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Chebyshev-like interpolation points on the unit disc based on quasi-random sampling

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Abstract

A simple heuristic construction of good interpolation points on the unit disc is proposed. First, a set of quasi-random uniformly distributed points in the unit square is generated. Consequently, this set is transformed onto the unit sphere using cylindrical Lambert map and, finally, the points of the upper hemisphere are orthogonally projected onto the unit disc. Numerical results concerning important properties of the proposed disc interpolation points are presented.

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1 Introduction

One of the fundamental problems in polynomial interpolation is efficient choice of interpolation points ensuring well-conditioned interpolation process and high accuracy of the resulting interpolating polynomial. To investigate this problem in the multidimensional case, let us consider the interpolation problem

$$P_N(\mathbf{x}_i) = f(\mathbf{x}_i), \quad i = 1, 2, ..., n$$

for a multivariate function $f(\mathbf{x})$ continuous on a compact multidimensional domain Ω and the interpolation points $\mathbf{x}_i \in \Omega$. Here, the lowercase letter n denotes the dimension of the space of all multivariate polynomials of total degree $\leq N$. Therefore, n is the total number of the polynomial basis functions $\phi_j(\mathbf{x})$ used in the interpolation process. Note that in the bivariate case the letters n and N are coupled by the equality $n = (N+1)(N+2)/2$.

Although in contrast to the one-dimensional case the construction of multivariate fundamental Lagrange polynomials is far from being an easy task, using suitable polynomial basis functions $\phi_j(\mathbf{x})$, we may write the desired interpolating polynomial $P_N(\mathbf{x})$ in the form

$$P_N(\mathbf{x}) = \sum_{j=1}^{n} c_j \phi_j(\mathbf{x}).$$

The unknown coefficients c_j are determined by solving the system of linear equations

$$M_n \mathbf{c} = \mathbf{f},$$

where $M_n(i,j) = \phi_j(\mathbf{x}_i), 1 \leq i, j \leq n$ is generalized Vandermonde matrix (collocation matrix), $\mathbf{c} = (c_1, c_2, ..., c_n)^T$ and $\mathbf{f} = (f(\mathbf{x}_1), f(\mathbf{x}_2), ..., f(\mathbf{x}_n))^T$.

As generally known, the requirements mentioned above are satisfied if the condition number of the collocation matrix M_n and the Lebesgue constant Λ_N are as small as possible. More precisely, if $\mathrm{cond}(M_n)$ is large, one must expect $\log_{10}(\mathrm{cond}(M_n))$ digits of the final output may be destroyed by the roundoff error. The requirement for the Lebesgue constant Λ_N to be as small as possible stems from the inequality

$$\|u - L_N\| \leq (1 + \Lambda_N)\|u - u_N\|,$$

where L_N and u_N are the Lagrange interpolating polynomial of u and the best polynomial approximation of u in the maximum norm, respectively. The Lebesgue constant Λ_N, dependent on the set of n interpolation points $\mathbf{x}_i$, is defined

$$\Lambda_N = \max_{\mathbf{x} \in \Omega} \sum_{i=1}^{n} |\ell_i(\mathbf{x})|.$$

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Here

$$\ell_i(\mathbf{x}) = \det(M_n^{(i)}(\mathbf{x}))/\det(M_n),$$

where $M_n^{(i)}(\mathbf{x})$ is created from M_n by replacing $\mathbf{x}_i$ with $\mathbf{x}$, denotes the $i-th$ fundamental Lagrange polynomial with the cardinal property $\ell_i(\mathbf{x}_j) = \delta_{i,j}$. The interpolation points for which Lebesgue constant attains its minimal value are called Lebesgue points of Ω.

The problem of choice of the optimal interpolation points in one dimension is well understood and well documented [31, 46]. On the other hand, the situation in the multidimensional case is essentially more complicated and efficient choice of the optimal or at least good interpolation points with respect to Λ_N (eventually with respect to cond(M_n)) for general domains is still an open question. Interpolation points are considered to be the good points if Λ_N grows at most polynomially. To clarify more precisely what means by good points for polynomial interpolation, see [6].

In the last years we see a few successful attempts to solve this problem in the multidimensional case as well. The first of these attempts, we comment here, was published in 2009 by Sommariva and Vianello [43]. The authors presented an elegant greedy algorithm selecting approximate Fekete points by QR factorization of a generalized Vandermonde matrix created for a finite set of carefully selected candidate points. Similarly, greedy algorithms selecting among suitable candidate points have also been used for computing good approximations of the Lebesgue points greedily minimizing a variant of a weighted Lebesgue function [33] and an approximate Lebesgue function [47]. The methods based on approximate Lebesgue function, where this function is represented by its values on a large finite set of *a priori* selected suitable points followed by nonlinear optimization were proposed by Briani *et al.* [12] and Gunzburger and Teckentrup [20]. The main difference between [12] and [20] consists in different evaluation of the fundamental Lagrage polynomials. In [12] the Vandermonde determinant representation followed by the MATLAB function 'fmincon' to solve the corresponding minimax problem is used. In [20] the use of an efficient algorithm of Sauer ad Xu [38] for evaluating $\ell_i(\mathbf{x})$ followed by the MATLAB function 'fminmax' is considered. Although all the mentioned methods require nontrivial computations mainly for large sets of interpolation points, e.g., $n > 1000$, they are able in principle to produce good interpolation points for arbitrarily shaped simply connected domains and probably for multiply connected domains as well.

Attempts have been made to construct good interpolation points on the unit disc given explicitly by a formula. Based on the results of [7], Carnicer and Godés [13] proposed points in the form $(r_\ell, \theta_{\ell j})$, where these points are equally placed on concentric circles situated inside the unit disc. However, in this construction it is necessary to optimize the values of r_ℓ in order to ensure the Lebesgue constant and condition numbers of the corresponding collocation matrices created by the Zernike polynomials are small. Similar construction of good disc interpolation points based on equally spaced points on inner concentric circles (this pattern of points generating nonsingular collocation matrices is known as Bos array [5, 6, 8]), is presented by Ramos-López *et al.* [36], where the values of r_ℓ are determined by minimizing the condition number of the corresponding collocation matrices built by the Zernike polynomials. As shown in numerical experiments [13, 36], both the mentioned methods produce good interpolation points at least for small orders of interpolating polynomials ($N \leq 40$).

In this paper we concentrate our effort to construct good interpolation points on the unit disc with respect to the conditioning of the collocation matrices M_n for more than 10 000 interpolation points. The quality of the generated point sets will be measured by four parameters - the smallest and largest singular values of M_n, their spectral condition numbers, and maximum norm error bounds of the interpolating polynomials of three functions of very different nature.

2 Construction of disc interpolation points

For more than 150 years after appearance of the initiatory paper of Pafnuty Lvovich Chebyshev [14] the scientific community is fascinated by a variety of formulas, relations and very useful properties of the Chebyshev polynomials and their zeros [30]. With respect to the intention of this paper to create good interpolation points on the unit disc, we need to mention only two of them

- **excellent interpolation properties** - as known [31, 46], the zeros of the Chebyshev polynomials of the first kind are near-optimal points for interpolation of smooth functions on the finite interval by algebraic polynomials and the corresponding Lebesgue constant is bounded by the inequality

$$\Lambda_N < \frac{2}{\pi}\log(N+1) + 0.9733...;$$

- **simple geometric construction** - these zeros may be created also as the orthogonal projection of points equally distributed on the upper circular arc over the interval $\langle -1, 1\rangle$ into the same interval [46], where

$$\zeta_i^n = \cos((2i-1)\pi/2n), \quad i = 1, 2, ..., n.$$

The perfect interpolation properties of the Chebyshev zeros and their simple geometric construction raise the following question.

How good can interpolation points on the unit disc be if they are created from equally distributed points on the upper unit hemisphere orthogonally projected into the unit disc ?

Since the terminology concerning spherical points is diverse, e.g., equally spaced, evenly spaced, uniformly distributed and well-separated spherical points, we give the following definition.

Definition 2.1. Let us have n different points $Q_1, Q_2, ..., Q_n$ placed on the sphere $\mathbb{S}^2$ and define

$$d_i = \min_{1\leq j\leq n, j\neq i} \| Q_i - Q_j \|, \quad 1 \leq i \leq n,$$

where $\| . \|$ denotes the usual Euclidean norm in $\mathbb{R}^3$. The points $Q_1, Q_2, ..., Q_n$ are called equally distributed on $\mathbb{S}^2$ if all d_i are equal.

However, there are exactly only five possibilities how to distribute points on the sphere equally. These possibilities are represented by the Platonic solids - tetrahedron, hexahedron (cube), octahedron, icosahedron, and dodecahedron, inscribed in the sphere under consideration. To avoid this obstacle, instead of equally distributed spherical points we will consider the so called uniformly distributed spherical points [4].

Definition 2.2. The sequence of spherical points $Q_1, Q_2, ...$ is called uniformly distributed, if we have, for each spherical cap A,

$$\lim_{n\to\infty}(\frac{1}{n}\#\{1 \le i \le n \ : \ Q_i \in A\}) = \frac{Area(A)}{4\pi},$$

where the symbol $\#X$ denotes the number of elements (or cardinality) of the set X.

Although the efficient construction of uniformly distributed spherical points is one of the challenging mathematical problems for the XXI. century [39], with history dating back to the famous paper of L. L. Thomson [45], this approach offer all the theoretical results and computational experiences achieved throughout the decades of research in this area. The variety of methods used to generate uniformly distributed spherical points is enormous and may roughly be divided into three categories

- **computational category** - the points are computed in order to optimize a suitable criterion;
- **closed form category** - the points are computed directly using simple formulas derived usually from different geometrical considerations, e.g., equal area partition of the sphere;
- **quasi-random number category** - the points are computed from a quasi-random uniformly distributed points in the unit square mapped onto the unit sphere by a suitable map.

In this section we review briefly some of them. A more exact presentation of the most popular uniformly distributed spherical point sets can be found in [21].

2.1 Computational category

One of the most famous problems generating the set of uniformly distributed spherical points (UDSPs) based on an optimization principle arises from electrostatics, where the arrangement of equal point charges on the sphere to be in equilibrium with respect to a Coulomb potential law is investigated. This problem (called Thomson's problem) is defined as follows

$$\text{UDSPs} = \arg\min_{\mathbf{Q}_j,\mathbf{Q}_k\in\mathbb{S}^2} \sum_{1\le j<k\le n} \| \mathbf{Q}_j - \mathbf{Q}_k \|^{-1},$$

where $\mathbb{S}^2$ denotes the unit sphere in the Euclidean space $\mathbb{R}^3$ [1, 11, 27, 37].

Another important problem leading to uniformly distributed spherical points is linked to the computation of point sets providing equal weights quadrature formulae on $\mathbb{S}^2$, the so called spherical t-design [11]. This problem is defined as to find a finite set of points $\mathbf{Q} = \{\mathbf{Q}_1, \mathbf{Q}_2, ..., \mathbf{Q}_M\} \subset \mathbb{S}^2$ satisfying

$$\int_{\mathbb{S}^2} p(\mathbf{x}) d\mu(\mathbf{x}) = \frac{4\pi}{M}\sum_{i=1}^{M} p(\mathbf{Q}_i)$$

for all $p \in P_t(\mathbb{S}^2)$, where μ is the surface measure on $\mathbb{S}^2$ and $P_t(\mathbb{S}^2)$ is the space of all spherical polynomials on $\mathbb{S}^2$ of total degree $\le t$. The solution of these problems for large n (say $n > 1000$) is computationally very expensive [2, 3, 15] and the corresponding methods are not considered in this paper.

2.2 Closed form category

The construction of different sets of uniformly distributed spherical points known in closed form (as a formula) has been initiated by serious demands in science and engineering, e.g., biomedical imaging [25], material sciences [26], astrophysics [18], and computer graphics [29], to have simply generated spherical points exhibiting some specific properties. Among many sets of such points the generalized spiral points [35, 37], HEALPix (Hierarchical Equal Area isoLatitude Pixelization) and Reuter points [19], Koaya points [26] and Fibonacci points [29] have been used in our computational experiments. Now, having a set of uniformly distributed spherical points, the corresponding Chebyshev-like disc interpolation points with the cartesian coordinates (x_k, y_k) are generated by those spherical points (x_k, y_k, z_k) for which $z_k > 0$.

Although the disc interpolation points proposed in [13, 36], mentioned in the introduction, need an initial optimization of some parameters, we can consider these point sets as generated by a formula as well.

The sets of spherical points of closed form mentioned above are easily computable for all n. On the other hand, the condition numbers of the corresponding collocation matrices M_n grow rapidly. The sets of points giving the smallest condition numbers of M_n are presented in Table 1. To create the collocation matrices M_n the Logan-Shepp algebraic polynomials (4) defined in Section 3 were used. An attempt has been made to improve the conditioning of M_n for the generalized spiral points of Rakhmanov, Saff, and Zhou [35] with respect to the parameter C (the authors recommend the value $C \equiv C_n = 3.6$), however, the improved values of $\text{cond}(M_n(C_n^{opt}))$ for $C_n^{opt} \in \langle 3.6, 10\rangle$ were not sufficiently small for our purposes. Moreover, the function $\omega(C_n) = \text{cond}(M_n(C_n))$ to be minimized exhibits chaotic multiminima behaviour that makes the optimization process very awkward.

Perhaps the most aesthetically influential uniformly distributed points on the sphere are represented by Fibonacci spherical points (the following derivation is presented in [29]) generated starting from the Fibonacci lattice on the unit square

$$x_j = \langle jF_{m-1}/F_m \rangle, \quad y_j = j/F_m, \quad 0 \le j < F_m,$$

where F_m are numbers of the Fibonacci sequence $\{0,1,1,2,3,5,8,13,...\}$ given by the recurrence relation

$$F_m = F_{m-1} + F_{m-2}, \quad m > 1.$$

Here $F_0 = 0, F_1 = 1$ and $\langle x \rangle$ denotes the fractional part of x. Consequently, the transformation of these points by the cylindrical Lambert map

$$(x,y) \to (2\cos(2\pi y)\sqrt{x-x^2}, 2\sin(2\pi y)\sqrt{x-x^2}, 1-2x) \tag{1}$$

produces the following set of points situated on the unit sphere

$$\theta_j = \arccos(1-2j/F_m), \quad \phi_j = 2\pi\langle jF_{m-1}/F_m \rangle \;\; 0 \le j < F_m. \tag{2}$$

Table 1: Comparison of the condition numbers of M_n generated by the Fibonacci disc points, Carnicar and Godés points (C&G), and OCS (optimal concentric sampling) points of Ramos-López *et. al.* The last three columns report the smallest condition numbers of M_n (one hundred samples for each n) for two kinds of quasi-random point sets created by the Matlab function 'rand' and a jittered sampling generated by the tensor product of one-dimensional equally spaced points.

N	n	Fibonacci	C&G	OCS	rand	n	jittered
10	66	1.9+01	2.1+01	1.2+01	4.7+02	64	1.3+02
20	231	8.3+01	9.8+01	4.5+01	1.2+04	225	8.8+02
30	496	9.6+02	5.4+02	2.5+02	9.8+04	484	4.9+03
40	861	9.9+03	5.1+03	1.8+03	3.8+05	841	4.8+03
50	1326	3.1+04	6.1+04	1.4+04	1.6+06	1296	7.9+03
60	1891	2.7+05	7.7+05	1.1+05	5.7+06	1936	2.0+04
70	2556	9.5+08	1.1+07	9.1+05	6.1+06	2601	4.3+04
80	3321	3.1+07	1.3+08	7.8+06	2.2+07	3364	5.7+04
90	4186	3.0+07	1.8+09	6.6+07	1.6+07	4225	7.7+04
100	5151	3.4+08	2.5+10	5.6+08	5.3+07	5184	9.4+04

Finally, considering the property of the Fibonacci sequence

$$\lim_{m\to\infty} F_m/F_{m-1} = (1+\sqrt{5})/2 \equiv \Phi,$$

the periodicity of the spherical coordinates and setting $F_m = n$, we achieve the spherical Fibonacci points

$$\theta_j = \arccos(1-2j/n), \quad \phi_j = 2j\pi\Phi^{-1}, \quad 0 \le j < n. \tag{3}$$

To obtain a more uniform distribution of points near the poles, Swinbank and Purser [42] consider $\theta_j = \arccos(1-(2j+1)/n)$. Although the point set (2), available only for the number of points n equal to a member of the Fibonacci sequence, belongs to the quasi-random point category, its simplified version (3), giving the Fibonacci spherical points for arbitrary n, may be considered to be of the closed form category for all n. The configurations of 1597 Fibonacci lattice points in $(0,0.5)\times(0,1)$ and the corresponding 1597 Fibonacci disc points are presented in Figure 1.

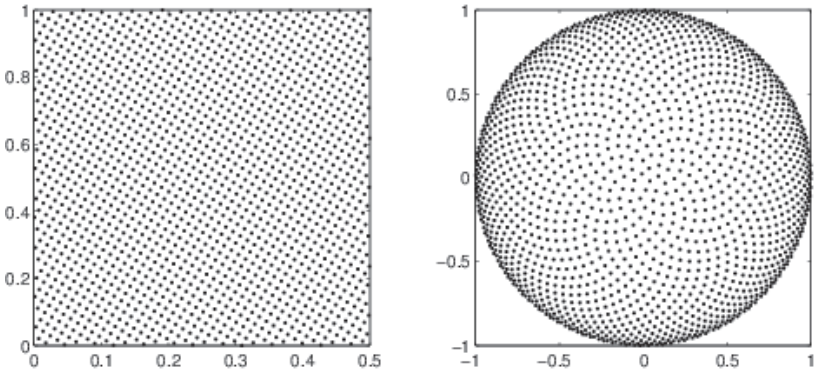

Figure 1: Configurations of the Fibonacci lattice points in $(0,0.5)\times(0,1)$ and the corresponding Fibonacci disc points for $n = 1597$.

Another sets of uniformly distributed spherical points available in closed form, very useful in the solution of global weather and climate prediction models, are generated by subdividing the faces of three most popular Platonic solids - cube (6 square faces), icosahedron (20 triangular faces), and dodecahedron (12 pentagonal faces) inscribed in the sphere under consideration [22, 24, 44]. The faces of the considered solid are divided into triangles or quadrilaterals and the centroids or edge midpoints of all faces are gnomically projected onto the spherical surface in order to create a new solid with more vertices. (The gnomic projection maps each point P of an inscribed solid onto spherical surface along the straight line drawn from the center of the considered sphere through the point P.) This process is repeated until the requested density of the generated spherical points

is achieved. Since this construction may be initiated also by the Archimedean solids, e.g., truncated icosahedron (a mix of pentagonal and hexagonal faces), and truncated cube (a mix of square and triangular faces), this approach offers enormous diversity of resulting spherical point sets. On the other hand, the repeated divisions of polyhedron faces causes the number of generated spherical points grows rapidly. Consequently, this approach is not able to generate sets of points for arbitrary value of n and, as such, this approach lies beyond the scope of this work.

2.3 Quasi-random number category

The Fibonacci lattice enable us to create not only the sets of nice and useful spherical points, however, it offers also very simple recipe how to generate variety of uniformly distributed spherical points if an arbitrary set of uniformly distributed points on $(0,1)\times(0,1)$ is considered and transformed by the cylindrical Lambert map (1) onto the unit sphere. To test the efficiency of this approach we can start our search for the optimal disc points with quasi-random points on the unit square created by the MATLAB function 'rand'. Although the distribution of these points exhibits very small uniformity, the smallest condition numbers of the corresponding interpolation matrices M_n (computed repeatedly for 100 samples of the point sets generated by 'rand') and shown in Table 1 behave better than those of the Fibonacci, Carnicer and Godés, and Ramos-López *et al.* points. Consequently, using a jittered sampling (perturbed regularly distributed points), exhibiting higher uniformity than the points generated by 'rand', the corresponding values of $\min_{1\le i\le 100}(\mathrm{cond}(M_n(i)))$ reported also in Table 1 exhibit very encouraging behavior - as n increases, the values of $\mathrm{cond}(M_n)$ increases more slowly than the values of all the point sets presented in Table 1. In this place it is worth to note that a simple quasi-random set of points may produce better conditioned collocation matrices than the sets of points generated by essentially more sophisticated constructions.

The natural question now arises.**Is there a set of uniformly distributed points in the unit square giving the corresponding collocation matrices M_n better conditioned than the jittered sampling reported in Table 1 ?**

As the uniformly distributed points represent one of the basic tools of very mature and well developed scientific discipline, the quasi-Monte Carlo methods [28, 34], the choice of these points is very rich. To avoid never ending computations with different sets of uniformly distributed points, we decided for the class of very popular and highly uniform Halton points. At this place it is worth to refresh at least the definition of the uniformly distributed sequences of real numbers [32].

Definition 2.3. The sequence of real numbers $x_1, x_2, \ldots$ is called uniformly distributed in $[0,\ 1]$ if we have, for each subinterval $[a,b)\subset[0,1]$,

$$\lim_{n\to\infty}\left(\frac{1}{n}\#\{x_1,x_2,\ldots,x_n\}\cap[a,b)\right)=b-a,$$

where the symbol $\#X$ denotes the number of elements (or cardinality) of the set X.

To create the Halton points on the unit square we have firstly to define a Van der Corput sequence of real numbers from $(0,1)$ based on the radical inverse function

$$\xi_p(n)=\sum_{i=0}^{n_p}a_i/p^{i+1},$$

where $a_i\in[0,p-1]$ are the coefficients of the p-adic expansion of n defined as

$$n=a_0+a_1p+\ldots+a_rp^r.$$

Here r denotes the maximum index for which a_i is not equal to zero, i.e., $n_p=r$. Then the p-adic Van der Corput sequence is defined as the sequence $x_n=\xi_p(n)$ for $n\ge 0$.

Therefore, the set of n Halton points on the unit square $\mathrm{H}_n(p,q)$ is defined as the set of points $(\xi_p(k),\xi_q(k)), k=1,\ldots,n$, where p,q are co-prime integers often called the bases. Now, the cylindrical Lambert map (1) transforms each set of n Halton points on the unit sphere. Finally, the corresponding Chebyshev-like Halton disc interpolation points $\mathrm{HD}_n(p_x,p_y)$ with the coordinates (x_k,y_k) are generated by those spherical points (x_k,y_k,z_k) for which $z_k>0$. This procedure may be slightly simplified if the Halton points are created on the rectangle $(0,0.5)\times(0,1)$. The Lambert map transforms this rectangle directly on the upper hemisphere.

3 Numerical experiments

The numerical experiments described in this section pursue two targets. First, a simple procedure how to find a set of conditionally near-optimal interpolation points on the unit disc is presented. Second, the accuracy of the polynomial interpolation based on the proposed Halton disc points $\mathrm{HD}_n(p_x,p_y)$ is compared with the accuracy based on approximate Fekete points (AFPs) of Sommariva and Vianello [9, 43] for three functions of different nature. All the presented results are obtained using the basis functions composed by the Logan-Shepp algebraic polynomials

$$\varphi_{L,\ell}(x,y)=\frac{1}{\sqrt{\pi}}\mathrm{U}_L\left(x\cos(\frac{\ell\pi}{L+1})+y\sin(\frac{\ell\pi}{L+1})\right),\quad \ell=0,1,\ldots,L, \tag{4}$$

where U_L is the degree L Chebyshev polynomial of the second kind [10]. These polynomials are orthonormal on the unit disc in the sense

$$\int_0^{2\pi}\int_0^1\varphi_{L,\ell}(r\cos(\theta),r\sin(\theta))\varphi_{K,k}(r\cos(\theta),r\sin(\theta))rdrd\theta=\delta_{L,K}\delta_{\ell,k}$$

for all indexes L,K and $0\le\ell\le L,\ 0\le k\le K$. The sets of disc interpolation points used in the presented computations are created for $n=100*2^k, k=0,1,2,\ldots,7$, points.

3.1 Computation of Halton-disc points

Usually, we need sets of interpolation points for *a priori* selected values of n. In this case it is possible to find a set of n good points among $\mathrm{HD}_n(p_x,p_y)$ for different values of co-prime integers $p_x,p_y \geq 2$. This approach is defined as the discrete optimization problem

$$(p_x^{opt},p_y^{opt}) = \arg\min_{(p_x,p_y)\in \mathbf{P}_K} \mathrm{cond}(M_n(\mathrm{HD}_n(p_x,p_y))),$$

where $\mathbf{P}_K$ denotes the set of co-prime integer pairs (p_x,p_y), e.g., $p_x = 2, p_y = 2i+1, i = 1,2,...,K$. Consequently, the resulting pair (p_x^{opt},p_y^{opt}) generates the desired Halton-disc points $\mathrm{HD}_n(p_x^{opt},p_y^{opt})$. Although to find the optimal set of points among all the reasonable candidates $\mathrm{HD}_n(p_x,p_y)$ for large n is computationally very intensive task, sets of good disc interpolation points have been found without massive computations as seen in Table 2. Table 3 shows the same parameters as Table 2 obtained for the sets $\mathrm{HD}_n(2,3)$.

Table 2: Condition numbers and the smallest and largest singular values of M_n generated by the optimized $\mathrm{HD}_n(p_x,p_y)$ point sets.

n	p_x^{opt}/p_y^{opt}	$\mathrm{cond}(M_n)$	$\sigma_{min}(M_n)$	$\sigma_{max}(M_n)$
100	2/99	4.9+1	0.6965	3.4+1
200	101/4	1.6+2	0.3925	6.1+1
400	4/202	4.9+2	0.2001	9.8+1
800	3/796	1.6+3	0.1010	1.6+2
1600	3/37	6.9+3	0.0456	3.1+2
3200	4/97	1.7+4	0.0317	5.4+2
6400	2/103	3.0+4	0.0261	7.8+2
12800	2/27	6.1+4	0.0242	1.5+3

Table 3: Condition numbers and the smallest and largest singular values of M_n generated by the $\mathrm{HD}_n(2,3)$ point sets.

n	$\mathrm{cond}(M_n)$	$\sigma_{min}(M_n)$	$\sigma_{max}(M_n)$
100	1.5+2	0.2607	3.8+1
200	1.1+3	0.0556	5.6+1
400	2.1+3	0.0552	1.1+2
800	4.0+3	0.0438	1.8+2
1600	7.9+3	0.0391	3.1+2
3200	2.8+5	0.0020	5.3+2
6400	8.9+5	0.0010	9.1+2
12800	2.6+5	0.0062	1.6+3

Table 4: Condition numbers and the smallest and largest singular values of M_n generated by the approximate Fekete disc points computed by the algorithm of Sommariva and Vianello applied to m Halton disc candidate points $\mathrm{HD}_m(2,3)$.

n	m	$\mathrm{cond}(M_n)$	$\sigma_{min}(M_n)$	$\sigma_{max}(M_n)$
100	3262	2.8+1	1.6271	4.5+1
200	7600	4.5+1	1.6473	7.3+1
400	6130	8.3+1	1.4124	1.2+2
800	7800	1.8+2	1.1032	2.0+2
1600	16900	3.8+2	0.8949	3.4+2
3200	11320	8.0+2	0.6762	5.4+2
6400	10140	2.2+3	0.3903	8.6+2
12800	21200	5.6+3	0.2807	1.6+3

3.2 Computation of AFPs

The crucial point in computing AFPs is the choice of suitable sets of candidate points with the property to mimic the distribution of the corresponding Fekete points as much as possible. Although there are many ways how to select suitable sets of candidate points, the proposed Halton-disc points are used owing to their simplicity and efficiency.

For the selected n and constant co-prime integers p_x,p_y, e.g., $p_x = 2, p_y = 3$, we solve the discrete optimization problem

$$m^{opt} = \arg\min_{m\in \mathrm{T}_K} \mathrm{cond}(M_n(\mathrm{AFPs}(m,n,p_x,p_y))),$$

where T_K denotes the finite increasing sequence of integers, e.g., $t_1 = 2n, t_{i+1} = t_i + h, 1 \leq h \leq 100, i = 1,2,...,K$ and $\mathrm{AFPS}(m,n,p_x,p_y)$ denotes the set of n AFPs created by the algorithm of Sommariva and Vianello using $\mathrm{HD}_m(p_x,p_y)$ as the candidate points. Consequently, the algorithm of Sommariva and Vianello applied to the set of Halton-disc points $\mathrm{HD}_{m^{opt}}(p_x,p_y)$ gives the desired AFPs with the properties presented in Table 4.

3.3 Interpolation: Halton-disc points versus AFPs

The values of cond(M_n) and $\sigma_1(M_n)$ presented in Tables 2, 3, and 4 manifest that the proposed Halton-disc points are able to generate well-conditioned collocation matrices. However, the crucial property of good interpolation points that remains to be investigated is either these points are able to produce also high accurate interpolants for a class of important functions. To verify this property, comparisons of the interpolation errors based on the Halton disc points and AFPs are made interpolating three functions depending on a parameter α managing the behaviour of the considered functions. The class of highly oscillating functions are represented by the function

$$u_1^{(\alpha)}(x,y) = \sin(\alpha x y).$$

For a sufficiently large α the function

$$u_2^{(\alpha)}(x,y) = 1 - \exp(-\alpha(1-x^2-y^2))$$

exhibits a rapid variation (boundary layer) near the boundary $\partial\Omega$. Finally, the function

$$u_3^{(\alpha)}(x,y) = \exp(-\alpha(x^2+y^2))$$

generates a mildly sharp spike at the origin. The interpolation error bounds for these three functions and for selected values of α are shown in Tables 5, 6, and 7.

Table 5: Interpolation error bounds $err_{1,n}^{\alpha}$ for Example 1 and $\alpha = 1, 50, 100$ (from top to bottom) computed by HD$_n(2,3)$ points, optimized HD$_n(p_x^{opt}, p_y^{opt})$ points, and AFPs.

n	100	200	400	800	1600	3200	6400	12800
(2,3)	1.6-07	9.6-13	8.3-15	7.0-15	1.7-14	4.3-14	2.9-13	9.4-14
optim	1.5-07	3.1-13	4.3-15	8.0-15	2.2-14	1.7-14	6.9-14	2.7-14
AFPs	9.8-08	6.2-14	1.5-15	1.6-15	1.6-15	3.1-15	5.9-15	1.1-14
(2,3)	>1	>1	>1	>1	>1	9.4-05	1.6-10	7.8-12
opt	>1	>1	>1	>1	>1	8.5-05	4.2-12	7.6-12
AFPs	>1	>1	>1	>1	0.5-00	1.5-05	5.5-13	1.5-12
(2,3)	>1	>1	>1	>1	>1	>1	>1	1.8-09
opt	>1	>1	>1	>1	>1	>1	6.8-01	1.9-09
AFPs	>1	>1	>1	>1	>1	>1	2.4-01	1.5-09

Table 6: Interpolation error bounds $err_{2,n}^{\alpha}$ for Example 2 and $\alpha = 10, 100, 200$ (from top to bottom) computed by HD$_n(2,3)$ points, optimized HD$_n(p_x^{opt}, p_y^{opt})$ points, and AFPs.

n	100	200	400	800	1600	3200	6400	12800
(2,3)	1.4-01	2.0-03	9.8-06	4.7-11	4.8-14	2.3-12	1.9-12	4.9-13
opt	4.1-02	2.1-03	6.1-06	5.8-11	4.6-14	7.7-14	1.1-13	1.4-13
AFPs	9.4-03	2.7-04	5.2-07	7.9-12	8.2-15	7.8-15	1.4-14	1.5-14
(2,3)	>1	>1	>1	2.4-01	5.6-03	6.1-05	3.8-11	4.3-13
opt	>1	>1	>1	2.6-01	8.5-03	5.6-06	6.8-12	2.0-13
AFPs	5.5-01	3.8-01	1.3-01	2.2-02	1.1-03	8.9-07	1.5-12	6.2-14
(2,3)	>1	>1	>1	>1	2.1-01	1.5-01	4.5-05	2.8-12
opt	>1	>1	>1	>1	3.6-01	6.7-03	3.5-06	1.9-12
AFPs	7.4-01	6.3-01	3.7-01	1.5-01	4.1-02	1.1-03	1.5-06	3.0-13

Table 7: Interpolation error bounds $err_{3,n}^{\alpha}$ for Example 3 and $\alpha = 10, 100, 200$ (from top to bottom) computed by HD$_n(2,3)$ points, optimized HD$_n(p_x^{opt}, p_y^{opt})$ points, and AFPs.

n	100	200	400	800	1600	3200	6400	12800
(2,3)	1.0-01	1.4-01	8.4-06	4.2-11	8.8-14	5.4-13	2.7-12	1.7-13
opt	4.2-02	2.6-03	5.8-06	6.5-11	3.5-14	3.8-14	1.2-13	1.9-13
AFPs	1.1-02	2.1-04	4.9-07	9.1-12	3.7-15	7.7-15	1.5-14	3.0-14
(2,3)	>1	>1	9.9-01	1.2-01	9.8-03	1.3-04	1.4-10	1.2-12
opt	>1	>1	>1	2.0-01	8.0-03	7.2-06	9.0-12	3.3-13
AFPs	7.2-01	3.4-01	1.9-01	3.2-02	6.6-04	8.8-07	7.0-13	3.9-14
(2,3)	>1	>1	>1	6.6-01	3.5-01	1.8-01	6.9-05	3.0-12
opt	>1	>1	>1	>1	2.9-01	8.8-03	4.2-06	2.0-12
AFPs	9.0-01	6.3-01	4.9-01	1.8-01	2.7-02	1.1-03	4.7-07	2.9-13

These tables are organized as follows. The first triplet of rows reports the interpolation errors obtained using HD$_n(2,3)$ points, optimized HD$_n(p_x^{opt}, p_y^{opt})$ points (reported in Table 2) and AFPs (reported in Table 4) for $\alpha = 1$ in order to check the roundoff

error propagation process. Then the second and third triplets of rows report the same kind of values for $\alpha = 50$ and $\alpha = 100$, respectively. The values of the reported maximum errors are of the form

$$err^{\alpha}_{\ell,n} = \max_{(x_i,y_j)\in\Omega_h} |u^{(\alpha)}_{\ell}(x_i,y_j) - u^{(\alpha)}_{\ell,n}(x_i,y_j)|,$$

where Ω_h is the set of discrete points (x_i, y_j) from the unit disc satisfying $x_{i+1} - x_i = y_{i+1} - y_i = h = 2/101$.

Finally, two figures showing the asymptotic behaviour of $\log_{10}(\text{cond}(M_n))$ for the proposed Halton disc interpolation points are presented. Figure 2 shows $\log_{10}(\text{cond}(M_n))$ for values of $n = 100, 101, 102, ..., 3000$, where M_n are generated using $\text{HD}_n(2,3)$. The same dependence is shown in Figure 3 for the optimized sets $\text{HD}_n(p_x^{opt}, p_y^{opt})$ and AFPs computed for $n = 100 * 2^k, k = 0, 1, 2, ..., 7$.

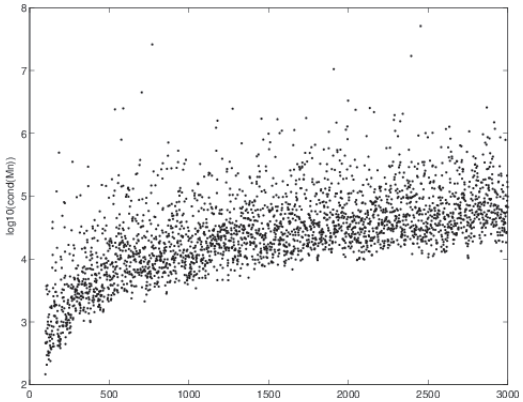

Figure 2: Dependence of $\log_{10}(\text{cond}(M_n))$ on n for the Halton disc points $\text{HD}_n(2,3)$ and $n = 100, 101, 102, ..., 3000$.

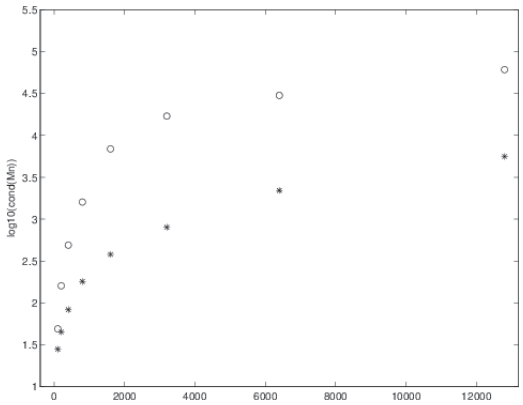

Figure 3: Dependence of $\log_{10}(\text{cond}(M_n))$ on n for the optimized Halton disc points $\text{HD}_n(px^{opt}, py^{opt})$ (marked by o), reported in Table 2, and AFPs (marked by *) reported in Table 4 for $n = 100 * 2^k, k = 0, 1, 2, ..., 7$.

These figures indicate that the function $\text{cond}(M_n)$ grows slowly and many sets of disc points $\text{HD}_n(p_x, p_y)$ for reasonably selected parameters p_x, p_y may create well-conditioned collocation matrices M_n suitable for practice also for $n > 12800$.

3.4 Acceleration of the search procedure

The complexity of the presented search procedure is dominated by repeated generation of the matrices M_n and computation of $\text{cond}(M_n)$ using the MATLAB function 'cond' requiring $O(n^3)$ arithmetic operations . Although the CPU times 0.5693 and 0.0014 seconds for computing M_{100} and $\text{cond}(M_{100})$, respectively, are negligible, the CPU times 1185.6 and 1961.1 seconds for computing M_{12800} and $\text{cond}(M_{12800})$ are horrible. Therefore, to find a high quality set of 12800 Halton disc interpolation points or more among a few hundreds or thousands of candidates $\text{HD}_n(p_x, p_y)$ is an unviable task for any personal computer.

To avoid this serious computational drawback, we can try to find the desired points $\text{HD}_n(p_x^{opt}, p_y^{opt})$ among the spherical points $\mathbf{Q}_i$ maximizing the sum

$$\sum_{1\le i<j\le n} \| \mathbf{Q}_i - \mathbf{Q}_j \|^{\alpha}, \tag{5}$$

where the symbol $\| . \|$ denotes the usual Euclidian norm in $\mathbb{R}^3$, $\mathbf{Q}_i$ are determined by $\text{H}_n(p_x, p_y)$ and $0 < \alpha < 2$. As noted by Stolarsky [40], this problem has been studied by various authors beginning probably with Pólya and Szegö in 1931. Due to the fact that the spherical points maximizing (5) are well-spaced [41], we may expect the corresponding disc interpolation points will generate well-conditioned collocation matrices. The computational experiments for $\alpha = 1$ reported in Table 8 show that

the points maximizing (5) create disc interpolation points giving collocation matrices with the smallest condition numbers (as presented in Table 2) only for $n = 100$, 200, and 3200. For other values of n the conditionally optimal point sets occur among a few (10, 20 or more) point sets giving the greatest values of the sum (5). This situation is presented in Table 8 for the remaining cases $n = 400$, 800, 1600, 6400, and 12800 and is described for the case $n = 12800$.

Let us denote by $\mathbf{V}$ the sequence of the values V_k of the sum (5) with $\alpha = 1$ computed for 1000 samples of 12800 spherical points. These points are generated by Halton points $\mathrm{H}_{12800}(2, p_y), p_y = 2k+1, k = 1, 2, ..., 1000$, placed in $(0, 0.5) \times (0, 1)$ and mapped on the upper unit hemisphere by the cylindrical Lambert map (1). The parameters p_y corresponding to 10 greatest values of the sequence $\mathbf{V}$ are reported in Table 8 together with the corresponding values of $\mathrm{cond}(M_{12800}(2, p_y))$. As seen in Table 8, the pair (2, 753) that generates spherical points giving the greatest value of $\mathbf{V}$ is different from the pair (2, 27) generating the collocation matrix $M_{12800}(2, 27)$ with the smallest condition number. In general, the point sample giving the largest value of the sum (5) is not usually identical to the sample giving the corresponding collocation matrix with the smallest condition number.

Table 8: The values of the parameter p_y corresponding to 10 greatest values of the sum (5) together with the values of $\mathrm{cond}(M_n(p_x, p_y))$ for n = 400, 800, 1600, 6400 and 12800 selected from 250, 500, 1000, 4000 and 1000 samples of spherical point sets, respectively.

p_y	200	400	398	402	50	396	100	134	198	**202**
$\mathrm{cond}(M_{400}(4, p_y))$	1.8+3	5.4+4	3.3+4	1.4+6	9.5+3	1.7+5	5.6+4	3.6+4	2.0+3	**4.9+2**
p_y	800	798	802	200	100	400	266	**796**	10	804
$\mathrm{cond}(M_{800}(3, p_y))$	3.1+3	2.0+3	6.3+3	8.4+3	6.0+3	4.8+4	5.2+4	**1.6+3**	8.2+3	8.2+3
p_y	1599	**1601**	1597	799	533	1603	801	11595	89	399
$\mathrm{cond}(M_{1600}(3, p_y))$	1.3+4	**1.2+4**	1.8+4	7.3+4	3.7+4	1.3+4	2.4+5	3.1+4	2.3+4	2.3+4
p_y	6399	**6401**	6397	711	2133	6403	6395	237	173	6405
$\mathrm{cond}(M_{6400}(2, p_y))$	2.0+5	**7.8+4**	6.2+5	5.1+5	1.3+5	1.0+5	1.7+5	9.0+4	1.6+5	2.6+5
p_y	753	1829	251	51	985	217	**27**	413	173	97
$\mathrm{cond}(M_{12800}(2, p_y))$	1.3+6	2.8+6	3.4+5	6.5+5	9.7+5	2.5+5	**6.1+4**	3.4+5	4.0+5	5.1+5

This means that the proposed search procedure is not able to find exactly what we are seeking. However, this procedure is able to identify set of a few point samples (e.g., 10, 20 or more) among those the desired sample generating collocation matrix with the smallest condition number may be found. Why 10 or 20 point samples are usually enough ? To answer this question we have to reorder the sequence $\mathbf{V}$ for $n = 800$ in decreasing order (see the middle plot in Figure 4) and use the same reordering to rearrange the sequence of the parameters p_y. The dependence of $\log_{10}(\mathrm{cond}(M_{800}(3, p_y)))$ on the rearranged sequence of p_y is shown in the right plot in Figure 4. In the left plot in Figure 4 dependence of the values of $\mathbf{V}$ on the values p_y before reordering, i.e., $p_y = 2k, k = 1, 2, ..., 500$, is shown.

Therefore, in virtue of the $\log_{10}(\mathrm{cond}(M_{800}(3, p_y)))$ behaviour we can consider point samples corresponding to 10 or 20 greatest values of (5) to be usually enough for all n. If not, the point set giving the smallest condition number among the selected 10 or 20 samples with respect to Stolarsky criterion (5) is good alternative to the optimal point set as seen comparing Table 2 and Table 8 for $n = 1600$ and 6400. Similar behaviour exhibits also the search procedure based on minimizing (5) for $\alpha = -1$.

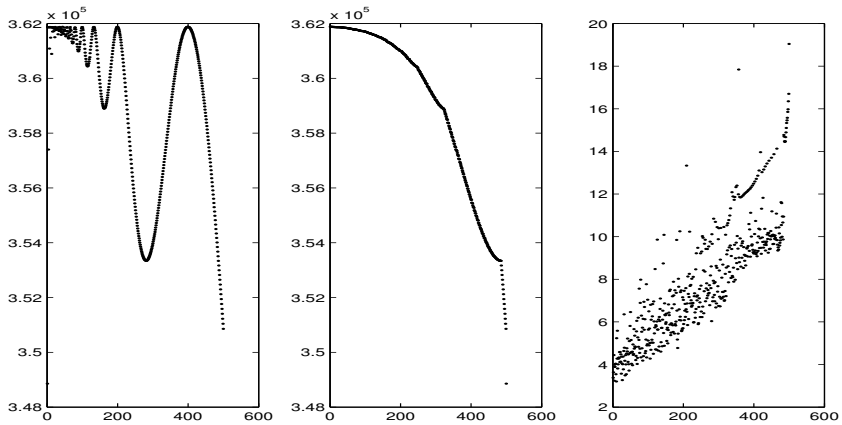

Figure 4: Illustrations (from left to right) of the values of the sum (5) for $n = 800$ and $p_y = 2k, 1 \le k \le 500$, the same values reordered in decreasing order, and the values $\log_{10}(\mathrm{cond}(M_{800}(3, p_y)))$ corresponding to the reordered values of p_y.

Now we can compare the CPU time needed to investigate one point sample with respect to the condition number criterion versus the Stolarsky criterion (5) for $n = 12800$. The achieved saving of CPU time can be seen comparing 3146.7 seconds needed for generating M_{12800} and computing $\mathrm{cond}(M_{12800})$ versus 7.5 seconds used for the computation of the sum (5). Consequently, the average speed-up in the investigation of one sample containing 12800 points is 420.

At the end of this paper the configurations of 12800 Halton disc interpolation points $\mathrm{HD}_{12800}(2, 753)$ (left) and $\mathrm{HD}_{12800}(2, 27)$ (right) generated in 0.0434 seconds for one set on a PC with the Intel Core i5-480M processor are presented in Figure 5.

4 Concluding remarks

Although the matrix conditioning and accuracy of the polynomial interpolation based on the presented Chebyshev-like Halton disc points are comparable with the outputs produced by AFPs, in spite of the use of the speed-up procedure presented above, the computational process needed to find the point set $\mathrm{HD}_n(p_x^{opt}, p_y^{opt})$ for very large n is still expensive and needs more efficient improvement. To minimize computational complexity of the method, it could be more advantageous to avoid the search procedure completely. The answer to the following question may be useful in this direction.

Is there a possibility to create a set of uniformly distributed points on the unit square for which the corresponding Chebyshev-like disc interpolation points will be comparable to AFPs for all n and a set of sampling parameters not dependent on n ?

Owing to the great variety of the sets of uniformly distributed points in the square, available in the theory and practice of the quasi-Monte Carlo methods, the positive answer to this question cannot be *a priori* rejected. For example, instead of the one-sequence construction of uniformly distributed points on the unit square, resulting in our case in the Halton square points, we can use hybrid sampling coupling together two different uniformly distributed sequences, e.g., Van der Corput, Weyl [16] and Zinterhoff [23] sequences. Moreover, we have at hand different sets of high-quality uniformly distributed points with blue-noise property generated by various techniques [48].

Finally, it would certainly be of interest to construct disc interpolation points for which $\Lambda_N, \mathrm{cond}(M_n)$ and CPU time needed to generate the desired points and corresponding collocation matrices M_n are minimized using one set of the disc algebraic polynomials - Zernike [10], Logan-Shepp [10] and Kornwinder ball polynomials [17].

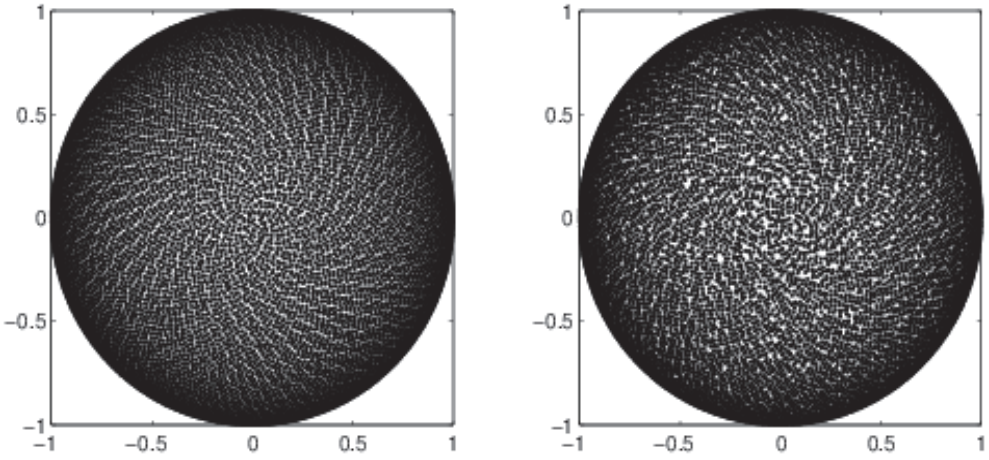

Figure 5: Configurations of the Halton disc interpolation points $\mathrm{HD}_{12800}(2,753)$ (left) and $\mathrm{HD}_{12800}(2,27)$ (right).

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