov's set then the logarithmic capacity $C(E) > 0$.

As an immediate corollary we get is that if $\sigma(x)$ is a Markov's set then $\mathrm{cap}(x) > 0$. But we can consider a Markov inequality with respect to any seminorm $\|\cdot\|$ in $\mathbb{A}$: say that $x \in \mathcal{M}(\alpha, A)$ if $\|P'(x)\| \le A(\deg P)^{\alpha}\|P(x)\|$ for all polynomials P, and we can put $m(x) = \inf\{\alpha : \ x \in \mathcal{M}(\alpha, A)\}$. If we replace Markov's inequality for $\sigma(x)$ by such a condition with respect to a norm in $\mathbb{A}$ then Białas-Cież theorem is no longer true (cf. [6], see also the example below): there is an x with $m(x) < \infty$ but $\mathrm{cap}(x) = 0$. It is an open question: if Markov's inequality is satisfied for x with respect to the spectral norm then $\mathrm{cap}(x) > 0$? In the counterexample in [6] the spectral norm is in fact only a seminorm and thus Markov's inequality can not be satisfied.

One can consider a much stronger version of Markov inequality related to the classical Vladimir Markov inequality $\|P^{(k)}\|_{[-1,1]} \le T_n^{(k)}(1)\|P\|_{[-1,1]}$, $\deg P \le n$. The paper [3] introduced the definition: $x \in \mathcal{VM}(\alpha, A)$ iff $\|P^{(k)}(x)\| \le A^k k! \binom{\deg P}{k}^{\alpha} \|P(x)\|$ for all P. In particular, $\|P^{(n)}\| \le A^n n! \|P(x)\|$ if $n = \deg P$, which gives $\mathrm{cap}(x) \ge 1/A > 0$. Unfortunately, in general A. Markov's inequality does not imply V. Markov's estimate. In the connection between these two Markov type conditions we can consider *Markov's factors*:

$$M_{n,k} = \sup\{\|P^{(k)}(x)\| : \ k \le \deg P \le n, \|P(x)\| = 1\}, \ \mu_{n,k} = M_{n,k}^{1/k} \text{ for } 1 \le k \le n.$$

If $K(x) := \sup\{\mu_{n,1}^{1-\log k/\log n}\mu_{n,n}^{\log k/\log n}/\mu_{n,k} : \ 1 \le k \le n, \ n \ge 2\} > 0$ then A. Markov's inequality for x and positivity of $\mathrm{cap}(x)$ give V. Markov's inequality. Separately the condition $K(x) > 0$ does not imply anything. There is the conjecture that it is satisfied in very general case, maybe always (cf. [4]).

Example 1.1. Let $\mathcal{A}$ be the Wiener algebra of absolutely convergent Fourier series and consider $\mathcal{P}(e^{it})$ the algebra generated by $x = e^{it}$ that is

$$\mathcal{P}(e^{it}) = \{P(e^{it}) : \ P \in \mathcal{P}(\mathbb{C})\}, \ \mathcal{P}_n(e^{it}) = \{P(e^{it}) : \ p \in \mathcal{P}_n(\mathbb{C})\}.$$

If $P(e^{it}) = \sum_{j=0}^{n} a_j e^{ijt}$, then

$$\|P\| = \sum_{j=0}^{n} |a_j| = \sum_{j=0}^{n} \frac{1}{j!}|P^{(j)}(0)| = \|(P^{(j)}(0)/j!)_{j\ge 0}\|_{\ell^1}$$

and

$$\|P\|_{\sigma} = \|P\|_{\mathbb{T}} = \sup\{|P(\zeta)| : \ |\zeta| = 1\}.$$

Obviously $\|P\|_{\sigma} \le \|P\|$ and one can estimate $\|P\| \le \sqrt{n+1}\|P\|_{\sigma}$.

Let us consider two generalizations of the above norm:

$$\|P\|_{1,p} = \| \left(P^{(j)}(0)/(j!)^p\right)_{j\ge 0} \|_{\ell^1}, \ p \ge 1,$$

and

$$\|P\|_{2,p} = \| \left(P^{(j)}(0)/j!\right)_{j\ge 0} \|_{\ell^p}, \ 1 \le p \le 2.$$

We have

$$\|P\|_{\sigma} \le (n+1)^{1-1/p}\|P\|_{2,p}, \ \|P\|_{2,p} \le (n+1)^{1/p-1/2}\|P\|_{\sigma},$$

which gives $\lim_{m\to\infty} \|P^m\|_{2,p}^{1/m} = \|P\|_{\sigma}$.

The norm $\|\cdot\|_{1,p}$ for $p > 1$ is submultiplicative and equals one for e (i.e. for the identically one function), but it behaves badly. We have $\lim_{m\to\infty} \|(e^{it})^m\|_{1,p}^{1/m} = 0$ which implies $\lim_{m\to\infty} \|P^m(e^{it})\|_{1,p}^{1/m} = |P(0)|$ and it is not a norm.

Now consider $\|P\| = \sum_{l=0}^{\infty} \alpha_l |a_l| = \sum_{l=0}^{\infty} \frac{\alpha_l}{l!}|P^{(l)}(0)|$, where all α_l are positive. Then

$$\mathrm{cap}_n(x) = \alpha_n^{1/n}, \ \mathrm{cap}(x) = \inf_{n\ge 1} \alpha_n^{1/n}, \ M_{n,k} = \max_{k\le l\le n} \frac{\alpha_{l-k}}{\alpha_l} l(l-1)\cdots(l-k+1),$$

$$\mu_{n,k} = M_{n,k}^{1/k} = \max_{k\le l\le n} \left(\frac{\alpha_{l-k}}{\alpha_l} l(l-1)\cdots(l-k+1)\right)^{1/k}.$$

If $\alpha_l = (1/l!)^{p-1}$, then $\mu_{n,k} = (n(n-1)\cdots(n-k+1))^{p/k} = \left(k!\binom{n}{k}\right)^{p/k}$, $\mu_{n,1} = n^p$, $\mu_{n,n} = n!^{p/n}$. Applying known bounds for binomial coefficients and factorials we obtain $K(x) \ge e^{-p}$.

In the paper we shall consider norms in the multivariate case. Supremum norms are most commonly used and L^2 norms are often considered (see e.g. [16], [17], [13]). Sometimes applicable norms are related to coefficients of polynomials, cf. [9] with applications in number theory.

Our considerations are organized in the following way. In section 2 we introduce a basic notion of a generalized Nikolski property and discuss different contexts where this property can be considered. In the next section we propose a new idea of asymptotic Markov's exponent and we shall prove the main results of this paper, in particular a minimality of this quantity in the class of spectral norms. It is illustrated by a few examples.

2 Nikolskii's property

Definition 2.1. Let $q = \|\cdot\|$ be a norm on $\mathbb{P}(\mathbb{C}^N)$. This norm has *Markov's property* (cf. [37] for more information and bibliography) if there exist nonnegative constants M, m such that *Markov's type inequality* is satisfied

$$\|D_j Q\| \le M(\deg Q)^m \|Q\|,\ j = 1, \dots, N,\ Q \in \mathbb{P}(\mathbb{C}^N).$$

The *Markov's exponent* (cf. [8], [5]) of q is defined as the infimum of all possible constants m in Markov's inequality for q or is set to $+\infty$ if q does not have Markov's property.

One of the most important applications of Markov's type inequality is possibility to construct so called optimal meshes (for polynomials) (see e.g. [35], [42]) that are applied in numerical analysis.

In connection with the above property the next property is essential to check Markov's inequality with a proper norm to show that Markov's inequality is also satisfied for a given, interesting for us, norm (cf. [7], [6] for more precise examples).

Definition 2.2. Let E be a compact subset of $\mathbb{C}^N$. A norm $q = \|\cdot\|$ on $\mathbb{P}(\mathbb{C}^N)$ is *E-admissible* or *has Nikolskii's property* if there exist constants: positive A, B and nonnegative a, b such that for every $P \in \mathbb{P}(\mathbb{C}^N)$ with $\deg P \ge 1$ we have

$$\|P\|_E \le A(\deg P)^a \|P\| \text{ and } \|P\| \le B(\deg P)^b \|P\|_E.$$

It is worth remarking that some authors refer to this as a Bernstein-Markov property.

If $q = \|\cdot\|$ is E-admissible then

$$\|P\|_E = \lim_{s\to\infty} \|P^s\|^{1/s}.$$

Since the supremum norm is the main example of a spectral norm (see [44]) we can generalize the above definition.

Definition 2.3. A norm $q = \|\cdot\|$ on $\mathbb{P}(\mathbb{C}^N)$ is *spectral admissible* or *has the generalized Nikolskii's property* (briefly, GNP) if there exist a spectral norm $\|\cdot\|_\sigma$ and constants: positive A, B and nonnegative a, b such that for every $P \in \mathbb{P}(\mathbb{C}^N)$ with $\deg P \ge 1$ we have

$$\|P\|_\sigma \le A(\deg P)^a \|P\| \text{ and } \|P\| \le B(\deg P)^b \|P\|_\sigma.$$

The spectral norm is given by the formula

$$q_\sigma(P) = \|P\|_\sigma = \lim_{s\to\infty} \|P^s\|^{1/s}.$$

By way of illustration, here are examples of such norms.

Example 2.1. Let E be a compact subset of $\mathbb{C}$ and $r > 0$ be fixed. Put (cf. [6])

$$\|P\| = \sum_{k=0}^{\infty} \frac{1}{k!} \|P^{(k)}\|_E\, r^k.$$

Then

$$\lim_{n\to\infty} \|P^n\|^{1/n} = \max_{|\zeta|\le r} \|P(x+\zeta)\|_E.$$

Moreover for $\|P\|_\sigma := \max_{|\zeta|\le r} \|P(x+\zeta)\|_E$ we have

$$\|P\|_\sigma \le \|P\| \le (\deg P + 1)\|P\|_\sigma.$$

Example 2.2. If μ is a probability measure on E, then for $1 \le s < \infty$ the norm

$$\|P\| := \|P\|_E + \left(\int_E |P(z)|^s d\mu(z) \right)^{1/s}$$

satisfies $\|P\|_E \le \|P\| \le 2\|P\|_E$, so it is E-admissible on $\mathbb{P}(\mathbb{C}^N)$ with

$$\lim_{n\to\infty} \|P^n\|^{1/n} = \max(\|P\|_E, \operatorname{ess\,sup}_E |P|) = \|P\|_E.$$

Example 2.3. In the classical case of the interval $[-1,1]$ we have S. M. Nikolskii's inequalities (cf. [33], [24], [39], [31], [18], [38])

$$\left(\frac{1}{2} \int_{-1}^{1} |P(x)|^p dx \right)^{1/p} \le \|P\|_{[-1,1]} \le (2(p+1)n^2)^{1/p} \left(\frac{1}{2} \int_{-1}^{1} |P(x)|^p dx \right)^{1/p}.$$

Proposition 2.1 (A generalization of Nikolskii's inequality). *Let μ be a probability measure on E such that for a system of orthonormal polynomials we have the inequality $\|P\|_E \le B(\deg P)^\beta$ with some positive β, which is equivalent to the fact that for each polynomial P, $\deg P \ge 1$,*

$$\|P\|_E \le B_1 (\deg P)^{\beta_1} \|P\|_2 \tag{1}$$

with some $B_1, \beta_1 > 0$. Then for all $p \ge 1$ the norm $\|P\|_p = \left(\int_E |P(z)|^p d\mu(z) \right)^{1/p}$ is an E-admissible norm.

Proof. Indeed, if $(P_\alpha)_{\alpha\in\mathbb{N}^N}$ is an orthonormal system such that $\deg P_\alpha = |\alpha|$ then for each polynomial $P=\sum_\alpha c_\alpha P_\alpha$ with $\deg P\ge 1$, $\|P\|_E \le \binom{n+N}{n}\max_{|\alpha|\le n}|c_\alpha|B|\alpha|^\beta$, where $c_\alpha = \int_E P(z)\overline{P_\alpha(z)}d\mu(z)$, so we can take $B_1 = B\frac{2^N}{N!}$, $\beta_1=\beta+N$.

Let us also note that the condition $\|P\|_E \le B_1(\deg P)^{\beta_1}\|P\|_2$ implies the inequality $\|P\|_E \le B_1^{2/s}(\lceil s\rceil)^{2\beta_1/s}(\deg P)^{\beta/s}\|P\|_s$, $s\ge 1$. In particular, $\|P\|_E = \|P\|_\infty = \operatorname{ess\,sup}_E |P|$. □

Theorem 2.2. *If μ is the normalized Lebesgue measure on a fat compact set $E\subset\mathbb{R}^N$ then Nikolskii's inequality implies Markov's property of E.*

Proof. This is a consequence of the main results of [2] where pluripotential method were used. There is not known more elementary proof of those cited results, maybe except in the one dimensional case, which was investigated by Totik (cf. [40]) □

Hence, if we want to show that a given compact subset of $\mathbb{R}^N$ possesses Markov's property, it suffices to show Nikolskii's inequality as in the example above. Generally, it is a very difficult task to check Markov's property. Recently, a nontrivial result in this topic has been obtained by R. Pierzchała [36]. His remarkable result relates to a class of sets with a *special parametric property* introduced by himself. This property implies Nikolskii's inequality. It was not known to him that such sets possess Markov's property (except of a proper subclass of UPC sets, where the situation was clear).

Corollary 2.3. *If E is a subset of $\mathbb{R}^N$ with the special parametric property of Pierzchala [36], then also Markov's property is satisfied.*

Theorem 2.4 (Goetgheluck [23], Zeriahi [43], Jonsson [28])**.** *Let $E\subset\mathbb{R}^N$ be a compact set and μ be a probability measure on E with the following density condition:*

$$\exists G,\gamma>0\ \forall x\in E, r>0\quad \mu(E\cap\overline{B}(x,r))\ge Gr^\gamma.$$

The assumption that E has Markov's property implies (1) for E. This assumption is also necessary in the case of (normalized) Lebesgue measure.

Remark 1. This method was used in the proof of Nikolskii's inequality in the classical case (cf. [33], [24], [39]) as well as in more general situations investigated by A. Zeriahi [43], P. Goetgheluck [23] and A. Jonsson [28] (cf. also [29]). Goetgheluck in [23] proved that each UPC set in $\mathbb{R}^N$ (this wide family of sets was introduced by W. Pawłucki and W. Pleśniak in [34]) satisfies the density condition and has Markov's property. Therefore each UPC set (in particular each compact fat subanalytic subset of $\mathbb{R}^N$, (cf. [34] for this deep result) satisfies the generalized Nikolskii's inequality with respect to the normalized Lebesgue measure μ and Markov's inequality in $L^p(\mu)$. Zeriahi and Goetgheluck gave an upper bound for Markov's exponent in L^p norms related to sets with cusps. But they never have calculated the exact value of Markov's exponent for the norms that they considered. Actually there are known first examples of sets with cusps for which Markov's exponent (in $L^p(\mu)$, $1\le p<\infty$) is calculated. These were given by T. Beberok (cf. [10]).

Proposition 2.5. *Let $E\subset\mathbb{R}^N$. Put*

$$\|P\| = \|P\|_E + \int_{int(E)} |D_jP(x)|dx. \tag{2}$$

Then this norm is E-admissible.

Proof. The above conclusion is a consequence of a very nontrivial inequality

$$\int_{int(E)} |D_jP(x)|dx \le \sqrt{N}\pi^N(\mathrm{diam}(E))^{N-1}(\deg P)\|P\|_E,$$

which follows from [2]. □

Example 2.4. Let $\|P\| = \sup_{x\in[-1,1]} |P(x)|\sqrt{1-x^2}$ be Schur's norm. Since

$$\|P\| \le \|P\|_{[-1,1]} \le (\deg P+1)\|P\|,$$

Schur's norm is $[-1,1]$-admissible. Similarly, if we put

$$\|P\|_\alpha = \sup_{x\in[-1,1]} |P(x)|(1-x^2)^\alpha,\ \alpha\ge\frac{1}{2},$$

then the norm $\|\cdot\|_\alpha$ is $[-1,1]$-admissible (see [1]). Moreover, if we replace the interval $[-1,1]$ by the closed unit ball $B := \{x\in\mathbb{R}^N : \|x\|_*\le 1\}$ with respect to a fixed norm $\|\cdot\|_*$ in $\mathbb{R}^N$ then the norm defined by

$$\|P\|_\alpha = \sup_{x\in B}|P(x)|(1-\|x\|_*^2)^\alpha$$

is B-admissible. A more general situation is contained in the following way (cf. [1]). Let Ω be a bounded, star-shaped (with respect to the origin) and symmetric domain in $\mathbb{R}^N$ and let $E=\overline{\Omega}$. Let $v\in S^{N-1}$ be a fixed direction. As in [1], we define $\rho_v(x) = \sup\{\tau\ge 0 : [x-\tau v, x+\tau v]\subset E\}$. Assume that $\rho_v(tx)\ge M(1-|t|)^m$, $t\in[-1,1]$, $x\in\partial E$. Then for any $\alpha>0$, the norm

$$\|P\|_\alpha = \sup\{|P(tx)|(1-|t|)^\alpha : x\in\partial E, t\in[-1,1]\}$$

is E-admissible.

Remark 2. The Schur inequality in Example 2.4 is a special case of the division type inequality, which is often called the Schur type inequality. It was proved in [12] that on the complex plane properties related to Markov's and Schur's inequalities are equivalent.

Proposition 2.6. *If we have some norms with GNP, we can easily construct many other norms with this property. For example, if q_1, q_2 have GNP (with spectral norms $q_{1,\sigma}, q_{2,\sigma}$) then $q(P) = (q_1(P)^p + q_2(P)^p)^{1/p}$, $1 \le p \le \infty$ has the GNP with $q_\sigma = \max(q_{1,\sigma}, q_{2,\sigma})$.*

Proof. By assumption, there exist positive constants $A_1, B_1, A_2, B_2, a_1, b_1, a_2, b_2$ such that

$$q_{1,\sigma}(P) \le A_1 n^{a_1} q_1(P),\ q_1(P) \le B_1 n^{b_1} q_{1,\sigma}(P),$$

$$q_{2,\sigma}(P) \le A_2 n^{a_2} q_2(P),\ q_2(P) \le B_2 n^{b_2} q_{2,\sigma}(P),$$

where $\deg P \le n$. Then

$$q_\sigma(P) = \max(q_{1,\sigma}(P), q_{2,\sigma}(P)) \le \max(A_1 n^{a_1} q_1(P), A_2 n^{a_2} q_2(P)) \le \max(A_1, A_2) n^{\max(a_1,a_2)} \max(q_1(P), q_2(P)) \le A n^a q(P)$$

with $A = \max(A_1, A_2)$, $a = \max(a_1, a_2)$.

On the other hand

$$q(P) \le \left(\left(B_1 n^{b_1} q_{1,\sigma}(P) \right)^p + \left(B_2 n^{b_2} q_{2,\sigma}(P) \right)^p \right)^{1/p} \le \max(B_1, B_2) n^{\max(b_1,b_2)} \left(q_{1,\sigma}(P)^p + q_{2,\sigma}(P)^p \right)^{1/p} \le B n^b q_\sigma(P)$$

with $B = 2^{1/p} \max(B_1, B_2)$, $b = \max(b_1, b_2)$. □

Proposition 2.7. *Let $\|\cdot\|_0$ be a spectral norm in $\mathbb{P}(\mathbb{C}^N)$ and let $\|\cdot\|_1$ be a GNP norm with respect to $\|\cdot\|_0$. If $\alpha_j \in \mathbb{Z}_+^N$, $j = 1, \dots, l$ are fixed then we can consider*

$$\|P\| = \|P\|_0 + \max_{1 \le j \le l} \|D^{\alpha_j} P\|_1.$$

We have $\lim_{s\to\infty} \|P^s\|^{1/s} = \|P\|_0$ but GNP will be satisfied if and only if we have a Markov-Nikolskii type bound

$$\max_{1 \le j \le l} \|D^{\alpha_j} P\|_1 \le C(\deg P)^\gamma \|P\|_0.$$

Proof. It is clear that $\liminf_{s\to\infty} \|P^s\|^{1/s} \ge \|P\|_0$. If $\|P\|_1 \le A(\deg P)^a \|P\|_0$ then we have $\|P^s\| \le \|P\|_0^s + A(s \deg P)^a \max_{1\le j\le l} \|D^{\alpha_j}(P^s)\|_0$.

As an application of the multivariate version of the *Faà di Bruno formula for the composition of two functions* (cf. [19] where the proper formula was discovered and proved) we get the following expression

$$D^\alpha(P^s) = \sum_{1 \le i \le |\alpha|} s(s-1)\cdots(s-i+1)P^{s-i} \sum_{l=1}^{|\alpha|} \sum_{p_l(\alpha,i)} \prod_{j=1}^{l} \frac{(D^{\ell_j} P)^{\kappa_j}}{(\kappa_j)!(\ell_j)!},$$

where $\kappa_j \in \mathbb{Z}_+$, $\ell_j \in \mathbb{Z}_+^N$ and

$$p_l(\alpha, i) = \left\{ (\kappa_1, \dots, \kappa_l, \ell_1, \dots, \ell_l) :\ k_j > 0,\ 0 \prec \ell_1 \prec \cdots \prec \ell_s,\ \sum_{r=1}^{l} \kappa_r = i,\ \sum_{r=1}^{l} \kappa_r \ell_r = \alpha \right\}.$$

Hence we derive for a fixed polynomial P of $\deg P > 0$

$$\begin{aligned}
\|P^s\| \le& \|P\|_0^s + A(s \deg P)^a \max_{1\le j\le l} \left(\|P\|_0^{s-|\alpha_j|} \sum_{1\le i\le|\alpha_j|} s^i \|P\|_0^{|\alpha_j|-i} \sum_{l=1}^{|\alpha|} \sum_{p_l(\alpha,i)} \prod_{j=1}^{l} \frac{(\|D^{\ell_j} P\|_0)^{\kappa_j}}{(\kappa_j)!(\ell_j)!} \right) \\
\le& \|P\|_0^s + A(s \deg P)^a \|P\|_0^s s^{\max_{1\le j\le l}|\alpha_j|} \max_{1\le j\le l} \left(\sum_{1\le i\le|\alpha_j|} \|P\|_0^{-i} \sum_{l=1}^{|\alpha|} \sum_{p_l(\alpha,i)} \prod_{j=1}^{l} \frac{(\|D^{\ell_j} P\|_0)^{\kappa_j}}{(\kappa_j)!(\ell_j)!} \right) \\
=& \|P\|_0^s + A(s \deg P)^a \|P\|_0^s s^{\max_{1\le j\le l}|\alpha_j|} C(P),
\end{aligned}$$

where the constant $C(P)$ depends on P. Applying elementary calculus arguments we easily obtain $\limsup_{s\to\infty} \|P^s\|^{1/s} \le \|P\|_0$ and thus $\lim_{s\to\infty} \|P^s\|^{1/s} = \|P\|_0$.

The second part of the Proposition is obvious. □

Example 2.5. Let us give two examples.

Let $\|P\| = \|P\|_{[-1,1]\cup\{2\}} + \|P'\|_{[-1,1]\cup\{2\}}$. The set $[-1,1]\cup\{2\}$ is not perfect. It is well known that each Markov's set is perfect. By this reason the considered norm does not satisfy GNP.

Now we define $\|P\| = \|P\|_E + \|\frac{\partial P}{\partial x}\|_E$, where $E = \{(x,y) \in \mathbb{R}^2 :\ |x| < 1,\ |y| \le \exp(-1/(1-|x|))\} \cup \{(-1,0),(1,0)\}$. Since (cf. [1])

$$\left\| \frac{\partial P}{\partial x} \right\|_E \le 2(\deg P)^2 \|P\|_E,$$

the norm $\|\cdot\|$ possesses GNP.

3 Asymptotic exponent in Markov's inequality

Let $\varphi = (\varphi_1, \dots, \varphi_N) \in \mathcal{C}^\infty(\mathbb{K}^N)^N$ (if $\mathbb{K} = \mathbb{C}$ we understand that $\varphi_j \in \mathcal{C}^\infty(\mathbb{R}^{2N})$). We assume that φ_j can take complex values. In particular, we can consider $\varphi_j \equiv v_j \in \mathbb{C}$ for $j \in \{1, \dots, N\}$ and then $\varphi = v \in \mathbb{C}^N$. Define

$$D = D_\varphi = \varphi_1 D_1 + \dots + \varphi_N D_N : \mathcal{C}^\infty(\mathbb{K}^N) \longrightarrow \mathcal{C}^\infty(\mathbb{K}^N)$$

and put $D^{(k)} = D \circ \dots \circ D$ k-times.

Let us recall a deep identity (cf. [32], [7], [6])

$$(D(f))^k = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} f^j D^{(k)}(f^{k-j}). \tag{3}$$

Definition 3.1. Let $q = \|\cdot\|$ be a norm in $\mathbb{P}(\mathbb{K}^N)$. If H is a homogenous polynomial of N variables of degree $k \geq 1$ then we consider a differential operator $D = H(D_1, \dots, D_N)$ and define

$$m(H,q) = \inf\{s > 0 : \exists M > 0 \; \forall P \in \mathbb{P}(\mathbb{K}^N) \; \|DP\| \leq M(\deg P)^s \|P\|\}.$$

For $\alpha \in \mathbb{N}^N$ and $H_\alpha(x) = x^\alpha$, $x \in \mathbb{K}^N$, we put $m(\alpha, q) = m(H_\alpha, q)$ and $m(q) = \max_{1 \leq j \leq N} m(e_j, q)$. For $k \geq 1$ we put $m_k(q) = \max\{m(\alpha, q) : \; |\alpha| = k\}$. In particular, $m_1(q) = m(q)$ is *Markov's exponent for a norm* q.

One can observe that for $m(x)$ defined in the introduction we have $m(x) = m(q)$, where $q(P) = \|P(x)\|$.

Remark 3. In the special case of $q(P) = \|P\|_E$, where E is a compact subset of $\mathbb{K}^N$, then we define $m(H,E) = m(H,q)$, $m(\alpha, E) = m(\alpha, q)$, $m_k(E) = m_k(q)$, $m(E) = m(q)$. Moreover the last one is Markov's exponent of E which was recalled in the first section and if $m(E) < \infty$ we say that E has *Markov's property*. Let us note the equality (for subsets of $\mathbb{R}^N$, cf. [6])

$$m(H_k, E) = k m(E),$$

where $H_k(x_1, \dots, x_N) = x_1^k + \dots + x_N^k$ (k is a fixed positive even integer).

Since

$$m(\alpha, q) \leq m(e_1, q)\alpha_1 + \dots + m(e_N, q)\alpha_N \leq \max_{1 \leq j \leq N} m(e_j, q)|\alpha| = m(q)|\alpha|,$$

we get the inequality

$$m_k(q) \leq k m(q) \;\Rightarrow\; \frac{1}{k} m_k(q) \leq m(q). \tag{4}$$

Remark 4. From [32] we have $\frac{1}{k} m_k(E) = m(E)$. Therefore

$$\lim_{k \to \infty} \frac{1}{k} m_k(E) = m(E).$$

Definition 3.2. Let q be a norm in $\mathbb{P}(\mathbb{K}^N)$. We define *the asymptotic exponent* for q,

$$m^*(q) := \limsup_{k \to \infty} \frac{1}{k} m_k(q).$$

Proposition 3.1. *Let us note a few basic properties of the above notion.*

(a) If q_1 and q_2 are two norms on $\mathbb{P}(\mathbb{K}^N)$ such that

$$q_1(P) \leq A(\deg P)^a q_2(P), \; q_2(P) \leq B(\deg P)^b q_1(P), \; \deg P \geq 1,$$

then $m^*(q_1) = m^*(q_2)$.

(b) In particular, if $q_1(P) = q_{2,\sigma}(P)$ (q_2 has the GNP with the spectral norm $q_{2,\sigma}$) then $m^*(q_2) = m^*(q_1)$.

(c) We have $m^*(q) \leq m(q)$. In general, these exponents do not need to be equal.

Proof. Fix an $\varepsilon > 0$, there exists $k_0(\varepsilon)$ such that

$$m_k(q_2) < k(m^*(q_2) + \varepsilon) \;\text{ for } k > k_0(\varepsilon).$$

Let $|\alpha| = k$, we have

$$q_1(D^\alpha P) \leq A(\deg P)^a q_2(D^\alpha P) \leq A(\deg P)^a (\deg P)^{m_k(q_2)} q_2(P) \leq AB(\deg P)^{a+b} (\deg P)^{k(m^*(q_2)+\varepsilon)} q_1(P).$$

Therefore

$$\frac{1}{k} m_k(q_1) \leq \frac{a+b}{k} + m^*(q_2) + \varepsilon.$$

By taking the lim sup of both sides of the above inequality as $k \to \infty$, we get

$$m^*(q_1) \leq m^*(q_2) + \varepsilon,$$

which by arbitrariness of ε gives us $m^*(q_1) \leq m^*(q_2)$. Similarly, $m^*(q_2) \leq m^*(q_1)$, hence $m^*(q_1) = m^*(q_2)$.

The inequality in the condition (c) follows easily from (4). □

Now, we give an example of the norms for which $m^*(q) < m(q)$. First, we need the following

Proposition 3.2. *For $\|\cdot\|_0$ a seminorm on $\mathbb{P}(\mathbb{C})$, $m > 0$ and $s \in \mathbb{N}_1$ we define the norm*

$$q_{m,s}(P) = \|P\|_{m,s} = \sum_{r=0}^{\infty} \frac{1}{((rs)!)^m} \|P^{(rs)}\|_0.$$

If for every $s \geq 2$ there exist positive constants A, B such that for every $j \in \{1, \ldots, s\}$ and $P \in \mathbb{P}(\mathbb{C})$, $\|P^{(j)}\|_0 \leq A\|P\|_0 + B\|P^{(s)}\|_0$, then $m_k(q_{m,s}) \leq sm\lceil \frac{k}{s} \rceil$ for every $k \in \mathbb{N}_1$.

Proof. For every $m > 0$, $t, s \in \mathbb{N}_1$, $j \in \{1, \ldots, s\}$ and $P \in \mathbb{P}(\mathbb{C})$ we obtain

$$\begin{aligned}
\|P^{(st+j)}\|_{m,s} =& \sum_{r=0}^{\infty} \frac{1}{((rs)!)^m} \|P^{(rs+st+j)}\|_0 \leq A \sum_{r=0}^{\infty} \frac{1}{((rs)!)^m} \|P^{(rs+st)}\|_0 + \max\{B, 1\} \sum_{r=0}^{\infty} \frac{1}{((rs)!)^m} \|P^{(rs+st+s)}\|_0 \\
\leq& A \sum_{r=0}^{[\frac{\deg P}{s}]} \frac{1}{((rs)!)^m} \|P^{(rs+st)}\|_0 + B' \sum_{r=0}^{[\frac{\deg P}{s}]} \frac{1}{((rs)!)^m} \|P^{(rs+st+s)}\|_0 \\
=& A \sum_{r=t}^{[\frac{\deg P}{s}]+t} \frac{1}{(((r-t)s)!)^m} \|P^{(rs)}\|_0 + B' \sum_{r=t+1}^{[\frac{\deg P}{s}]+t+1} \frac{1}{(((r-t-1)s)!)^m} \|P^{(rs)}\|_0 \\
\leq& (A + B')(\deg P)^{s(t+1)m} \sum_{r=0}^{\infty} \frac{1}{((rs)!)^m} \|P^{(rs)}\|_0 = (A + B')(\deg P)^{s(t+1)m} \|P\|_{m,s}
\end{aligned}$$

with $B' = \max\{B, 1\}$. □

Example 3.1. Let us consider the norms $q_{m,s}$ defined as in Proposition 3.2 with seminorm $\|P\|_0 = \sum_{l=0}^{s-1} \frac{1}{l!}|P^{(l)}(0)|$, $s \in \mathbb{N}_1$. Then for every $P \in \mathbb{P}(\mathbb{C})$ and $j \in \{1, \ldots, s-1\}$ we have

$$\begin{aligned}
\|P^{(j)}\|_0 =& \sum_{l=0}^{s-1} \frac{1}{l!} |P^{(j+l)}(0)| = \sum_{l=j}^{s-1} \frac{l!}{l!(l-j)!} |P^{(l)}(0)| + \sum_{l=0}^{j-1} \frac{l!}{l!(s+l-j)!} |P^{(s+l)}(0)| \\
\leq& (s-1)^{s-1} \sum_{l=0}^{s-1} \frac{1}{l!} |P^{(l)}(0)| + \sum_{l=0}^{s-1} \frac{1}{l!} |P^{(s+l)}(0)| \leq (s-1)^{s-1} \|P\|_0 + \|P^{(s)}\|_0
\end{aligned}$$

From Proposition 3.2 for $m > 0$ and $s \in \mathbb{N}_1$ we obtain $m_k(q_{m,s}) \leq sm\lceil \frac{k}{s} \rceil$.

On the other hand for every $m > 0$ and $s, n \in \mathbb{N}_1$ we have

$$\|x^{sn}\|_{m,s} = \sum_{r=0}^{\infty} \left(\frac{1}{(rs)!} \right)^m \sum_{l=0}^{s-1} \frac{1}{l!} |(x^{sn})^{(rs+l)}(0)| = \frac{1}{(sn)!^{m-1}}.$$

and

$$\|(x^{sn})^{(st+j)}\|_{m,s} = \sum_{r=0}^{\infty} \left(\frac{1}{(rs)!} \right)^m \sum_{l=0}^{s-1} \frac{1}{l!} |(x^{sn})^{(rs+st+j+l)}(0)| = \frac{(sn)!}{(s-j)!(sn-st-s)!^m}.$$

Hence for every $k \in \mathbb{N}_1$ we have $m_k(q_{m,s}) = sm\lceil \frac{k}{s} \rceil$, where for $x \in \mathbb{R}$, $\lceil x \rceil$ is the smallest integer greater than or equal to x. From this it follows that $m^*(q_{m,s}) = m$ and $m(q_{m,s}) = sm$.

We now formulate the main results of this paper.

Theorem 3.3. *Let q be a GNP norm with the spectral norm q_σ. Then*

$$m^*(q) = \lim_{k \to \infty} \frac{1}{k} m_k(q) = m(q_\sigma).$$

In particular, $m(q_\sigma) \leq m(q)$.

Proof. Firstly, we prove that $m_k(q_\sigma) = km(q_\sigma)$, $k \geq 1$. If for every $j \in \{1, \ldots, N\}$ there exist positive constants M_j, m_j such that for every polynomial $P \in \mathbb{P}_n(\mathbb{K}^N)$,

$$\|D_j P\| \leq M_j n^{m_j} \|P\|$$

then for $\alpha \in \mathbb{N}_0^N$ such that $|\alpha| = k$ we have

$$\|D^{(\alpha_1, \ldots, \alpha_N)} P\| \leq M_1^{\alpha_1} \cdot \ldots \cdot M_N^{\alpha_N} n^{\alpha_1 m_1 + \ldots + \alpha_N m_N} \|P\| \leq (\max_{j \in \{1, \ldots, N\}} M_j)^k n^{km} \|P\|,$$

where $m = \max_{j \in \{1, \ldots, N\}} m_j$. Hence $m_k(q) \leq km(q)$ for every spectral admissible norm q.

On the other hand

$$(D_jP)^k = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}P^j\frac{\partial^k}{\partial x_j^k}P^{k-j}.$$

The norm q_σ is spectral and so by the Theorem in [21] it is submultiplicative. Hence, if $\varepsilon > 0$ is fixed, then

$$\|(D_jP)\|_\sigma^k \le C(\varepsilon)\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\|P\|_\sigma^j(n(k-j))^{m_k(q_\sigma)+\varepsilon}\|P\|_\sigma^{k-j} \le C(\varepsilon)\frac{2^k}{k!}(nk)^{m_k(q_\sigma)+\varepsilon}\|P\|_\sigma^k,$$

where $C(\varepsilon)$ is a constant that depends on ε. Therefore $m(q_\sigma) \le m_k(q_\sigma)/k + \varepsilon/k$. Letting $\varepsilon \to 0+$ we get the inequality $m(q_\sigma) \le m_k(q_\sigma)/k$ and finally $m_k(q_\sigma) = km(q_\sigma),\ k \ge 1$.

Now, let $s > m(\alpha, q_\sigma)$. Then

$$\|D^\alpha P\| \le B(\deg P)^b\|D^\alpha P\|_\sigma \le BM_s(\deg P)^{b+s}\|P\|_\sigma \le BM_sA(\deg P)^{b+a+s}\|P\|.$$

Thus, we have

$$m(\alpha, q) \le b + a + s \ \Rightarrow\ m(\alpha, q) \le b + a + m(\alpha, q_\sigma)$$

and therefore $m_k(q) \le b + a + m_k(q_\sigma) = b + a + km(q_\sigma)$. Hence

$$m^*(q) = \limsup_{k\to\infty}\frac{1}{k}m_k(q) \le m(q_\sigma).$$

Analogously, let $s > m(\alpha, q)$. Then

$$\|D^\alpha P\|_\sigma \le A(\deg P)^a\|D^\alpha P\| \le AM_s'(\deg P)^{a+s}\|P\| \le ABM_s'(\deg P)^{a+b+s}\|P\|_\sigma.$$

Therefore $m(\alpha, q_\sigma) \le a + b + s$ and $m(\alpha, q_\sigma) \le a + b + m(\alpha, q)$. Hence $km(q_\sigma) = m_k(q_\sigma) \le a + b + m_k(q)$ which shows that

$$m(q_\sigma) \le \liminf_{k\to\infty}\frac{1}{k}m_k(q) \le \limsup_{k\to\infty}\frac{1}{k}m_k(q) \le m(q_\sigma).$$

□

Corollary 3.4. *Let q be an E-admissible norm. Then*

$$m^*(q) = \lim_{k\to\infty}\frac{1}{k}m_k(q) = m(E).$$

In particular, $m(E) \le m(q)$.

Finally, we have the following important corollary

Corollary 3.5. *(a) If a norm q has the GNP with the spectral norm q_σ then*

$$m^*(q) = m(q) \quad\Leftrightarrow\quad m(q) = m(q_\sigma).$$

(b) If for a norm $q = \|\cdot\|$ we have Markov's inequality

$$\|D_jP\| \le M(\deg P)^{m(q_\sigma)}\|P\|,\ j = 1, \ldots, N$$

then the exponent $m(q_\sigma)$ is the best possible. In particular, $m(q) = m(q_\sigma)$.

(c) If E is an UPC subset of $\mathbb{R}^N$, then $m_p(E) \ge m(E)$, where $m_p(E)$ is Markov's exponent with respect to the Lebesgue measure.

From Theorem 3.3 we have $m^*(q) = m(q_\sigma)$ which gives (a). We can also appeal to Theorem 3.3 to see that the condition (b) is met. The third conclusion follows from Remark 1. The second statement is a useful tool to find the best exponent.

Remark 5. In papers where Markov's inequality in L^p norms was proved with the best possible exponent, usually it was difficult and time-consuming to prove the optimality of the exponent, which is Markov's exponent for such kinds of norms (cf. [27], [22], [13], [16], [20], [30], [41]). By applying the above corollary it is done automatically.

Let us consider another (simple) example. By Bernstein's inequality

$$\|\sqrt{1-x^2}P'(x)\|_{[-1,1]} \le (\deg P)\|P\|_{[-1,1]}$$

and by Schur's inequality

$$\|P\|_{[-1,1]} \le (\deg P + 1)\|\sqrt{1-x^2}P(x)\|_{[-1,1]}$$

we get Markov's inequality with respect to Schur's norm

$$\|\sqrt{1-x^2}P'(x)\|_{[-1,1]} \le \deg P(\deg P + 1)\|\sqrt{1-x^2}P(x)\|_{[-1,1]},$$

with exponent 2. In view of Corollary 3.5, it is the best possible exponent.

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